Došlá and Marini *Boundary Value Problems* (2015) 2015:95
DOI 10.1186/s13661-015-0355-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Positive decaying solutions for differential equations with phi-Laplacian

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Abstract

We solve a nonlocal boundary value problem on the half-close interval $[1,\infty)$ associated to the differential equation $(a(t)|x'|^\alpha \operatorname{sgn} x')' + b(t)|x|^\beta \operatorname{sgn} x = 0$, in the superlinear case $\alpha < \beta$. By using a new approach, based on a special energy-type function E, the existence of slowly decaying solutions is examined too.

MSC: 34B40; 34B15; 34B18

Keywords: nonlinear boundary value problem; globally positive solution; decaying solution; oscillatory solution

1 Introduction

Consider the Emden-Fowler type differential equation

$$\left(t^{2\alpha}\left|x'\right|^\alpha \operatorname{sgn} x'\right)' + b(t)|x|^\beta \operatorname{sgn} x = 0, \tag{1}$$

where $0 < \alpha < \beta$ and b is a positive continuous function on $[1,\infty)$ satisfying

$$J_b = \int_1^\infty b(s)\,ds = \infty. \tag{2}$$

Jointly with (1) consider also the more general equation

$$\left(a(t)\left|x'\right|^\alpha \operatorname{sgn} x'\right)' + b(t)|x|^\beta \operatorname{sgn} x = 0, \tag{3}$$

where a is a positive continuous function on $[1,\infty)$ such that

$$J_a = \int_1^\infty a^{-1/\alpha}(s)\,ds < \infty. \tag{4}$$

Equations (1) and (3) arise in the study of radially symmetric solutions of elliptic differential equations with phi-Laplacian operator in $\mathbb{R}^3$; see, *e.g.*, [1, 2].

By a solution of (3) we mean a differentiable function x on an interval $I_x \subseteq [1,\infty)$, such that $a(\cdot)|x'(\cdot)|^\alpha$ is continuously differentiable and satisfies (3) on I_x. In addition, x is called *local solution* if I_x is bounded and *proper solution* if I_x is unbounded and $\sup\{|x(t)| : t \geq T\} > 0$ for any large $T \geq 1$. As usual, a proper solution of (3) is said to be *oscillatory* if it has a sequence of zeros tending to infinity, otherwise it is said to be *nonoscillatory*. Equation (3) is said to be *oscillatory* if any its proper solution is oscillatory.

Define

$$A(t) = \int_t^\infty a^{-1/\alpha}(s)\,ds.$$

Let $\mathbb{P}$ be the class of eventually positive proper solutions x of (3). In view of (4), the class $\mathbb{P}$ can be divided into three subclasses, according to the asymptotic behavior of x as $t \to \infty$, see, *e.g.*, [3], Lemma 1.1. More precisely, any proper solution $x \in \mathbb{P}$ satisfies one of the following asymptotic properties:

$$\lim_{t\to\infty} x(t) = \ell_x, \quad 0 < \ell_x < \infty, \tag{5}$$

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} \frac{x(t)}{A(t)} = \infty, \tag{6}$$

$$\lim_{t\to\infty} \frac{x(t)}{A(t)} = \ell_x, \quad 0 < \ell_x < \infty, \tag{7}$$

where ℓ_x is a positive constant depending on x.

Let $x, y \in \mathbb{P}$ satisfy (6), (7), respectively. Then x, y tend to zero as $t \to \infty$ and $0 < y(t) < x(t)$ for large t. Hence, proper solutions of (3) satisfying (6) are called *slowly decaying solutions*, and proper solutions satisfying (7) *strongly decaying solutions*.

Here, we consider the nonlocal BVP on the half-line $[1,\infty)$

$$\begin{cases} (a(t)|x'|^\alpha \operatorname{sgn} x')' + b(t)|x|^\beta \operatorname{sgn} x = 0 \quad \alpha < \beta, \\ x(t) > 0, \qquad x'(t) < 0, \\ \lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} x(t)/A(t) = \infty. \end{cases} \tag{8}$$

Using a suitable change of variable and certain monotonicity properties of a energy-type function E, we prove that (8) has infinitely many solutions. Consequently, we get also a global multiplicity existence result for slowly decaying solutions of (3), which are positive decreasing on the whole interval $[1,\infty)$. We recall that in the superlinear case $\alpha < \beta$, sufficient conditions for existence of slowly decaying solutions are difficult to establish, due to the problem to find sharp upper and lower bounds; see, *e.g.*, [4], p.241, [5], p.3.

