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Global existence and uniform boundedness of smooth solutions to a parabolic-parabolic chemotaxis system with nonlinear diffusion

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Abstract

This paper is devoted to the following quasilinear chemotaxis system: $\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(u\chi(v)\nabla v) + uf(u), & x\in\Omega, t>0, \\ v_t = \Delta v - ug(v), & x\in\Omega, t>0, \end{cases}$ under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. The given functions $D(s)$, $\chi(s)$, $g(s)$, and $f(s)$ are assumed to be sufficiently smooth for all $s \geq 0$ and such that $f(s) \leq \kappa - \mu s^{\tau}$. It is proved that the corresponding initial boundary value problem possesses a unique global classical solution for any $\mu > 0$ and $\tau \geq 1$, which is uniformly bounded in $\Omega \times (0, +\infty)$. Moreover, when $\kappa = 0$, the decay property of the solution is also discussed in this paper.

MSC: 35K55; 35Q92; 35Q35; 92C17

Keywords: global existence; uniform boundedness; nonlinear diffusion; chemotaxis

1 Introduction

In this paper, we consider the fully parabolic chemotaxis system:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(u\chi(v)\nabla v) + uf(u), & x\in\Omega, t>0, \\ v_t = \Delta v - ug(v), & x\in\Omega, t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t>0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. Equations (1.1) are a generalized version of the classical Keller-Segel model [1] and describe the directed movement of cells as a response to gradients of the concentration of a chemical signal substance (*e.g.*, oxygen) in the environment, where the chemical signal substance is consumed rather than produced by the cells themselves. In this case, $u = u(x,t)$ denotes the density of the cells, and $v = v(x,t)$ denotes the concentration of the oxygen, the function $D(u)$ describes the density-dependent motility of the cells, and the logistic source $uf(u)$ models proliferation and death of the cells, $\chi(v)$ and $g(v)$ denote a chemotactic sensitivity function and consumption rate of the oxygen by the cells, respectively. From a physical point of view, migration of the cells should be regarded as movement in a porous medium, and so we are led to consider the cell motility as a nonlinear function $D(u)$ of the cell density. Precisely, we will assume that the diffusion coefficient D satisfies

$$\begin{cases} D(s) \in C^2([0,\infty)), \qquad D(0) > 0, \\ D(s) \geq c_D s^{m-1} \quad \text{on } (0,+\infty) \end{cases} \tag{1.2}$$

with $m \geq 1$ and $c_D > 0$. Moreover, we assume that

$$\begin{cases} \chi(s) \in C^2([0,+\infty)) \text{ is nonnegative} \quad \text{and} \quad \chi'(s) \geq 0 \quad \text{on } [0,+\infty), \\ g(s) \in C^2([0,+\infty)) \text{ satisfies } g(0) = 0 \quad \text{and} \quad g(s) \geq 0 \quad \text{on } (0,+\infty). \end{cases} \tag{1.3}$$

As to the source term f we require that

$$\begin{cases} f \in C^1([0,\infty)), \\ f(s) \leq \kappa - \mu s^\tau \quad \text{with } \kappa \geq 0, \mu > 0, \tau \geq 1 \text{ on } [0,+\infty). \end{cases} \tag{1.4}$$

The initial data u_0, v_0 are supposed throughout this paper to satisfy

$$\begin{cases} u_0 \in W^{1,\theta}(\Omega) \quad \text{for some } \theta > N, u_0 \geq 0, x \in \Omega, \\ v_0 \in W^{1,\infty}(\Omega), \qquad v_0 \geq 0, \quad x \in \Omega. \end{cases} \tag{1.5}$$

To motivate our study, let us first recall the following related models to (1.1):

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u\nabla v) + uf(u), \quad x \in \Omega, t > 0, \\ v_t = \Delta v - \alpha_1 v + \beta_1 u, \quad x \in \Omega, t > 0, \end{cases} \tag{1.6}$$

where the chemical signal substance is produced by the cells themselves. This system has been widely studied during the past decades. It was known that in the linear diffusion case $D \equiv 1$ and $f(u) \equiv 0$, solutions may blow up in finite time when $N \geq 2$ [2–4]; however, in the case $f(u) \leq a - bu$, arbitrarily small $b > 0$ guarantee the global existence and boundedness of solutions when $N = 2$ [5], and that appropriately large $\frac{b}{\chi}$ preclude blow-up in the case $N \geq 3$ [6]. In the nonlinear case [7, 8] provide uniform-in-time boundedness of solutions in (1.6) under the condition that the logistic term $f(u)$ vanishes, and that in this respect the condition $m > 2 - \frac{2}{N}$ found in work [7] is optimal, because in [9] it has been shown that if $D(u) = (u+1)^{2-\frac{2}{N}-\varepsilon}$ for some $\varepsilon > 0, f(u) \equiv 0$, and Ω is a ball, then (1.7) possesses some unbounded solutions. Moreover, in [10], the authors consider a more general version of (1.6). For more related work one can refer to [11, 12]. Next we mention some results about the signal is consumed by the cells. The following chemotaxis-(Navier)-Stokes model which is a generalized version of the model proposed in[13], describes the motion of oxygen-driven swimming cells in an incompressible fluid, which is closely related to (1.1)

$$\begin{cases} u_t + w \cdot \nabla u = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u\chi(v)\nabla v), \quad x \in \Omega, t > 0, \\ v_t + w \cdot \nabla v = \Delta v - ug(v), \quad x \in \Omega, t > 0, \\ w_t + \kappa(w \cdot \nabla w) = \eta \Delta w - \nabla P + u\nabla\phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot w = 0. \end{cases} \tag{1.7}$$

Here, w denotes the velocity field of the fluid subject to an incompressible Navier-Stokes equation with pressure P and viscosity η, and a gravitational force $\nabla\phi$. In (1.7), both cells and oxygen are transported with the fluid. There are many results about the mathematical analysis on (1.7). See [14–19] *etc.* If the flow of the fluid is ignored (*i.e.*, $w \equiv 0$) or the fluid is stationary, then w is decoupled from (1.7), which yields

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u\chi(v)\nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v - ug(v), & x \in \Omega, t > 0. \end{cases} \tag{1.8}$$

When $D(u) \equiv 1$, the model (1.8) is originally proposed by Keller and Segel with $g(v) = v^\gamma$ and $\chi(v) = \frac{1}{v} > 0$ to describe the bacterial wave propagation, where the chemical is consumed by bacterial. For this model, many progresses are made in recent years, *e.g.*, see a review paper [20] and references therein. Moreover, when $g(v) = v$ and $\chi(v) = \text{const.} := \chi > 0$, Tao [21] proved that if $\|v_0\|_{L^\infty(\Omega)}$ is sufficiently small, then the corresponding initial boundary value problem possesses a unique global solution that is uniformly bounded. Moreover, the same result was obtained in [22] for the model with logistic source. For large initial data, Tao and Winkler [23] showed that the problem has a global weak solution which is eventually bounded and smooth under the assumptions that $\chi(v) \equiv 1$ and $\Omega \subset \mathbb{R}^3$ is a bounded convex domain. When $D(u) \geq c_D(u+1)^{m-1}$ with $m > 2 - \frac{2}{N}$, Wang *et al.* [24] showed that the corresponding initial boundary value problem possesses a unique global classical solution that is uniformly bounded provided that $\Omega \subset \mathbb{R}^N$ is a bounded convex domain and some other technical conditions are fulfilled. Recently, the result on global existence is relaxed to $m > 2 - \frac{6}{n+4}$ in [25], and the result on uniformly boundedness is relaxed to $m > 2 - \frac{n+2}{2n}$ in [26].

