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Homogeneous Dirichlet condition of an anisotropic degenerate parabolic equation

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Abstract

Consider the following anisotropic degenerate parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right) + \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in \Omega \times (0,T),$$

with the homogeneous Dirichlet boundary value. If the equation is not only degenerate in the interior of Ω, but also on the boundary $\partial\Omega$, the paper discusses how to quote the suitable partly boundary condition to assure the well-posedness of an entropy solution of the equation. In particular, it is possible that the solution of the equation is free from the limitation of the boundary condition.

MSC: 35L65; 35L85; 35R35

Keywords: anisotropic degenerate parabolic equation; boundary condition; entropy solution

1 Introduction

The paper is to consider the anisotropic degenerate parabolic equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right) + \frac{\partial b_i(u)}{\partial x_i}, \quad \text{in } Q_T = \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain and the boundary $\partial\Omega = \Sigma$ is C^2, (a^{ij}) is a symmetric matrix with nonnegative characteristic values, *i.e.*, for any $\xi \in \mathbb{R}^N$,

$$a^{ij} = a^{ji}, \qquad a^{ij}\xi_i\xi_j \geq 0,$$

the pairs of the indices i, j imply the sum from 1 to N. Moreover, we assume that

$$a^{ij}(0) = 0. \tag{1.2}$$

Equation (1.1) arises in many applications, *e.g.*, heat flow in materials with temperature dependent on conductivity, flow in a porous medium,

$$\frac{\partial u}{\partial t} = \Delta u^m. \tag{1.3}$$

It also arises in the boundary layer theory,

$$w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w = 0, \tag{1.4}$$

where A, B are two known functions derived from the Prandtl system, one can refer to [1] for details. Here and in what follows, we say that equation (1.1) is strongly degenerate if there are interior points in the set $\{s \in \mathbb{R} : (a^{ij}(s))$ is a degenerate matrix$\}$. Clearly, equation (1.1) is of hyperbolic-parabolic mixed type and might have a discontinuous solution. The posedness of the Cauchy problem of equation (1.1) has been deeply investigated (see [2–14] *etc.*). At the same time, Li and Wang [15] studied the well-posedness for anisotropic degenerate parabolic equation (1.1) with inhomogeneous boundary condition on a bounded rectangle by using the kinetic formulation which was introduced in [16]. Kobayasi and Ohwa [17] considered the entropy solutions of equation (1.1) with the homogeneous Dirichlet boundary value in an arbitrary bounded domain. Since the entropy solutions defined in [15, 17] are only in the L^∞ space, the existence of the trace (defined in a traditional way, which was called the strong trace in [17]) on the boundary is not guaranteed, the appropriate definition of entropy solutions is quoted, and a new definition of the trace of the solution on the boundary, defined in an integral formula sense, is introduced; they called it the weak trace. So, not only is Definition 1.1 in what follows different from the definitions of entropy solutions in [15, 17], but the trace of the solution in our paper is also in the traditional way.

In fact, if we want to consider the initial boundary value problem of equation (1.1), the initial value condition is always required

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.5}$$

But can we give Dirichlet homogeneous boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) = \Sigma \times (0,T) \tag{1.6}$$

as usual?

Clearly, if (1.2) and (1.6) are both true, equation (1.1) is not only degenerate in the interior of Ω, but also degenerate on the boundary Σ of Ω. If equation (1.1) is weakly degenerate, we can give the boundary value (1.6) as usual. But if equation (1.1) is strongly degenerate, we shall show that only a portion of the boundary should be given the boundary value. Let us give a basic review of the history of the corresponding problem and show what we consider now.

The memoir by Tricomi [18], as well as subsequent investigations of equations of mixed type, elicited interest in the general study of elliptic equations degenerating on the boundary of the domain. The paper by Keldyš [19] played a significant role in the development of the theory. It was this paper that first brought to light the fact that in the case of elliptic equations degenerating on the boundary, under definite assumptions, a portion of the boundary may be free from the prescription of boundary conditions. Later, Fichera [20, 21] and Oleĭnik [22, 23] developed and perfected the general theory of second order equation with a nonnegative characteristic form, which in particular contains those degenerating on the boundary.