Observe that necessary and sufficient conditions for existence of solutions of (3), which satisfy (5) or (7), can easily be produced; see, *e.g.*, [3, 6] or [7], Section 14. Moreover, the same is true for slowly decaying solutions in the sublinear case $\alpha > \beta$; see, *e.g.*, [2, 8]. In the opposite situation, that is, in the superlinear case $\alpha < \beta$, in spite of many examples of equations of type (3) having solutions of type (6), which can be easily produced, until now no general sufficient conditions for their existence are known.

The paper is completed by the solvability of a special BVP, in which also the initial starting point is fixed. Moreover, some examples and suggestions for future research complete the paper.

Our results are also motivated by the papers [9, 10], in which the special case $\alpha = \beta$ is considered. More precisely in [9, 10] necessary and sufficient conditions for existence of slowly decaying solutions for the half-linear equation

$$\big(a(t)\big|y'\big|^{\alpha}\operatorname{sgn} y'\big)' + b(t)|y|^{\alpha}\operatorname{sgn} y = 0, \tag{9}$$

are established, according to $\alpha < 1$ or $\alpha > 1$, respectively.

Recently, BVPs on infinite intervals, associated to equations with phi-Laplacian have been considered in [11, 12]. The case of nonlocal BVPs for the generalized Laplacian, has been studied, *e.g.*, in [1, 13]. Finally, we refer the reader to [4, 14] for other references on this topic.

2 Preliminaries

We start with a change of the independent variable in (3), which will be useful.

Lemma 1 *Consider the transformation*

$$s(t) = \left(\int_t^{\infty} \frac{1}{a^{1/\alpha}(\tau)}\,d\tau\right)^{-1}, \qquad u(s) = x\big(t(s)\big), \tag{10}$$

where $t(s)$ is the inverse of the function $s(t)$. Then x is a solution of the BVP (8) *if and only if u is a solution of*

$$\frac{d}{ds}\big(s^{2\alpha}\big|\dot{u}(s)\big|^{\alpha}\operatorname{sgn}\dot{u}\big) + \frac{a^{1/\alpha}(s)b(s)}{s^2}|u|^{\beta}\operatorname{sgn} u = 0 \tag{11}$$

on $[s_1, \infty)$, $s_1 = s(1) > 0$, and satisfies for $s \geq s_1$

$$u(s) > 0, \qquad \dot{u}(s) < 0, \qquad \lim_{s\to\infty} u(s) = 0, \qquad \lim_{s\to\infty} su(s) = \infty,$$

where $\cdot$ denotes the derivative with respect to s.

Moreover, condition (4) *is satisfied for* (11).

Proof Using (10), we have

$$x'(t) = \dot{u}\frac{ds}{dt} = \frac{s^2}{a^{1/\alpha}(t)}\dot{u}(s) \tag{12}$$

and (3) is transformed into the equation

$$\frac{d}{ds}\big(\big|\dot{u}(s)\big|^{\alpha}\operatorname{sgn}\dot{u}\big) + \frac{2\alpha}{s}\big|\dot{u}(s)\big|^{\alpha}\operatorname{sgn}\dot{u} + \frac{a^{1/\alpha}(s)b(s)}{s^{2(\alpha+1)}}|u|^{\beta}\operatorname{sgn} u = 0, \tag{13}$$

see also [15], p.946, with minor changes. Multiplying (13) by $s^{2\alpha}$, we obtain (11). Moreover, for (11) assumptions (2) and (4) are satisfied because

$$J_a = \int_{s_1}^{\infty} \frac{1}{s^2}\,ds < \infty,$$

and

$$\int_{s_1}^{\infty} \frac{a^{1/\alpha}(s)b(s)}{s^2}\,ds = \int_1^{\infty} b(t)\,dt = \infty.$$

□

Thus, in view of Lemma 1, in the sequel we will study the existence of solutions x of (1) which satisfy on $[1,\infty)$ the boundary conditions

$$x(t) > 0, \qquad x'(t) < 0,$$
$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} x(t)/A(t) = \lim_{t\to\infty} tx(t) = \infty.$$

If x is a solution of (1), then denote by $x^{[1]}$ its quasiderivative, that is,

$$x^{[1]}(t) = t^{2\alpha}\left|x'(t)\right|^{\alpha} \operatorname{sgn} x'(t). \tag{14}$$

Moreover, set

$$Z = \int_1^{\infty} s^{-2}\left(\int_1^s b(r)\,dr\right)^{1/\alpha} ds, \qquad Y = \int_1^{\infty} s^{-\beta} b(s)\,ds,$$

and

$$B(t) = \int_1^t b(s)\,ds + c, \tag{15}$$

where c is an arbitrary positive constant. Hence $B(t) > 0$ for $t \geq 1$.