The aim of this paper is to study the global existence and boundedness of the solutions for the parabolic-parabolic chemotaxis system with linear or nonlinear diffusion and logistic source (1.1). Moreover, when $\kappa = 0$, *i.e.*, cells are *a priori* unable to reproduce themselves, such as broadcast spawning phenomena discussed in [27, 28], the decay property of the solution component u is also obtained. Our main result is stated as follows.

Theorem 1.1 *Let $m \geq 1$, $\tau \geq 1$, $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Suppose that $D(s)$, $\chi(s)$, $f(s)$, and $g(s)$ satisfy* (1.2)-(1.4). *Then for each (u_0, v_0) fulfilling* (1.5), *the problem* (1.1) *possesses a unique classical solution which is global in time and uniformly bounded in $\Omega \times (0,\infty)$. Furthermore, if $\kappa = 0$, then the component u of global classical solution has the following decay property*:

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \nearrow \infty. \tag{1.9}$$

Remark 1.1 In Theorem 1.1, the smallness of $\|v_0\|_{L^\infty(\Omega)}$ is canceled, which is vital in [22] and [21]. Moreover, the convexity of the domain Ω is also canceled, which is needed in [23, 24] *etc.*

2 Preliminaries

We first state the local existence of classical solutions of (1.1). The proof is based on an appropriate fixed point argument. One can refer to [18, 22, 29] *etc.* for more details.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with smooth boundary. Suppose that $D(s)$, $\chi(s)$, $f(s)$, and $g(s)$ satisfy* (1.2)-(1.4), *and assume that the initial data $(u_0, v_0) \in (W^{1,\theta}(\Omega))^2$ (for some $\theta > N$). Then the model* (1.1) *has a unique local-in-time nonnegative classical solution $(u,v) \in (C([0,T^*); W^{1,\theta}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0,T^*)))^2$. Here, T^* denotes the maximal existence time. Moreover, if $T^* < \infty$, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T^*. \tag{2.1}$$

The following L^1 estimate can easily be checked.

Lemma 2.2 *Assume that $v_0 \in W^{1,\infty}(\Omega)$. Then the solution (u,v) of* (1.1) *satisfies the following properties*:

$$\|u(\cdot,t)\|_{L^1(\Omega)} \leq M_1 := \max\left\{\|u_0\|_{L^1(\Omega)}, \left(\frac{\kappa}{\mu}\right)^{\frac{1}{\tau}} |\Omega|\right\} \quad \text{for all } t \in [0,T^*) \tag{2.2}$$

and

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in [0,T^*). \tag{2.3}$$

Proof Integrating the first equation in (1.1) over Ω, and utilizing the Hölder inequality, yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\,dx &\leq \kappa \int_\Omega u\,dx - \mu \int_\Omega u^{1+\tau}\,dx \\ &\leq \kappa \int_\Omega u\,dx - \frac{\mu}{|\Omega|^\tau}\left(\int_\Omega u\,dx\right)^{1+\tau} \quad \text{for all } t \in (0,T^*).\end{aligned}$$

Then $y(t) := \int_\Omega u\,dx$ satisfies

$$y'(t) \leq \kappa y(t) - \frac{\mu}{|\Omega|^\tau} y(t)^{1+\tau} \quad \text{for all } t \in (0,T^*).$$

Thus, a standard ODE argument implies that $y(t) \leq \max\{\|u_0\|_{L^1(\Omega)}, (\frac{\kappa}{\mu})^{\frac{1}{\tau}}|\Omega|\}$, for all $t \in (0,T^*)$. Since $g \geq 0$, (2.3) immediately results from the parabolic maximum principle applied to the second equation in (1.1). □

Next we establish an elementary inequality, which will be used in Lemma 3.2.

Lemma 2.3 *Let $\mu > 0$, $\tau \geq 1$, $\kappa \geq 0$, and $A > 0$. Then there exists a positive constant $C^* := C(\kappa,\mu,A)$ such that*

$$Az^2 + \kappa(1+z)\ln(1+z) - \mu z^{1+\tau}\ln(1+z) + (1+z)\ln(1+z) - z \leq C^* \quad \text{for all } z > 0. \tag{2.4}$$

Proof Define

$$\varphi(z) := Az^2 + \kappa(1+z)\ln(1+z) - \mu z^{1+\tau}\ln(1+z) + (1+z)\ln(1+z) - z.$$

Then it satisfies

$$\varphi(0) = 0 \quad \text{and} \quad \lim_{z\to\infty} \frac{\varphi(z)}{z^{1+\tau}\ln(1+z)} = -\mu.$$

Thus for some $z_0 > 0$ we have $\varphi < 0$ on (z_0, ∞). In view of the continuity of φ on $[0, \infty)$, we have

$$\varphi(z) \le \max_{z\in[0,z_0]} \varphi(z) \quad \text{for all } z > 0,$$

which implies (2.4). □

3 Proof of Theorem 1.1

3.1 Proof of global existence

In this section, we focus on proof of global existence in Theorem 1.1. Inspired by [30], we will first establish estimate on $\int_\Omega u^{m+1}\,dx + \int_\Omega |\nabla v|^4\,dx$ to obtain estimate on $\int_\Omega u^p\,dx$ for any $p > 1$. To achieve this purpose, we need a series of *a priori* estimates.

Lemma 3.1 *Let the same assumptions as in Theorem* 1.1 *hold. Then the solution of* (1.1) *satisfies*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p\,dx + \frac{p(p-1)c_D}{2}\int_\Omega u^{p+m-3}|\nabla u|^2\,dx \\
&\quad\le \frac{p(p-1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D}\int_\Omega u^{p-m+1}|\nabla v|^2\,dx \\
&\qquad + \kappa p\int_\Omega u^p\,dx - \mu p\int_\Omega u^{p+\tau}\,dx
\end{aligned} \tag{3.1}$$

for any $p > m$ and each $t \in (0, T^*)$. Particularly, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^{m+1}\,dx + \frac{m(m+1)c_D}{2}\int_\Omega u^{2m-2}|\nabla u|^2\,dx \\
&\quad\le \frac{m(m+1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D}\int_\Omega u^2|\nabla v|^2\,dx \\
&\qquad + \kappa(m+1)\int_\Omega u^{m+1}\,dx - \mu(m+1)\int_\Omega u^{m+1+\tau}\,dx.
\end{aligned} \tag{3.2}$$