The equation considered by Fichera and Oleĭnik is linear and the second order derivatives of coefficients of principal part are bounded. From Fichera-Oleĭnik theory, for a linear degenerate elliptic equation,

$$a^{rs}(x)\frac{\partial^2 u}{\partial x_r\,\partial x_s}+b_r(x)\frac{\partial u}{\partial x_r}+c(x)u=f(x),\quad x\in\widetilde{\Omega}\subset\mathbb{R}^{N+1},\tag{1.7}$$

where the pairs of the indices r, s imply the sum from 1 to $N+1$. If one wants to consider the boundary value problem of (1.7), one needs and only needs to give a partly boundary condition. In detail, let $\{n_s\}$ be the unit inner normal vector of $\partial\widetilde{\Omega}$ and denote

$$\Sigma_2=\{x\in\partial\widetilde{\Omega}:a^{rs}n_rn_s=0,(b_r-a^{rs}_{x_s})n_r<0\},$$
$$\Sigma_3=\{x\in\partial\widetilde{\Omega}:a^{rs}n_sn_r>0\}.$$

Then, to ensure the posedness of equation (1.7), Fichera-Oleĭnik theory tells us that the suitable boundary condition is

$$u|_{\Sigma_2\cup\Sigma_3}=g(x).\tag{1.8}$$

In particular, if the matrix (a^{rs}) is positive definite, (1.8) is just the usual Dirichlet boundary condition.

Now, for equation (1.3), or the general equation

$$u_t=\triangle A(u),\tag{1.9}$$

with the existence of A^{-1}, in other words, equation (1.9) is weakly degenerate, then let $v=A(u)$, $u=A^{-1}(v)$,

$$\triangle v-(A^{-1}(v))_t=0.\tag{1.10}$$

According to Fichera-Oleĭnik theory, we know that we can give the Dirichlet homogeneous boundary condition (1.6). For equation (1.4), if the domain $\Omega=\{0<\tau<T,0<\xi<X,0<\eta<1\}$, then comparing (1.4) with (1.7), according to Fichera-Oleĭnik theory, the initial and the boundary value conditions for w have the form

$$w|_{\tau=0}=w_0(\xi,\eta),\qquad w|_{\eta=1}=0,\qquad (\nu ww_\eta-\nu_0w+c(\tau,\xi))|_{\eta=0}=0,\tag{1.11}$$

where ν is the viscous coefficient, ν_0 and $c(\tau,\xi)$ are known functions; one can refer to [1] for details.

But if equation (1.1) is strongly degenerate, then the inverse matrix $A^{-1}=(a_{ij})^{-1}$ is not existential, we cannot deal with it as (1.10). Rewrite equation (1.1) as

$$\frac{\partial u}{\partial t}=a^{ij}(u)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+a^{ij'}(u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+b_i'(u)\frac{\partial u}{\partial x_i},\quad \text{in } Q_T=\Omega\times(0,T),\tag{1.12}$$

the domain is a cylinder $\Omega \times (0,T)$. If we let $t = x_{N+1}$ and regard the degenerate parabolic equation (1.12) as the form of a 'linear' degenerate elliptic equation as in (1.7), then

$$(\widetilde{a}^{rs})_{(N+1)\times(N+1)} = \begin{pmatrix} a^{ij} & 0 \\ 0 & 0 \end{pmatrix}.$$

If $a^{ij}(0) = 0$, which means that equation (1.12) is not only strongly degenerate in the interior of Ω, but also degenerate on the boundary $\partial\Omega$, then Σ_3 is an empty set, while

$$\widetilde{b}_s(x,t) = \begin{cases} b_i'(u) + a^{ij'}(u)\frac{\partial u}{\partial x_j}, & 1 \le s = i \le N, \\ -1, & s = N+1. \end{cases}$$

Under this observation, according to Fichera-Oleĭnik theory, the initial value condition (1.5) is always needed, but on the lateral boundary $\partial\Omega \times (0,T)$, by $a^{ij}(0) = 0$, the partly boundary on which we should give the boundary value is

$$\begin{aligned} \Sigma_p &= \left\{ x \in \partial\Omega : \left(b_i'(0) + a^{ij'}(0)\frac{\partial u}{\partial x_j}\bigg|_{x\in\partial\Omega} - a^{ij'}(0)\frac{\partial u}{\partial x_j}\bigg|_{x\in\partial\Omega} \right) n_i < 0 \right\} \\ &= \left\{ x \in \partial\Omega : b_i'(0) n_i < 0 \right\}, \end{aligned} \tag{1.13}$$

where $\{n_i\}$ is the unit inner normal vector of $\partial\Omega$.