The following result is needed in the following.

Lemma 2

(i) *Equation* (1) *has solutions* $x \in \mathbb{P}$ *which satisfy* (5), *if and only if* $Z < \infty$.

(ii) *Equation* (1) *has solutions* $x \in \mathbb{P}$ *which satisfy* $\lim_{t\to\infty} tx(t) = \ell_x$, $0 < \ell_x < \infty$ *if and only if* $Y < \infty$. *Moreover, for any* ℓ_x, $0 < \ell_x < \infty$, *there exists* $x \in \mathbb{P}$ *such that* $\lim_{t\to\infty} tx(t) = \ell_x$.

(iii) *Equation* (1) *is oscillatory if and only if* $Y = \infty$.

Proof Claims (i) and (ii) follow from [3], Theorems 1.1, 1.2, with minor changes. Claim (iii) follows from [3], Theorem 2.2. □

3 The main result

Our main result deals with the solvability of the nonlocal BVP

$$\begin{cases} (t^{2\alpha}|x'|^{\alpha} \operatorname{sgn} x')' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad t \geq 1, \alpha < \beta, \\ x(t) > 0, \qquad x'(t) < 0, \\ \lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} tx(t) = \infty, \end{cases} \tag{BVP}$$

under the additional assumption

$$b \in C^1[1,\infty). \tag{16}$$

Remark 1 In the superlinear case $\alpha < \beta$, in virtue of (16), any local solution of (1) is a solution, *i.e.* it is continuable to infinity and is proper; see, *e.g.*, [16], Theorem 3.2, or [15], Appendix A. Notice also that, under the weaker assumption $b(t) \geq 0$, $\sup\{b(t) : t \geq T\} > 0$

for any $T \geq 1$, there may exist equations of type (1) with uncontinuable solutions; see, *e.g.*, [16], p.343.

The following holds.

Theorem 1 *Assume* $Z = \infty$. *If* (16) *is satisfied and the function*

$$G(t) = \frac{1}{t^2 b(t)} B^{\gamma}(t) \quad \textit{is nonincreasing for } t \geq 1, \tag{17}$$

where

$$\gamma = \frac{1 + \alpha\beta + 2\alpha}{\alpha(\beta + 1)}, \tag{18}$$

then (BVP) *has infinitely many solutions*.

To prove this result, several auxiliary results are needed. Define for any solution x of (1) the energy-type function

$$E_x(t) = B(t)\left|x(t)\right|^{\beta+1} + x(t)x^{[1]}(t) + k\frac{t^{2\alpha}}{b(t)}B(t)\left|x'(t)\right|^{\alpha+1}, \tag{19}$$

where the quasiderivate $x^{[1]}$ is defined in (14) and

$$k = \frac{\alpha(\beta + 1)}{\alpha + 1}. \tag{20}$$

Lemma 3 *For any solution* x *of* (1) *we have*

$$\left|x'(t)\right|^{\alpha} x''(t) = \frac{1}{\alpha}\frac{|x'(t)|}{t^{2\alpha}}\frac{d}{dt}x^{[1]}(t) - 2t^{-1}\left|x'(t)\right|^{\alpha+1}\operatorname{sgn} x'(t). \tag{21}$$