Proof Multiplying $(1.1)_1$ by u^{p-1} and integrating the resulted equation over Ω, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx + (p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2\,dx \\
&\quad= (p-1)\int_\Omega u^{p-1}\chi(v)\nabla u\cdot\nabla v\,dx + \int_\Omega u^p f(u)\,dx.
\end{aligned} \tag{3.3}$$

We now estimate the last three terms of the above equality. Indeed, by (1.2) we have

$$(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2\,dx \ge (p-1)c_D\int_\Omega u^{p+m-3}|\nabla u|^2\,dx,$$

and by the Young inequality, we get

$$\begin{aligned}
&(p-1)\int_\Omega u^{p-1}\chi(v)\nabla u\cdot\nabla v\,dx\\
&\quad\le (p-1)\|\chi\|_{L^\infty(\Omega)}\int_\Omega u^{p-1}|\nabla u||\nabla v|\,dx\\
&\quad\le \frac{(p-1)c_D}{2}\int_\Omega u^{p+m-3}|\nabla u|^2\,dx+\frac{(p-1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D}\int_\Omega u^{p-m+1}|\nabla v|^2\,dx.
\end{aligned}$$

For the last term, in view of (1.5), we have

$$\int_\Omega u^p f(u)\,dx\le\kappa\int_\Omega u^p\,dx-\mu\int_\Omega u^{p+\tau}\,dx.$$

Summarily, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p\,dx+\frac{p(p-1)c_D}{2}\int_\Omega u^{p+m-3}|\nabla u|^2\,dx\\
&\quad\le\frac{p(p-1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D}\int_\Omega u^{p-m+1}|\nabla v|^2\,dx\\
&\qquad+\kappa p\int_\Omega u^p\,dx-\mu p\int_\Omega u^{p+\tau}\,dx.
\end{aligned}$$

Taking $p=m+1$, one can easily deduce (3.2). This completes the proof of Lemma 3.1. □

We then establish a coupled estimate on $\int_\Omega[(1+u)\ln(1+u)-u]\,dx+\int_\Omega|\nabla v|^2\,dx$.

Lemma 3.2 *Let the same assumptions as in Theorem* 1.1 *hold. Then there exists a positive constant C such that*

$$\int_\Omega\bigl[(1+u)\ln(1+u)-u\bigr]\,dx\le C\quad\textit{for all } t\in\bigl(0,T^*\bigr)\tag{3.4}$$

and

$$\int_\Omega|\nabla v|^2\,dx\le C\quad\textit{for all } t\in\bigl(0,T^*\bigr).\tag{3.5}$$

Proof We shall divide the proof into three steps.

Step 1. First testing $(1.1)_1$ against $\ln(1+u)$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\bigl[(1+u)\ln(1+u)-u\bigr]\,dx+\int_\Omega\frac{D(u)}{1+u}|\nabla u|^2\,dx\\
&\quad=\int_\Omega\frac{u}{1+u}\chi(v)\nabla u\cdot\nabla v\,dx+\int_\Omega uf(u)\ln(1+u)\,dx
\end{aligned}\tag{3.6}$$

for all $t\in(0,T^*)$. To estimate each term on the right side of (3.6), we first note that

$$\ln(1+u)\le u\quad\text{for all } u\ge 0.$$

Then utilizing the Young inequality and integration by parts, we get

$$\begin{aligned}
&\int_\Omega \frac{u}{1+u}\chi(v)\nabla u\cdot\nabla v\,dx\\
&\quad= \int_\Omega\bigl[\ln(1+u)-u\bigr]\chi(v)\Delta v\,dx + \int_\Omega\bigl[\ln(1+u)-u\bigr]\chi'(v)|\nabla v|^2\,dx\\
&\quad\le \frac{\|\chi\|^2_{L^\infty}}{2}\int_\Omega\bigl[\ln(1+u)-u\bigr]^2\,dx + \frac{1}{2}\int_\Omega|\Delta v|^2\,dx\\
&\quad\le \frac{\|\chi\|^2_{L^\infty}}{2}\int_\Omega u^2\,dx + \frac{1}{2}\int_\Omega|\Delta v|^2\,dx \quad \text{for all } t\in\bigl(0,T^*\bigr),
\end{aligned}\tag{3.7}$$

where we have used $\chi'(v)\ge 0$. For the last term on the right of (3.6), according to (1.4), we obtain

$$\begin{aligned}
&\int_\Omega uf(u)\ln(1+u)\,dx\\
&\quad\le \int_\Omega u\bigl(\kappa-\mu u^\tau\bigr)\ln(1+u)\,dx\\
&\quad= \int_\Omega\bigl[\kappa u\ln(1+u)-\mu u^{1+\tau}\ln(1+u)\bigr]\,dx \quad \text{for all } t\in\bigl(0,T^*\bigr).
\end{aligned}\tag{3.8}$$

Then combining (3.7) and (3.8) along with (3.6), we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\bigl[(1+u)\ln(1+u)-u\bigr]\,dx + \int_\Omega\frac{D(u)}{1+u}|\nabla u|^2\,dx\\
&\quad\le \frac{\|\chi\|^2_{L^\infty}}{2}\int_\Omega u^2\,dx + \frac{1}{2}\int_\Omega|\Delta v|^2\,dx\\
&\qquad+ \int_\Omega\bigl[\kappa u\ln(1+u)-\mu u^{1+\tau}\ln(1+u)\bigr]\,dx \quad \text{for all } t\in\bigl(0,T^*\bigr).
\end{aligned}\tag{3.9}$$

Step 2. In order to cancel $\frac{1}{2}\int_\Omega|\Delta v|^2\,dx$ on the right of (3.9), we first test $(1.1)_2$ against $-\Delta v$ to find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla v|^2\,dx ={}& -\int_\Omega|\Delta v|^2\,dx + \int_\Omega ug(v)\Delta v\,dx + \int_\Omega v\Delta v\,dx\\
&- \int_\Omega v\Delta v\,dx \quad \text{for all } t\in\bigl(0,T^*\bigr).
\end{aligned}$$

Then by integration by parts and the Young inequality, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla v|^2\,dx \le{}& -\int_\Omega|\Delta v|^2\,dx + \|g\|_{L^\infty}\int_\Omega u|\Delta v|\,dx\\
&- \int_\Omega|\nabla v|^2\,dx + \frac{\int_\Omega|\Delta v|^2\,dx}{4} + \int_\Omega|v|^2\,dx\\
\le{}& -\frac{\int_\Omega|\Delta v|^2\,dx}{2} + \|g\|^2_{L^\infty}\int_\Omega u^2\,dx - \int_\Omega|\nabla v|^2\,dx\\
&+ \|v_0\|^2_{L^\infty(\Omega)}|\Omega| \quad \text{for all } t\in\bigl(0,T^*\bigr),
\end{aligned}\tag{3.10}$$

where (2.3) has been used.