Though (1.13) seems reasonable and beautiful, whether the term $\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega}$ has an explicit definition is unclear, unless equation (1.12) has a classical solution. In fact, due to the strongly degenerate property of (a^{ij}), equation (1.12) generally only has a weak solution. In our paper, we consider the solution of equation (1.12) in BV sense, and we cannot define the trace of $\frac{\partial u}{\partial x_i}$ on $\partial\Omega$, which means that we also cannot define

$$\Sigma_p = \left\{ x \in \partial\Omega : \left(b_i'(0) + a^{ij'}(0)\frac{\partial u}{\partial x_j}\bigg|_{x\in\partial\Omega} - a^{ij'}(0)\frac{\partial u}{\partial x_j}\bigg|_{x\in\partial\Omega} \right) n_i < 0 \right\}.$$

Fortunately, only if $b_i(s)$ is derivable, then

$$\Sigma_p = \left\{ x \in \partial\Omega : b_i'(0) n_i < 0 \right\} \tag{1.14}$$

has a definite sense. Our paper will show that Σ_p defined in (1.14) can be given the boundary condition in some way.

It is well known that the BV functions are the weakest functions which have the traces in the usual way. At the same time, in order to get the uniqueness, we need to consider the entropy solution of equation (1.1) instead of the general weak solution.

The existence will be proved by means of the method of parabolic regularization, namely the solution of our problem will be obtained as a limit point of the family $\{u_\varepsilon\}$ of solutions of the regularized problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j} \right) + \varepsilon\Delta u + \frac{\partial b_i(u)}{\partial x_i}, \quad \text{in } Q_T = \Omega \times (0,T), \tag{1.15}$$

with compatible initial boundary values (1.4)-(1.5).

In order to prove the compactness of $\{u_\varepsilon\}$, we need some estimates on $\{u_\varepsilon\}$. However, in the present case of strong degeneration, it is difficult to estimate $|\operatorname{grad} u_\varepsilon|_{L^1(Q_T)}$. In addition, since for the limit function u of certain subsequence of $\{u_\varepsilon\}$, $\widehat{a^{ij}(u)}\frac{\partial u}{\partial x_j}$ need not have the trace $\gamma(\widehat{a^{ij}(u)}\frac{\partial u}{\partial x_j})$ on Σ, we have to make a detour to avoid $\gamma(\widehat{a^{ij}(u)}\frac{\partial u}{\partial x_j})$ in defining solution, where $\widehat{a^{ij}(u)}$ is the composite means function of BV function $a^{ij}(u)$, which will be defined in detail in what follows. By combining this inspiring idea of [24] with that of [9], we shall give a new entropy solution of equation (1.1). Let us give some preparedness.

For any $\eta > 0$, $\forall k \in R$, let $\vec{n} = \{n_i\}$ be the inner unit normal vector of Σ, and

$$\Sigma_{1\eta k} = \{x \in \Sigma, S_\eta(k)[b_i(0) - b_i(k)]n_i > 0\}, \tag{1.16}$$

$$\Sigma_{2\eta k} = \{x \in \Sigma, S_\eta(k)[b_i(0) - b_i(k)]n_i \le 0\}. \tag{1.17}$$

Clearly, $\Sigma = \Sigma_{1\eta k} \cup \Sigma_{2\eta k}$, and let

$$\Sigma_1 = \bigcup_{\forall \eta > 0, \forall k \in \mathbb{R}} \Sigma_{1\eta k}, \qquad \Sigma_2 = \Sigma \setminus \Sigma_1. \tag{1.18}$$

Now, if $\Sigma_1 \neq \emptyset$, we can give the boundary value condition as

$$\gamma u|_{\Sigma_1} = 0. \tag{1.19}$$

In fact, by the definition of $\Sigma_{1\eta k}$, we know that

$$0 < S_\eta(k)[b_i(0) - b_i(k)]n_i(x,t) = -kS_\eta(k)b_i'(\zeta)n_i(x,t),$$

where $\zeta \in (k, 0)$. If we let $\eta \to 0$, then

$$b_i'(\zeta)n_i(x,t) < 0.$$

Let $k \to 0$. We know that

$$b_i'(0)n_i(x,t) < 0,$$

which is in accordance with (1.14).

Let us consider $\Sigma_1 \neq \emptyset$ firstly. As for the case of $\Sigma_1 = \emptyset$, no boundary value condition is necessary. In other words, the solution of equation (1.1) is completely controlled by the initial value condition. We shall discuss the problem in this case at the end of the paper.