Proof Since $x^{[1]}$ is continuously differentiable on $[t_x, \infty)$, $t_x \geq 1$, and $t^{-2\alpha}x^{[1]}(t) = |x'(t)|^{\alpha} \times \operatorname{sgn} x'(t)$, the function x' is continuously differentiable on $[t_x, \infty)$ as well. If $x'(t) = 0$, then the identity (21) is valid. Now, assume $x'(t) \neq 0$. We have

$$\begin{aligned}
\left|x'(t)\right|\frac{d}{dt}x^{[1]}(t) &= \left|x'(t)\right|\left(2\alpha t^{2\alpha-1}\left|x'(t)\right|^{\alpha}\operatorname{sgn} x'(t) + \alpha t^{2\alpha}\left|x'(t)\right|^{\alpha-1}x''(t)\right) \\
&= 2\alpha t^{2\alpha-1}\left|x'(t)\right|^{\alpha+1}\operatorname{sgn} x'(t) + \alpha t^{2\alpha}\left|x'(t)\right|^{\alpha}x''(t),
\end{aligned}$$

from which the assertion follows. □

Lemma 4 *Assume* (16) *and* (17). *Then for any solution* x *of* (1) *we have for* $t \geq 1$

$$\frac{d}{dt}E_x(t) \leq 0.$$

Proof Let φ be a continuously differentiable function on $[1, \infty)$. Then for any positive constant σ the function $|\varphi(t)|^{\sigma+1}$ is continuously differentiable and

$$\frac{d}{dt}\left|\varphi(t)\right|^{\sigma+1} = (\sigma + 1)\left|\varphi(t)\right|^{\sigma}\varphi'(t)\operatorname{sgn}\varphi(t).$$

Using this equality, we have for $t \geq 1$

$$\begin{aligned}\frac{d}{dt}\left(\frac{t^{2\alpha}}{b(t)}B(t)\left|x'(t)\right|^{\alpha+1}\right) &= \frac{d}{dt}\left(\frac{1}{t^2 b(t)}B(t)t^{2\alpha+2}\left|x'(t)\right|^{\alpha+1}\right)\\ &= B(t)t^{2\alpha+2}\left|x'(t)\right|^{\alpha+1}\frac{d}{dt}\left(\frac{1}{t^2 b(t)}\right) + \frac{1}{t^2}t^{2\alpha+2}\left|x'(t)\right|^{\alpha+1}\\ &\quad + (\alpha+1)\frac{B(t)}{t^2 b(t)}\left(2t^{2\alpha+1}\left|x'(t)\right|^{\alpha+1} + t^{2\alpha+2}\left|x'(t)\right|^{\alpha}x''(t)\operatorname{sgn} x'(t)\right).\end{aligned}$$

Hence we get

$$\begin{aligned}\frac{d}{dt}E_x(t) &= (\beta+1)\left|x(t)\right|^{\beta}x'(t)B(t)\operatorname{sgn} x(t) + t^{2\alpha}\left|x'(t)\right|^{\alpha}x'(t)\operatorname{sgn} x'(t)\\ &\quad + kB(t)t^{2\alpha+2}\left|x'(t)\right|^{\alpha+1}\frac{d}{dt}\left(\frac{1}{t^2 b(t)}\right) + k\frac{1}{t^2}t^{2\alpha+2}\left|x'(t)\right|^{\alpha+1}\\ &\quad + k(\alpha+1)\frac{B(t)}{t^2 b(t)}\left(2t^{2\alpha+1}\left|x'(t)\right|^{\alpha+1} + t^{2\alpha+2}\left|x'(t)\right|^{\alpha}x''(t)\operatorname{sgn} x'(t)\right)\end{aligned}$$

or

$$\frac{d}{dt}E_x(t) = t^{2\alpha+2}g(t) + h(t),$$

where

$$g(t) = (1+k)\frac{1}{t^2}\left|x'(t)\right|^{\alpha+1} + kB(t)\left|x'(t)\right|^{\alpha+1}\frac{d}{dt}\left(\frac{1}{t^2 b(t)}\right)$$

and

$$\begin{aligned}h(t) &= (\beta+1)\left|x(t)\right|^{\beta}x'(t)B(t)\operatorname{sgn} x(t)\\ &\quad + k(\alpha+1)\frac{B(t)}{t^2 b(t)}\left(2t^{2\alpha+1}\left|x'(t)\right|^{\alpha+1} + t^{2\alpha+2}\left|x'(t)\right|^{\alpha}x''(t)\operatorname{sgn} x'(t)\right).\end{aligned}$$