Step 3. Adding (3.10) to (3.9) yields

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega\left[(1+u)\ln(1+u)-u\right]dx+\frac{1}{2}\int_\Omega|\nabla v|^2\,dx\right\}\\&\qquad+\int_\Omega\frac{D(u)}{1+u}|\nabla u|^2\,dx+\int_\Omega|\nabla v|^2\,dx\\&\quad\le C_2\int_\Omega u^2\,dx+\int_\Omega\left[\kappa u\ln(1+u)-\mu u^{1+\tau}\ln(1+u)\right]dx\\&\qquad+\|v_0\|^2_{L^\infty(\Omega)}|\Omega|\quad\text{for all } t\in\left(0,T^*\right)\end{aligned}$$

with $C_2=\frac{\|\chi\|^2_{L^\infty(\Omega)}+2\|g\|^2_{L^\infty(\Omega)}}{2}$, which is a positive constant according to (1.3) and (2.3). Adding $\int_\Omega[(1+u)\ln(1+u)-u]\,dx$ to both sides of this and dropping the nonnegative term $\int_\Omega\frac{D(u)}{1+u}|\nabla u|^2\,dx$ on the left, we obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega\left[(1+u)\ln(1+u)-u\right]dx+\frac{1}{2}\int_\Omega|\nabla v|^2\,dx\right\}\\&\qquad+\int_\Omega|\nabla v|^2\,dx+\int_\Omega\left[(1+u)\ln(1+u)-u\right]dx\\&\quad\le C_2\int_\Omega u^2\,dx+\int_\Omega\left[(1+u)\ln(1+u)-u\right]dx\\&\qquad+\int_\Omega\left[\kappa u\ln(1+u)-\mu u^{1+\tau}\ln(1+u)\right]dx+\|v_0\|^2_{L^\infty(\Omega)}|\Omega|\end{aligned}\tag{3.11}$$

for all $t\in(0,T^*)$. In view of Lemma 2.3, there exists a positive constant C^* such that

$$\int_\Omega\left\{C_2u^2+\kappa u\ln(1+u)-\mu u^{1+\tau}\ln(1+u)+(1+u)\ln(1+u)-u\right\}dx\le C^*|\Omega|.$$

Thus $y(t):=\int_\Omega[(1+u)\ln(1+u)-u]\,dx+\frac{1}{2}\int_\Omega|\nabla v|^2\,dx$ satisfies

$$\frac{dy}{dt}+y\le C^*|\Omega|+\|v_0\|^2_{L^\infty(\Omega)}|\Omega|\quad\text{for all } t\in\left(0,T^*\right).$$

By standard ODE argument, we can derive

$$y(t)\le\max\left\{y(0),C^*|\Omega|+\|v_0\|^2_{L^\infty(\Omega)}|\Omega|\right\}\quad\text{for all } t\in\left(0,T^*\right),$$

which implies (3.4) and (3.5). □

In order to obtain the coupled estimate on $\int_\Omega u^{m+1}\,dx+\int_\Omega|\nabla v|^4\,dx$, we then derive the following energy estimate on $\int_\Omega|\nabla v|^4\,dx$.

Lemma 3.3 *Let the same assumptions as in Theorem* 1.1 *hold. Then there exists a positive constant C_1 such that the solution of* (1.1) *satisfies*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega|\nabla v|^4\,dx+\frac{1}{2}\int_\Omega\left|\nabla|\nabla v|^2\right|^2dx\\&\quad\le(4+N)\|g\|^2_{L^\infty(\Omega)}\int_\Omega u^2|\nabla v|^2\,dx+C_1\quad\textit{for all } t\in\left(0,T^*\right).\end{aligned}\tag{3.12}$$

Proof Differentiating equation $(1.1)_2$, we obtain

$$\left(|\nabla v|^2\right)_t = 2\nabla v\cdot\nabla\Delta v - 2\nabla v\cdot\nabla\big(ug(v)\big),$$

which, together with the point-wise identity $2\nabla v\cdot\nabla\Delta v = \Delta|\nabla v|^2 - 2|D^2v|^2$, yields

$$\left(|\nabla v|^2\right)_t = \Delta|\nabla v|^2 - 2\left|D^2v\right|^2 - 2\nabla v\cdot\nabla\big(ug(v)\big). \tag{3.13}$$

Multiplying both sides of (3.13) by $2|\nabla v|^2$ and integrating over Ω, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega|\nabla v|^4\,dx + 2\int_\Omega\left|\nabla|\nabla v|^2\right|^2dx + 4\int_\Omega|\nabla v|^2\left|D^2v\right|^2dx\\ &\quad= -4\int_\Omega|\nabla v|^2\nabla v\cdot\nabla\big(ug(v)\big)\,dx + 2\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}\,dx\quad\text{for all } t\in\left(0,T^*\right).\end{aligned}\tag{3.14}$$

For the first term on the right-hand side of (3.14), we can use integration by parts and the Young inequality to obtain

$$\begin{aligned}&-4\int_\Omega|\nabla v|^2\nabla v\cdot\nabla\big(ug(v)\big)\,dx\\ &\quad= 4\int_\Omega ug(v)|\nabla v|^2\Delta v\,dx + 4\int_\Omega ug(v)\nabla v\cdot\nabla|\nabla v|^2\,dx\\ &\quad\le \frac{4}{N}\int_\Omega|\nabla v|^2|\Delta v|^2\,dx + \int_\Omega\left|\nabla|\nabla v|^2\right|^2dx + (4+N)\int_\Omega u^2g^2(v)|\nabla v|^2\,dx\\ &\quad\le 4\int_\Omega|\nabla v|^2\left|D^2v\right|^2dx + \int_\Omega\left|\nabla|\nabla v|^2\right|^2dx + (4+N)\|g\|^2_{L^\infty(\Omega)}\int_\Omega u^2|\nabla v|^2\,dx,\end{aligned}\tag{3.15}$$

where we have used the fact $|\Delta v|^2\le N|D^2v|^2$. For the second term on the right-hand side of (3.14), thanks to the boundedness of $\int_\Omega|\nabla v|^2\,dx$, by the same procedure as (3.7)-(3.10) in [31], we deduce that there exists a positive constant C_1 such that

$$2\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}\,dx\le\frac12\int_\Omega\left|\nabla|\nabla v|^2\right|^2dx + C_1\quad\text{for all } t\in\left(0,T^*\right). \tag{3.16}$$

Combining (3.15) and (3.16) with (3.14), we obtain (3.12) immediately. □

Corollary 3.1 *Let the same assumptions as in Theorem* 1.1 *hold. Then for the same constant* C_1 *as in Lemma* 3.3, *the solution of* (1.1) *carries the property*

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega u^{m+1}\,dx + \int_\Omega|\nabla v|^4\,dx\right\} + \left\{\int_\Omega u^{m+1}\,dx + \int_\Omega|\nabla v|^4\,dx\right\}\\ &\qquad+\frac{m(m+1)c_D}{2}\int_\Omega u^{2m-2}|\nabla u|^2\,dx + \frac12\int_\Omega\left|\nabla|\nabla v|^2\right|^2dx\\ &\quad\le\left(\frac{m(m+1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D} + (4+N)\|g\|^2_{L^\infty(\Omega)}\right)\int_\Omega u^2|\nabla v|^2\,dx\\ &\qquad+\big(1+\kappa(m+1)\big)\int_\Omega u^{m+1}\,dx - \mu(m+1)\int_\Omega u^{m+1+\tau}\,dx + \int_\Omega|\nabla v|^4\,dx + C_1.\end{aligned}\tag{3.17}$$