Let $S_\eta(s) = \int_0^s h_\eta(\tau)\,d\tau$ for small $\eta > 0$, where $h_\eta(s) = \frac{2}{\eta}(1 - \frac{|s|}{\eta})_+$. Obviously, $h_\eta(s) \in C(\mathbb{R})$, and

$$h_\eta(s) \ge 0, \qquad |sh_\eta(s)| \le 1, \qquad |S_\eta(s)| \le 1; \qquad \lim_{\eta \to 0} S_\eta(s) = \operatorname{sgn} s, \qquad \lim_{\eta \to 0} sS_\eta'(s) = 0.$$

Definition 1.1 If $\Sigma_1 \neq \emptyset$, a function u is said to be the entropy solution of equation (1.1)-(1.5)-(1.19), if

1. $u \in BV(Q_T) \cap L^\infty(Q_T)$, and there exist the functions $g^i \in L^2(Q_T)$, $i = 1, 2, \dots, N$, such that

$$\iint_{Q_T} g^i(x,t)\varphi(x,t)\,dx\,dt = \iint_{Q_T} \widehat{\gamma}^{ij}(u)\varphi(x,t)\frac{\partial u}{\partial x_j}\,dx\,dt, \tag{1.20}$$

where $\varphi(x,t) \in L^2(Q_T)$, (γ^{ij}) is the square root of (a^{ij}), and

$$\widehat{\gamma}^{ij}(u) = \int_0^1 \gamma^{ij}\big(su^+ + (1-s)u^-\big)\,ds.$$

2. For any $\varphi_1, \varphi_2 \in C^2(\overline{Q_T})$, $\varphi_1 \geq 0$, $\nabla\varphi_1|_\Sigma = 0$, $\varphi_1|_{\partial\Omega\times[0,T]} = \varphi_2|_{\partial\Omega\times[0,T]}$, and $\operatorname{supp}\varphi_2$, $\operatorname{supp}\varphi_1 \subset \overline{\Omega} \times (0,T)$, for any $k \in \mathbb{R}$, for any small $\eta > 0$, u satisfies

$$\begin{aligned}
&\iint_{Q_T}\Bigg[I_\eta(u-k)\varphi_{1t} - B_\eta^i(u,k)\varphi_{1x_i} + A_\eta^{ij}(u,k)\frac{\partial\varphi_1}{\partial x_i}\frac{\partial\varphi_1}{\partial x_j} - S_\eta'(u-k)\sum_{j=1}^N\big|g^j\big|^2\varphi_1\Bigg]\,dx\,dt\\
&\quad + S_\eta(k)\iint_{Q_T}\Bigg[u\varphi_{2t} - \big(b_i(u)-b_i(0)\big)\varphi_{2x_i} + A^{ij}(u)\frac{\partial\varphi_2}{\partial x_i}\frac{\partial\varphi_2}{\partial x_j}\Bigg]\,dx\,dt\\
&\quad + S_\eta(k)\int_0^T\int_{\Sigma_{1\eta k}}\big[(b_i(0)-b_i(k)\big]n_i\varphi_1\,dt\,d\sigma \geq 0.
\end{aligned} \tag{1.21}$$

3. The boundary value is satisfied in the sense of the trace

$$\gamma u|_{\Sigma_{1\eta k}} = 0. \tag{1.22}$$

4. The initial value is satisfied in the sense of the following equality:

$$\lim_{t\to 0}\int_\Omega \big|u(x,t) - u_0(x)\big|\,dx = 0, \quad \text{a.e. } x \in \Omega, \tag{1.23}$$

where the pairs of equal indices imply a summation from 1 up to N, and

$$B_\eta^i(u,k) = \int_k^u b_i'(s)S_\eta(s-k)\,ds, \qquad I_\eta(u-k) = \int_0^{u-k} S_\eta(s)\,ds,$$
$$A_\eta^{ij}(u,k) = \int_k^u a^{ij}(s)S_\eta(s-k)\,ds, \qquad A^{ij}(u) = \int_0^u a^{ij}(s)\,ds.$$

Clearly, let $\eta \to 0$ in (1.21). We can see that if u is the entropy solution in Definition 1.1, then it is an entropy solution defined in [2, 3, 7] *etc.*

We shall prove the following theorems.