From (20) we obtain

$$1+k = \gamma k, \qquad k(\alpha+1) = \alpha(\beta+1).$$

Thus, we get

$$g(t) = k\left(\gamma\frac{1}{t^2}\left|x'(t)\right|^{\alpha+1} + B(t)\left|x'(t)\right|^{\alpha+1}\frac{d}{dt}\left(\frac{1}{t^2 b(t)}\right)\right) \tag{22}$$

and

$$\begin{aligned}\frac{h(t)}{(\beta+1)B(t)} &= \left|x(t)\right|^{\beta}x'(t)\operatorname{sgn} x(t) + \frac{2\alpha}{b(t)}t^{2\alpha-1}\left|x'(t)\right|^{\alpha+1}\\ &\quad + \frac{\alpha}{b(t)}t^{2\alpha}\left|x'(t)\right|^{\alpha}x''(t)\operatorname{sgn} x'(t).\end{aligned}$$

In view of (17), we have

$$B^{-\gamma+1}(t)\left|x'(t)\right|^{\alpha+1}\frac{d}{dt}\left(\frac{1}{t^2 b(t)}B^{\gamma}(t)\right)$$
$$= B(t)\left|x'(t)\right|^{\alpha+1}\frac{d}{dt}\left(\frac{1}{t^2 b(t)}\right) + \gamma\frac{1}{t^2}\left|x'(t)\right|^{\alpha+1} \le 0,$$

and so, from (22) we obtain

$$g(t) \le 0.$$

In order to complete the proof, it is sufficient to show that $h(t) = 0$. Using Lemma 3, we have

$$\frac{h(t)}{(\beta+1)B(t)} = \left|x(t)\right|^{\beta} x'(t)\operatorname{sgn} x(t) + \frac{2\alpha}{b(t)}t^{2\alpha-1}\left|x'(t)\right|^{\alpha+1}$$
$$- x'(t)\left|x(t)\right|^{\beta}\operatorname{sgn} x(t) - \frac{2\alpha}{b(t)}t^{2\alpha-1}\left|x'(t)\right|^{\alpha+1} = 0,$$

thus the assertion follows. □

Lemma 5 *Assume* (16). *Then* (1) *has proper solutions* x *for which* $E_x(1) < 0$.

Proof Consider on $[0,\infty)$ the scalar function

$$\phi(u) = c_1 u^{\alpha+1} - m u^{\alpha} + c m^{\beta+1}, \tag{23}$$

where

$$c_1 = \frac{\alpha(\beta+1)}{\alpha+1}\frac{c}{b(1)} \tag{24}$$

and m is a positive parameter. A standard calculation shows that when m is sufficiently small, then ϕ attains negative values in a neighborhood of the point

$$\overline{u} = \frac{\alpha m}{(\alpha+1)c_1}. \tag{25}$$

Indeed, consider the local solution x of (1) with the initial condition

$$x(1) = m, \qquad x'(1) = -\overline{u}. \tag{26}$$

In view of Remark 1, x is continuable to infinity and proper. From (19) we obtain

$$E_x(1) = c m^{\beta+1} - m\left(\frac{\alpha m}{(\alpha+1)c_1}\right)^{\alpha} + k\frac{c}{b(1)}\left(\frac{\alpha m}{(\alpha+1)c_1}\right)^{\alpha+1},$$

or, in view of (20),

$$E_x(1) = \phi(\overline{u}).$$

Using (25), we get

$$\begin{aligned} E_x(1) = \phi(\overline{u}) &= cm^{\beta+1} - \left(\frac{\alpha}{(\alpha+1)c_1}\right)^{\alpha} m^{\alpha+1} + c_1\left(\frac{\alpha}{(\alpha+1)c_1} m\right)^{\alpha+1} \\ &= m^{\alpha+1}\left[cm^{\beta-\alpha} + \left(\frac{1}{(\alpha+1)c_1}\right)^{\alpha}\left(-1 + \frac{\alpha}{\alpha+1}\right)\right] \\ &= m^{\alpha+1}\left[cm^{\beta-\alpha} - \frac{1}{\alpha+1}\left(\frac{1}{(\alpha+1)c_1}\right)^{\alpha}\right]. \end{aligned}$$

From this and $\beta > \alpha$, choosing m sufficiently small such that

$$0 < m^{\beta-\alpha} < \frac{1}{c}\frac{1}{\alpha+1}\left(\frac{1}{(\alpha+1)c_1}\right)^{\alpha}, \tag{27}$$

we get $\phi(\overline{u}) < 0$, which is the assertion. □

Proof of Theorem 1 From Lemma 4, the function E_x is nonincreasing on $[1,\infty)$ for any solution x of (1).