Proof Adding (3.12) to (3.2) yields

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u^{m+1}\,dx+\int_\Omega|\nabla v|^4\,dx\right\}\\
&\qquad+\frac{m(m+1)c_D}{2}\int_\Omega u^{2m-2}|\nabla u|^2\,dx+\frac{1}{2}\int_\Omega\bigl|\nabla|\nabla v|^2\bigr|^2\,dx\\
&\quad\le\left(\frac{m(m+1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D}+(4+N)\|g\|^2_{L^\infty(\Omega)}\right)\int_\Omega u^2|\nabla v|^2\,dx\\
&\qquad+\kappa(m+1)\int_\Omega u^{m+1}\,dx-\mu(m+1)\int_\Omega u^{m+1+\tau}\,dx+C_1.
\end{aligned}\tag{3.18}$$

Adding $\int_\Omega u^{m+1}\,dx+\int_\Omega|\nabla v|^4\,dx$ to both sides of this, then yields (3.17). □

Now we are ready to establish the estimate on $\int_\Omega u^{m+1}\,dx+\int_\Omega|\nabla v|^4\,dx$.

Lemma 3.4 *Let $m>2-\frac{4}{N+2}$, $N\ge 1$. Then there exists a positive constant C such that*

$$\int_\Omega u^{m+1}\,dx\le C\quad\textit{for all } t\in(0,T^*)\tag{3.19}$$

and

$$\int_\Omega|\nabla v|^4\,dx\le C\quad\textit{for all } t\in(0,T^*).\tag{3.20}$$

Proof In order to obtain the estimate on the couple of $\int_\Omega u^{m+1}\,dx+\int_\Omega|\nabla v|^4\,dx$, we need to estimate the integrals on the right of inequality (3.17). We first utilize the Gagliardo-Nirenberg inequality and (3.5) to estimate $\int_\Omega|\nabla v|^4\,dx$:

$$\begin{aligned}
\int_\Omega|\nabla v|^4\,dx&=\bigl\||\nabla v|^2\bigr\|^2_{L^2(\Omega)}\\
&\le c_1\bigl\|\nabla|\nabla v|^2\bigr\|^{2\lambda}_{L^2(\Omega)}\bigl\||\nabla v|^2\bigr\|^{2(1-\lambda)}_{L^1(\Omega)}+c_1\bigl\||\nabla v|^2\bigr\|^2_{L^1(\Omega)}\\
&\le c_2\bigl\|\nabla|\nabla v|^2\bigr\|^{2\lambda}_{L^2(\Omega)}+c_2\\
&\le\frac{1}{4}\bigl\|\nabla|\nabla v|^2\bigr\|^2_{L^2(\Omega)}+c_3\quad\text{for all } t\in(0,T^*).
\end{aligned}\tag{3.21}$$

Here in the last inequality we have used the Young inequality, because $\lambda=\frac{N}{2+N}\in(0,1)$. Similarly, invoking the Young inequality with $\epsilon_2>0$ to estimate $\int_\Omega u^2|\nabla v|^2\,dx$, we have

$$\begin{aligned}
&\left(\frac{m(m+1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D}+(4+N)\|g\|^2_{L^\infty(\Omega)}\right)\int_\Omega u^2|\nabla v|^2\,dx\\
&\quad\le\epsilon_2\int_\Omega\bigl(|\nabla v|^2\bigr)^{\frac{2N+2}{N}}\,dx+\left(\frac{m(m+1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D}+(4+N)\|g\|^2_{L^\infty(\Omega)}\right)^{\frac{2N+2}{N+2}}\\
&\qquad\times\epsilon_2^{-\frac{N}{N+2}}\int_\Omega u^{\frac{4N+4}{N+2}}\,dx\quad\text{for all } t\in(0,T^*).
\end{aligned}\tag{3.22}$$

Further utilize the Gagliardo-Nirenberg inequality and (3.5) to estimate

$$\begin{aligned}\epsilon_2\int_\Omega\left(|\nabla v|^2\right)^{\frac{2N+2}{N}}dx&=\epsilon_2\left\||\nabla v|^2\right\|_{L^{\frac{2N+2}{N}}(\Omega)}^{\frac{2N+2}{N}}\\&\le\epsilon_2c_1\left\|\nabla|\nabla v|^2\right\|_{L^2(\Omega)}^2\left\||\nabla v|^2\right\|_{L^1(\Omega)}^{\frac{2}{N}}+c_1\left\||\nabla v|^2\right\|_{L^1(\Omega)}^{\frac{2N+2}{N}}\\&\le\epsilon_2c_2\left\|\nabla|\nabla v|^2\right\|_{L^2(\Omega)}^2+c_2\quad\text{for all }t\in\left(0,T^*\right).\end{aligned}\tag{3.23}$$

Since $m+\tau+1>\frac{4N+4}{N+2}$, utilizing the Young inequality we have

$$\begin{aligned}&\left(\frac{m(m+1)\|\chi\|_{L^\infty(\Omega)}^2}{2c_D}+(4+N)\|g\|_{L^\infty(\Omega)}^2\right)^{\frac{2N+2}{N+2}}\epsilon_2^{-\frac{N}{N+2}}\int_\Omega u^{\frac{4N+4}{N+2}}\,dx\\&\quad\le\frac{\mu(m+1)}{2}\int_\Omega u^{m+1+\tau}\,dx+c(\epsilon_2)\end{aligned}\tag{3.24}$$

for all $t\in(0,T^*)$. Taking $\epsilon_2=\frac{1}{4c_2}$, then from (3.22)-(3.24), we have

$$\begin{aligned}&\left(\frac{m(m+1)\|\chi\|_{L^\infty(\Omega)}^2}{2c_D}+(4+N)\|g\|_{L^\infty(\Omega)}^2\right)\int_\Omega u^2|\nabla v|^2\,dx\\&\quad\le\frac{1}{4}\left\|\nabla|\nabla v|^2\right\|_{L^2(\Omega)}^2+\frac{\mu(m+1)}{2}\int_\Omega u^{m+1+\tau}\,dx+C_2\end{aligned}\tag{3.25}$$

for all $t\in(0,T^*)$. For the third term $[\kappa(m+1)+1]\int_\Omega u^{m+1}\,dx$, utilizing the Young inequality we have

$$\left[\kappa(m+1)+1\right]\int_\Omega u^{m+1}\,dx\le\frac{\mu(m+1)}{2}\int_\Omega u^{m+1+\tau}\,dx+C_3\quad\text{for all }t\in\left(0,T^*\right).\tag{3.26}$$

Since $\frac{m(m+1)c_D}{2}\int_\Omega u^{2m-2}|\nabla u|^2\,dx$ is nonnegative, combining (3.24), (3.25), and (3.26) with (3.17) yields

$$\frac{d}{dt}\left\{\int_\Omega u^{m+1}\,dx+\int_\Omega|\nabla v|^4\,dx\right\}+\left\{\int_\Omega u^{m+1}\,dx+\int_\Omega|\nabla v|^4\,dx\right\}\le C_4\tag{3.27}$$

for all $t\in(0,T^*)$, where $C_4=C_1+C_2+C_3$. Thus an ODE comparison shows that $y(t):=\int_\Omega u^{m+1}\,dx+\int_\Omega|\nabla v|^4\,dx$ satisfies

$$y(t)\le\max\left\{y(0),C_4\right\}\quad\text{for all }t\in\left(0,T^*\right),\tag{3.28}$$

which implies (3.19) and (3.20). □

We note $m>1$ if $N=2$ in Lemma 3.4. However, if $m=1$, $N=2$, and $\tau=1$, *i.e.*, $m+\tau+1=\frac{4N+4}{N+2}=3$, the Young inequality will fail to lead to (3.24). In this case, we need to utilize the following generalization of the Gagliardo-Nirenberg inequality for the general case when $r>0$ (*cf.* [32], Lemma A.5, for a detailed proof), which extends the standard case when $r\ge1$ in [33], to build a bound for $\int_\Omega u^3\,dx$.

Lemma 3.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $p \in (1,\infty)$ and $r \in (0,p)$. Then there exists $C > 0$ such that for each $\eta > 0$ one can pick $C_\eta > 0$ with the property that*

$$\|u\|_{L^p(\Omega)}^p \le \eta \|\nabla u\|_{L^2(\Omega)}^{p-r} \big\| u \ln |u| \big\|_{L^r(\Omega)}^r + C\|u\|_{L^r(\Omega)}^p + C_\eta \tag{3.29}$$

holds for all $u \in W^{1,2}(\Omega)$.

Lemma 3.6 *Let $m = 1$, $N = 2$. Then there exists a positive constant C such that*

$$\int_\Omega u^2\,dx \le C \quad \textit{for all } t \in (0, T^*) \tag{3.30}$$

and

$$\int_\Omega |\nabla v|^4\,dx \le C \quad \textit{for all } t \in (0, T^*). \tag{3.31}$$

Proof By the same procedure as Lemma 3.4, we only need to handle

$$\left(\frac{m(m+1)\|\chi\|_{L^\infty(\Omega)}^2}{2c_D} + (4+N)\|g\|_{L^\infty(\Omega)}^2\right)^{\frac{2N+2}{N+2}} \epsilon_2^{-\frac{N}{N+2}} \int_\Omega u^{\frac{4N+4}{N+2}}\,dx$$

in (3.22). Invoking Lemma 3.5 along with (3.4) and Lemma 2.2, we deduce

$$\begin{aligned}
&\left(\frac{m(m+1)\|\chi\|_{L^\infty(\Omega)}^2}{2c_D} + (4+N)\|g\|_{L^\infty(\Omega)}^2\right)^{\frac{2N+2}{N+2}} \epsilon_2^{-\frac{N}{N+2}} \int_\Omega u^{\frac{4N+4}{N+2}}\,dx \\
&\quad = \left(\frac{\|\chi\|_{L^\infty(\Omega)}^2}{c_D} + 6\|g\|_{L^\infty(\Omega)}^2\right)^{\frac{3}{2}} \epsilon_2^{-\frac{1}{2}} \int_\Omega u^3\,dx \\
&\quad \le \left(\frac{\|\chi\|_{L^\infty(\Omega)}^2}{c_D} + 6\|g\|_{L^\infty(\Omega)}^2\right)^{\frac{3}{2}} \epsilon_2^{-\frac{1}{2}} \int_\Omega (u+1)^3\,dx \\
&\quad \le \left(\frac{\|\chi\|_{L^\infty(\Omega)}^2}{c_D} + 6\|g\|_{L^\infty(\Omega)}^2\right)^{\frac{3}{2}} \\
&\qquad \times \epsilon_2^{-\frac{1}{2}} \left[\epsilon_2 \|\nabla u\|_{L^2(\Omega)}^2 \big\|(u+1)\ln(u+1)\big\|_{L^1(\Omega)} + C\|u+1\|_{L^1(\Omega)}^3 + C_{\epsilon_2}\right] \\
&\quad \le c_3\sqrt{\epsilon_2}\|\nabla u\|_{L^2(\Omega)}^2 + c_4(\epsilon_2) \quad \text{for all } t \in (0, T^*).
\end{aligned} \tag{3.32}$$

Taking $\epsilon_2 \le \min\{\frac{1}{4c_2}, \frac{c_D^2}{4c_3^2}\}$, from (3.22), (3.23), and (3.32) we then deduce

$$\begin{aligned}
&\left(\frac{m(m+1)\|\chi\|_{L^\infty(\Omega)}^2}{2c_D} + (4+N)\|g\|_{L^\infty(\Omega)}^2\right) \int_\Omega u^2|\nabla v|^2\,dx \\
&\quad = \left(\frac{\|\chi\|_{L^\infty(\Omega)}^2}{c_D} + 6\|g\|_{L^\infty(\Omega)}^2\right) \int_\Omega u^2|\nabla v|^2\,dx \\
&\quad \le \frac{1}{4}\big\|\nabla|\nabla v|^2\big\|_{L^2(\Omega)}^2 + \frac{c_D}{2}\|\nabla u\|_{L^2(\Omega)}^2 + c_5 \quad \text{for all } t \in (0, T^*).
\end{aligned} \tag{3.33}$$

Combing (3.21), (3.26), (3.33), with (3.17) yields

$$\frac{d}{dt}\left\{\int_\Omega u^2\,dx + \int_\Omega |\nabla v|^4\,dx\right\} + \left\{\int_\Omega u^2\,dx + \int_\Omega |\nabla v|^4\,dx\right\} \le C_5 \tag{3.34}$$

for all $t \in (0, T^*)$, which implies (3.30) and (3.31). □

Lemma 3.4 and Lemma 3.6 result in the following useful corollary that will be used in the proof of Lemma 3.7 below.