Fixed m satisfying (27), consider the local solution x of (1) with the initial condition (26), where $\overline{u}$ is given by (25). In view of Remark 1, this solution is also continuable to infinity and proper. Moreover, it is uniquely determined, because in the superlinear case the uniqueness of solutions with respect to the initial conditions holds; see, *e.g.*, [7]. Moreover, in virtue of the proof of Lemma 5, we have $E_x(1) < 0$, and so, from Lemma 4, we obtain

$$E_x(t) \le E_x(1) < 0 \quad \text{on } [1,\infty). \tag{28}$$

Let us show that x and x' cannot have zeros for $t \ge 1$. By contradiction, if there exists $t_1 > 1$ such that $x(t_1) = 0$, then, in virtue of the uniqueness with respect to the initial data, we have $x'(t_1) \ne 0$. Hence $E(t_1) > 0$, which contradicts (28). Similarly, if there exists $t_2 \ge 1$ such that $x'(t_2) = 0$, we obtain $E(t_2) > 0$, which is again a contradiction.

Thus, x is nonoscillatory. Moreover, in view of Lemma 2, we have

$$\lim_{t\to\infty} x(t) = 0.$$

Hence, x is positive decreasing in the half-line $[1,\infty)$, that is,

$$x(t) > 0, \qquad x'(t) < 0 \quad \text{for } t \ge 1.$$

From (1), the quasiderivative $x^{[1]}$ is negative decreasing, *i.e.*

$$\lim_{t\to\infty} -x^{[1]}(t) = \ell_x, \quad 0 < \ell_x \le \infty.$$

If $\ell_x < \infty$, we get $\lim_{t\to\infty} x(t)x^{[1]}(t) = 0$, and from (19) we obtain $\liminf_{t\to\infty} E_x(t) \ge 0$, that is, a contradiction with (28). Hence $\ell_x = \infty$, *i.e.*

$$\lim_{t\to\infty} t^2 x'(t) = -\infty.$$

Using the l'Hospital rule we get

$$\lim_{t\to\infty} tx(t) = \infty.$$

Hence, x is a solution of (BVP). Since there are infinitely many solutions which satisfy (26) with the choice of m taken with (27), the proof is now complete. □

From Theorem 1 and its proof, we get the following.

Corollary 1 *Under assumptions of Theorem* 1, (1) *has infinitely many slowly decaying solutions, which are positive decreasing on the whole interval* $[1,\infty)$. *Moreover,* (1) *has also infinitely many strongly decaying solutions and every nonoscillatory solution of* (1) *tends to zero as* $t \to \infty$.

Proof In virtue of Theorem 1 and its proof, the boundary value problem (BVP) is solvable by every solution x which satisfies (26) and (27). Clearly, these solutions are slowly decaying solutions.

Consequently, (1) has nonoscillatory solutions and, in view of Lemma 2(iii) we get $Y < \infty$. Then the existence of infinitely many strongly decaying solutions follows from Lemma 2(ii). □

When the monotonicity condition (17) is valid only for large t, reasoning as in the proof of Theorem 1, we obtain the following.

Corollary 2 *Assume* $Z = \infty$. *If* (16) *is satisfied and the function* G, *given in* (17) *is nonincreasing for any large* t, *then* (1) *has infinitely many slowly decaying solutions, which are eventually positive decreasing. Moreover,* (1) *has also infinitely many strongly decaying solutions and every nonoscillatory solution of* (1) *tends to zero as* $t \to \infty$.

Finally, when also the initial starting point is fixed, we have the following.

Corollary 3 *Assume* $Z = \infty$. *If* (16) *and* (17) *are satisfied, then the BVP*

$$\begin{cases} (t^{2\alpha}|x'|^{\alpha}\operatorname{sgn} x')' + b(t)|x|^{\beta}\operatorname{sgn} x = 0, & t \geq 1, \alpha < \beta, \\ x(1) = x_1, \qquad x(t) > 0, \qquad x'(t) < 0, \\ \lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} tx(t) = \infty, \end{cases}$$

has infinitely many solutions for every initial data x_1 *such that*

$$0 < (x_1)^{\beta-\alpha} < \left(\frac{1}{c}\right)^{\alpha+1} \frac{1}{\alpha+1}\left(\frac{b(1)}{\alpha(\beta+1)}\right)^{\alpha}. \tag{29}$$

Proof The assertion follows by a reasoning as in the proof of Theorem 1 and choosing $m = x_1$ in (26). Taking into account that m satisfies (27) and c_1 is given by (24), we get (29). The details are left to the reader. □

Remark 2 It is worth to note that the condition (17) may depend on the choice of the constant c in (15), *i.e.* on the choice of a primitive to b.

If (17) is satisfied for a fixed B, then (17) remains to hold for $B(t) + \overline{c}$, where $\overline{c} > 0$. Indeed, setting $\Psi(t) = t^{-2}b^{-1}(t)$, from $G'(t) \le 0$ we get

$$0 \ge \Psi'(t) + \gamma\Psi(t)b(t)\frac{1}{B(t)} = \Psi'(t) + \gamma\Psi(t)b(t)\frac{1}{B(t) + \overline{c}}\frac{B(t) + \overline{c}}{B(t)}$$

or

$$0 \ge \Psi'(t) + \gamma\Psi(t)b(t)\frac{1}{B(t) + \overline{c}}$$

i.e. (17) is satisfied also for $B(t) + \overline{c}$ with $\overline{c} > 0$. However, if (17) is valid for B given by (15), then it is possible that (17) is not valid for $\widetilde{B}(t) = B(t) + \tilde{c}$, where $0 < \tilde{c} < c$. This fact is illustrated below in Example 2.

4 Oscillation and nonoscillation

In this section we discuss assumptions of Theorem 1, jointly with some consequences to the oscillation. Assumption (16) guarantees the continuability at infinity of any solution of (1) and its role is discussed in Remark 1. Concerning the condition $Z = \infty$, a consequence of a result in [17] shows that it is a necessary condition for the solvability of (BVP). The following holds.

Theorem 2 *If $Z < \infty$, then* (1) *does not have solutions $x \in \mathbb{P}$ such that*

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} tx(t) = \infty.$$

Proof Let x be a solution of (1) and set

$$y(t) = -t^{2\alpha}\left|x'(t)\right|^{\alpha}\operatorname{sgn} x'(t). \tag{30}$$

A standard calculation shows that y is a solution of equation

$$\left(\frac{1}{b^{1/\beta}(t)}\left|y'\right|^{1/\beta}\operatorname{sgn} y'\right)' + \frac{1}{t^2}|y|^{1/\alpha}\operatorname{sgn} y = 0, \tag{31}$$

where $1/\beta < 1/\alpha$. From [17], Theorem 4, (31) does not admit eventually positive solutions y such that

$$\lim_{t\to\infty} y(t) = \infty, \qquad \lim_{t\to\infty}\frac{y'(t)}{b(t)} = 0,$$

that is, in view of (30), (1) does not have solutions $x \in \mathbb{P}$ such that

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} t^2x'(t) = -\infty.$$

Since the l'Hospital rule gives

$$\lim_{t\to\infty} tx(t) = -\lim_{t\to\infty} t^2x'(t),$$

the assertion follows. □

Now, we discuss the monotonicity condition (17). We start by recalling the following nonoscillation result, which is an extension of a well-known Kiguradze criterion [18], Theorem 18.7.

Theorem 3 ([7], Theorem 14.3) *If there exists a positive number ε such that the function*

$$\bar{G}(t) = \frac{1}{t^2 b(t)} \left(\int_1^t b(s)\, ds \right)^{\gamma+\varepsilon} \quad \textit{is nonincreasing for large } t, \tag{32}$$

where γ is given in (18), *then all solutions of* (1) *are nonoscillatory.*

A standard calculation shows that if (32) holds for $t \geq t_0 \geq 1$, then (17) is valid on the same interval $[t_0, \infty)$ as well. Thus, in view of Corollary 2, we can obtain an existence result for slowly decaying solutions x of (1). Nevertheless, condition (17) can be valid in a larger interval than $[t_0, \infty)$. The next example illustrates this fact.