Corollary 3.2 *Let $N = 2$, $m \ge 1$, and assume that the initial data (u_0, v_0) satisfies* (1.5). *Then there exists a positive constant C such that for any constant $k > 1$ there exists $C(k) > 0$ such that*

$$\int_\Omega |\nabla v|^k \le C(k) \quad \textit{for all } t \in \bigl(0, T^*\bigr). \tag{3.35}$$

Proof Since $m + 1 \ge N = 2$, then by (3.19), (3.20), (3.30), (3.31), and the standard parabolic regularity theory (*cf.* [12], Lemma 4.1 or [34], Lemma 1), we can immediately obtain (3.35). □

Lemma 3.7 *Let the same assumptions as in Theorem* 1.1 *hold. Then for any $p > 1$ there exists a positive constant $C(p)$ such that*

$$\int_\Omega u^p\,dx \le C(p) \quad \textit{for all } t \in \bigl(0, T^*\bigr). \tag{3.36}$$

Proof Let us recall to (3.1) once again. Adding $\int_\Omega u^p\,dx$ to the both sides of (3.1) and neglecting the nonnegative term $\frac{p(p-1)c_D}{2}\int_\Omega u^{p+m-3}|\nabla u|^2\,dx$ on the left, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p\,dx + \int_\Omega u^p\,dx \\
&\quad \le \frac{p(p-1)\|\chi\|^2_{L^\infty(\Omega)}}{2c_D}\int_\Omega u^{p-m+1}|\nabla v|^2\,dx \\
&\qquad + (\kappa p + 1)\int_\Omega u^p\,dx - \mu p\int_\Omega u^{p+\tau}\,dx
\end{aligned} \tag{3.37}$$

for any $p > m$ and each $t \in (0, T^*)$. Utilizing (3.35) and the Young inequality we have

$$\begin{aligned}
&\frac{p(p-1)\chi^2}{2c_D}\int_\Omega u^{p-m+1}|\nabla v|^2\,dx \\
&\quad \le \frac{\mu p}{2}\int_\Omega u^{p+\tau}\,dx + c_4\int_\Omega |\nabla v|^{\frac{2(p+\tau)}{\tau+m-1}} \\
&\quad \le \frac{\mu p}{2}\int_\Omega u^{p+\tau}\,dx + c_2 C\left(\frac{2(p+\tau)}{\tau+m-1}\right) \quad \text{for all } t \in \bigl(0, T^*\bigr),
\end{aligned} \tag{3.38}$$

with some $c_2 > 0$ and $C(\frac{2(p+\tau)}{\tau+m-1})$ defined by Corollary 3.2, as well as

$$[\kappa p + 1]\int_\Omega u^p\,dx \le \frac{\mu p}{2}\int_\Omega u^{p+\tau}\,dx + \tilde{C}_1 \quad \text{for all } t \in \bigl(0, T^*\bigr). \tag{3.39}$$

Collecting (3.37)-(3.39), we thus deduce that $y(t) := \int_\Omega u^p\,dx$ satisfies the differential inequality

$$\frac{d}{dt}\int_\Omega u^p\,dx + \int_\Omega u^p\,dx \le C_6 \quad \text{for all } t \in (0, T^*), \tag{3.40}$$

with $C_6 = c_2 C(\frac{2(p+\tau)}{\tau+m-1}) + \tilde{C}_1$. Upon an ODE comparison, this implies

$$y(t) \le \max\{y(0), C_6\} \quad \text{for all } t \in (0, T^*).$$

Thus (3.36) holds for any $p > m$. Since $\int_\Omega u\,dx \le M_1$, utilizing the interpolation inequality, (3.36) holds also for any $1 < p \le m$. This completes the proof of Lemma 3.7. □

Once the uniform estimate of $\|u(\cdot,t)\|_{L^k(\Omega)}$ has been established, we can use the classical Alikakos iteration method to obtain the uniform bound of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$.

Lemma 3.8 *Let the same assumptions as in Theorem* 1.1 *hold. Then there exists a positive constant C such that the solution component u of* (1.1) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t \in (0, T^*). \tag{3.41}$$

We are now in the position to prove the global existence in Theorem 1.1.

Proof of global existence The existence of global classical solution to equations (1.1) is an immediate consequence of Lemma 3.8 and the extensibility criterion (2.1). □

3.2 Proof of decay property in Theorem 1.1

In this short subsection, we discuss the decay property in the limit case $\kappa = 0$. Our approach is inspired by that in [35].

Lemma 3.9 *Suppose that* $f(s)$ *satisfies* (1.4) *with* $\kappa = 0$. *Then*

$$\int_0^\infty \int_\Omega u^{\tau+1}\,dx\,dt < \frac{1}{\mu}\|u_0\|_{L^1(\Omega)} \tag{3.42}$$

and

$$\int_\Omega u(x,t)\,dx \le \|u_0\|_{L^1(\Omega)}|\Omega|\left(|\Omega|^\tau + \mu\tau t\|u_0\|^\tau_{L^1(\Omega)}\right)^{-\frac{1}{\tau}} \quad \textit{for all } t > 0. \tag{3.43}$$

Proof Integrating (1.1) in space we obtain under the assumption $\kappa = 0$,

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega uf(u)\,dx \le -\mu\int_\Omega u^{\tau+1}\,dx \le -\frac{\mu}{|\Omega|^\tau}\left(\int_\Omega u\,dx\right)^{1+\tau} \quad \text{for all } t > 0,$$

which implies both (3.42) and (3.43). □

Lemma 3.10 *Let* $\kappa = 0$, $\mu > \frac{\|u_0\|_{L^1(\Omega)}}{(\tau+1)\|v_0\|_{L^1(\Omega)}}$, *and assume that*

$$\int_0^\infty \int_\Omega g(x,t)^{\frac{\tau+1}{\tau}}\,dx < \frac{\tau+1}{\tau}\left(\|v_0\|_{L^1(\Omega)} - \frac{1}{(\tau+1)\mu}\|u_0\|_{L^1(\Omega)}\right). \tag{3.44}$$

Then there exists a positive constant C appropriately small such that

$$\int_\Omega v(x,t)\,dx \ge C \quad \textit{for all } t>0. \tag{3.45}$$

Proof Integrating the second equation in (1.1) in space and applying the Young inequality, yield

$$\frac{d}{dt}\int_\Omega v\,dx = -\int_\Omega ug(v)\,dx \ge -\frac{1}{\tau+1}\int_\Omega u^{\tau+1}\,dx - \frac{\tau}{\tau+1}\int_\Omega g(v)^{\frac{\tau+1}{\tau}}\,dx. \tag{3.46}$$

In view of (3.42), (3.46) implies

$$\int_\Omega v\,dx \ge \|v_0\|_{L^1(\Omega)} - \frac{1}{(\tau+1)\mu}\|u_0\|_{L^1(\Omega)} - \frac{\tau}{\tau+1}\int_0^\infty\int_\Omega g(v)^{\frac{\tau+1}{\tau}}\,dx \quad \text{for all } t>0. \tag{3.47}$$

Therefore, (3.44) asserts the positivity of the right-hand side of (3.47). We thus obtain the desired result (3.45). □

Lemma 3.11 *There exist* $\alpha\in(0,1)$ *and* $C>0$ *such that*

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} \le C \quad \textit{for all } t>0. \tag{3.48}$$

Proof Rewriting the first equation of (1.1) in the form

$$u_t = \nabla\cdot\big(D(u)\nabla u - u\chi(v)\nabla v\big) + uf(u), \quad x\in\Omega, t>0. \tag{3.49}$$

Utilizing the Young inequality we can estimate

$$\big(D(u)\nabla u - u\chi(v)\nabla v\big)\cdot\nabla u \ge \frac{D(u)}{2}|\nabla u|^2 - \frac{u^2\chi^2(v)|\nabla v|^2}{2D(u)}$$

and evidently

$$\big|D(u)\nabla u - u\chi(v)\nabla v\big| \le D(u)|\nabla u| + u\chi(v)|\nabla v|$$

in $\Omega\times(0,\infty)$. As Lemma 3.8 and Corollary 3.2 imply u and ∇v are bounded in $L^\infty((0,\infty);L^k(\Omega))$ for any $k\in(1,\infty)$, and that u is a bounded solution of (3.49), the Hölder continuity of u, *i.e.*, (3.48), immediately results from a known result on parabolic Hölder regularity ([36], Theorem 1.3). □

Proof of decay property in Theorem 1.1 In view of the Hölder continuity of u, *i.e.*, (3.48), Arzelà-Ascoli theorem asserts $(u(\cdot,t))_{t>1}$ is relatively compact in $C^0(\bar\Omega)$. Thus, the decay property (3.43) implies that $\|u(\cdot,t)\|_{L^\infty(\Omega)}\to 0$ as $t\to\infty$. □

Remark 3.1 The decay property in Theorem 1.1 reveals the fact that if the proliferation of cells is ignored, then the cells will not survive. However, Lemma 3.10 shows that if the death rate of the cells is large and the consumption rate of oxygen is small enough, then there is always oxygen remaining.

Since the global existence and decay property both have been proved, we then have completed the proof of Theorem 1.1.

Competing interests
The author declares to have no competing interests.

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References

1. Keller, EF, Segel, LA: Initiation of slime mold aggregation viewed as an instability. J. Theor. Biol. **26**, 399-415 (1970)
2. Herrero, MA, Velázquez, JJL: A blow-up mechanism for a chemotaxis model. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) **24**, 633-683 (1997)
3. Nagai, T: Blow-up of nonradial solutions to parabolic-elliptic systems modelling chemotaxis in two-dimensional domains. J. Inequal. Appl. **6**, 37-55 (2001)
4. Winkler, M: Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system. J. Math. Pures Appl. **100**, 748-767 (2013). arXiv:1112.4156v1
5. Osaki, K, Tsujikawa, T, Yagi, A, Mimura, M: Exponential attractor for a chemotaxis-growth system of equations. Nonlinear Anal., Theory Methods Appl. **51**, 119-144 (2002)
6. Winkler, M: Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source. Commun. Partial Differ. Equ. **35**, 1516-1537 (2010)
7. Tao, Y, Winkler, M: Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity. J. Differ. Equ. **252**, 692-715 (2012)
8. Sugiyama, Y: Time global existence and asymptotic behavior of solutions to degenerate quasilinear parabolic systems of chemotaxis. Differ. Integral Equ. **20**, 133-180 (2007)
9. Winkler, M: Does a 'volume-filling effect' always prevent chemotactic collapse? Math. Methods Appl. Sci. **33**, 12-24 (2010)
10. Li, X, Xiang, Z: Boundedness in quasilinear Keller-Segel equations with nonlinear sensitivity and logistic source. Discrete Contin. Dyn. Syst., Ser. A **35**(8), 3503-3531 (2015). doi:10.3934/dcds.2015.35.3503
11. Hillen, T, Painter, KJ: A user's guide to PDE models for chemotaxis. J. Math. Biol. **58**, 183-217 (2009)
12. Horstmann, D, Winkler, M: Boundedness vs. blow-up in a chemotaxis system. J. Differ. Equ. **215**, 52-107 (2005)
13. Tuval, I, Cisneros, L, Dombrowski, C, Wolgemuth, CW, Kessler, JO, Goldstein, RE: Bacterial swimming and oxygen transport near contact lines. Proc. Natl. Acad. Sci. USA **102**, 2277-2282 (2005)
14. Duan, R, Xiang, Z: A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion. Int. Math. Res. Not. **2014**, 1833-1852 (2014)
15. Francesco, M, Lorz, A, Markowich, PA: Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior. Discrete Contin. Dyn. Syst., Ser. A **28**, 1437-1453 (2010)
16. Liu, JG, Lorz, A: A coupled chemotaxis-fluid model: global existence. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **28**, 643-652 (2011)
17. Lorz, A: Coupled chemotaxis fluid model. Math. Models Methods Appl. Sci. **20**, 987-1004 (2010)
18. Tao, Y, Winkler, M: Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion. Discrete Contin. Dyn. Syst., Ser. A **32**(5), 1901-1914 (2012)
19. Tao, Y, Winkler, M: Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **30**, 157-178 (2013)
20. Wang, ZA: Mathematics of traveling waves in chemotaxis. Discrete Contin. Dyn. Syst., Ser. B **18**(3), 601-641 (2013)
21. Tao, Y: Boundedness in a chemotaxis model with oxygen consumption by bacteria. J. Math. Anal. Appl. **381**, 521-529 (2011)
22. Wang, L, Khan, SUD, Khan, SUD: Boundedness in a chemotaxis system with consumption of chemoattractant and logistic source. Electron. J. Differ. Equ. **2013**, 209 (2013)
23. Tao, Y, Winkler, M: Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant. J. Differ. Equ. **252**, 2520-2543 (2012)
24. Wang, L, Mu, C, Zhou, S: Boundedness in a parabolic-parabolic chemotaxis system with nonlinear diffusion. Z. Angew. Math. Phys. **65**(6), 1137-1152 (2014)
25. Wang, L, Mu, C, Lin, K, Zhao, J: Global existence to a higher-dimensional quasilinear chemotaxis system with consumption of chemoattractant. Z. Angew. Math. Phys. (2015). doi:10.1007/s00033-014-0491-9
26. Wang, Y, Xiang, Z: Global existence and boundedness in a higher-dimensional quasilinear chemotaxis system. Preprint
27. Espejo, E, Suzuki, T: Reaction terms avoiding aggregation in slow fluids. Nonlinear Anal., Real World Appl. **21**, 110-126 (2015)
28. Kiselev, A, Ryzhik, L: Biomixing by chemotaxis and enhancement of biological reactions. Commun. Partial Differ. Equ. **37**, 298-312 (2012)
29. Tao, Y, Winkler, M: A chmotaxis-haptotaxis model: the roles of nonlinear diffusion and logistic source. SIAM J. Math. Anal. **43**, 685-704 (2011)
30. Liu, D, Tao, Y: Global boundedness in a fully parabolic attraction-repulsion chemotaxis model. Math. Methods Appl. Sci. (2014). doi:10.1002/mma.3240
31. Ishida, S, Seki, K, Yokota, T: Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains. J. Differ. Equ. **256**, 2993-3010 (2014)

32. Tao, Y, Winkler, M: Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant. J. Differ. Equ. **257**, 784-815 (2014)
33. Biler, P, Hebisch, W, Nadzieja, T: The Debye system: existence and large time behavior of solutions. Nonlinear Anal. **23**, 1189-1209 (1994)
34. Kowalczyk, R, Szymanska, Z: On the global existence of solutions to an aggregation model. J. Math. Anal. Appl. **343**, 379-398 (2008)
35. Tao, Y, Winkler, M: Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system. Z. Angew. Math. Phys. (2015). doi:10.1007/s00033-015-0541-y
36. Porzio, MM, Vespri, V: Holder estimates for local solutions of some doubly nonlinear degenerate parabolic equations. J. Differ. Equ. **103**(1), 146-178 (1993)