Example 1 Consider the equation

$$\left(t \left| x'(t) \right|^{1/2} \operatorname{sgn} x' \right)' + x^3 = 0. \tag{33}$$

We have $Z = \infty$ and $\gamma = 7/4$. Moreover, choosing $c = 1$ in (15), for the function G given in (17) we have $G(t) = t^{-1/4}$. Then (17) is satisfied for $t \geq 1$ and Theorem 1 is applicable. Analogously, (32) is verified for any large t and $0 < \varepsilon < 1/4$, because

$$\bar{G}(t) = t^{-2}(t-1)^{7/4}(t-1)^{\varepsilon}.$$

On the other hand, we have

$$\bar{G}'(t) = t^{-2}(t-1)^{3/4}(t-1)^{\varepsilon} \left(7/4 + \varepsilon - 2t(t-1) \right)$$

and so (32) is not valid in a right neighborhood of $t = 1$. By Theorem 3, all solutions of (33) are nonoscillatory and by Lemma 2 tend to zero as $t \to \infty$. Moreover, in view of Theorem 1, (33) has both slowly decaying solutions and strongly decaying solutions. Finally, slowly decaying solutions are globally positive on the whole interval $[1, \infty)$.

Example 2 Consider the equation

$$\left(t^{2/3} \left| x'(t) \right|^{1/3} \operatorname{sgn} x' \right)' + x^3(t) = 0. \tag{34}$$

We have $Z = \infty$ and $\gamma = 2$. Choosing $c = 1$ in (15), we obtain $G(t) = 1$ and Theorem 1 can be applied. Indeed, as it is easy to verify, the function $x(t) = 2^{-1} t^{-1/4}$ is a slowly decaying solution of (34). Moreover, (34) has both slowly decaying solutions and strongly decaying solutions and slowly decaying solutions are globally positive on the whole interval $[1, \infty)$. Observe that if we choose $c = 1/2$ in (15), then the corresponding function G is increasing and (17) is not satisfied.

Furthermore, the function $\bar{G}$ given in (32) is increasing for large t and any $\varepsilon > 0$. Hence, Theorem 3 cannot be used. Then it is a question whether (34) admits or does not admit oscillatory solutions.

When $\alpha = 1$, the coexistence between oscillatory solutions and nonoscillatory solutions can be obtained by using Lemma 2 and a result from [19], Theorem 1. The following holds.

Theorem 4 *Consider the equation*

$$\left(t^2 x'\right)' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \tag{35}$$

where $\beta > 1$ *and* $b(t) > 0$ *for* $t \geq 1$. *Assume* $Y < \infty$. *If the function*

$$H(t) = t^{(1-\beta)/2} b(t)$$

is nondecreasing for large t and $\lim_{t\to\infty} H(t) = \infty$, *then* (35) *has infinitely many oscillatory solutions and infinitely many strongly decaying solutions. Moreover, every nonoscillatory solution of* (35) *tends to zero as* $t \to \infty$.

Proof In view of [19], Theorem 1, any solution x of (35) which satisfies $x(t_0)x'(t_0) > 0$ at some $t_0 \geq 1$, is oscillatory. The remaining part of the statement follows from Lemma 2. □

The following example shows that both types of nonoscillatory decaying solutions can coexist with oscillatory solutions.

Example 3 Consider the equation

$$\left(t^2 x'(t)\right)' + t^{4/3} x^3 = 0. \tag{36}$$

The assumptions in Theorem 4 are satisfied. Hence, (36) has oscillatory solutions and infinitely many strongly decaying solutions. Moreover, every nonoscillatory solution tends to zero as $t \to \infty$, according to Lemma 2, because $Z = \infty$. Furthermore, it is easy to verify that the function

$$x(t) = \frac{\sqrt{2}}{3} t^{-2/3}$$

is a slowly decaying solution of (36). Thus, for (36) slowly decaying solutions and strongly decaying solutions coexist with oscillatory solutions. Observe that for the function G given in (17) we have

$$G(t) = t^{-10/3} \left(\frac{3}{7} t^{7/3} + k \right)^{3/2},$$

where k is a suitable constant such that $k > -3/7$. A standard calculation shows that (17) is not satisfied and Theorem 1 cannot be applied. A similar argument shows that also (32) fails.

Open problems Example 3 suggests that for the existence of at least one slowly decaying solution, the assumption on monotonicity in (17) could be relaxed. Moreover, Example 3 (and Theorem 4) deal with the case $\alpha = 1$, that is, when the differential operator is the Sturm-Liouville disconjugate operator. Does the coexistence between oscillatory solutions and decaying solutions, illustrated in Example 3, occur also when $\alpha \neq 1$ (and $\beta > \alpha$)?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final draft.

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Acknowledgements
The first author is supported by Grant P201/11/0768 of the Czech Science Foundation.

Received: 4 February 2015 Accepted: 19 May 2015 Published online: 12 June 2015

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