Theorem 1.2 *Suppose that* $A^{ij}(s)$ *is* C^3, $b_i(s)$ *is* C^2, *and* $u_0(x) \in L^\infty(\Omega) \cap C^2(\Omega)$, *and suppose that*

$$a^{ij}(0) = 0.$$

Then equation (1.1) *with initial boundary value conditions* (1.5), (1.19) *has an entropy solution in the sense of Definition* 1.1.

Theorem 1.3 *Suppose that $A^{ij}(s)$ and $b_i(s)$ are C^1. Let u, v be solutions of equation* (1.1) *with different initial values $u_0(x), v_0(x) \in L^\infty(\Omega)$, respectively. Suppose that*

$$\gamma u(x,t) = f(x,t), \qquad \gamma v = g(x,t), \quad (x,t) \in \Sigma \times (0,T), \tag{1.24}$$

and in particular

$$\gamma u = \gamma v = 0, \quad x \in \Sigma_1, \tag{1.25}$$

suppose that the distance function $d(x) = \operatorname{dist}(x, \Sigma) < \lambda$ satisfies

$$|d_{x_i x_j}| \le c, \tag{1.26}$$

where λ is a small enough constant, and $\Omega_\lambda = \{x \in \Omega, d(x, \partial\Omega) < \lambda\}$. Then

$$\int_\Omega \bigl|u(x,t) - v(x,t)\bigr|\,dx \le \int_\Omega |u_0 - v_0|\,dx + \operatorname*{ess\,sup}_{(x,t)\in\Sigma_2\times(0,T)} \bigl|f(x,t) - g(x,t)\bigr|, \tag{1.27}$$

where $(x,t) \in \mathbb{R}^{N+1}$, $\operatorname{ess\,sup}_{(x,t)\in\Sigma_2\times(0,T)} |f(x,t) - g(x,t)|$ is in the sense of N-dimensional Hausdorff measure.

2 The proof of the existence

Without loss of generality, one may assume that $u \in BV(Q_T)$ is an almost everywhere continuous function on Q_T.

Let Γ_u be the set of all jump points of $u \in BV(Q_T)$, let $\nu = (\nu_1, \nu_2, \dots, \nu_N, \nu_{N+1})$ be the normal of Γ_u at $X = (x,t)$, $u^+(X)$, and let $u^-(X)$ be the approximate limits of u at $X \in \Gamma_u$ with respect to $(\nu, Y - X) > 0$ and $(\nu, Y - X) < 0$, respectively. For a continuous function $p(u,x,t)$ and $u \in BV(Q_T)$, define

$$\widehat{p}(u,x,t) = \int_0^1 p\bigl(\tau u^+ + (1-\tau)u^-, x, t\bigr)\,d\tau, \tag{2.1}$$

which is called the composite mean value of p. For a given t, we denote Γ_u^t, H^t, $(\nu_1^t, \dots, \nu_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, Hausdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$, respectively. Moreover, if $f(s) \in C^1(\mathbb{R})$, $u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i}, \quad i = 1, 2, \dots, N. \tag{2.2}$$

Lemma 2.1 [25] *Assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set, and let $f_k, f \in L^q(\Omega)$, as $k \to \infty$, $f_k \rightharpoonup f$ weakly in $L^q(\Omega)$, $1 \le q < \infty$. Then*

$$\liminf_{k\to\infty} \|f_k\|_{L^q(\Omega)}^q \ge \|f\|_{L^q(\Omega)}^q.$$

Lemma 2.2 [24] *Let u_ε be a solution of equation* (1.15) *with initial boundary value* (1.5), (1.6). *If the assumptions of Theorem* 1.2 *are true, then*

$$\varepsilon \int_{\Sigma} \left| \frac{\partial u_\varepsilon}{\partial n} \right| d\sigma \leq c_1 + c_2 \left(|\nabla u_\varepsilon|_{L^1(\Omega)} + \left| \frac{\partial u_\varepsilon}{\partial t} \right|_{L^1(\Omega)} \right)$$

with constants c_i, $i = 1, 2$, *independent of* ε.

Under the assumptions of A, b_i and u_0 in Theorem 1.2, it is well known that there is a classical solution u_ε of the initial boundary value problem (1.15)-(1.5)-(1.6), *e.g.*, one can refer to Chapter 8 of [26].

We need to make some estimates for u_ε of (1.15). Firstly, since $u_0(x) \in L^\infty(\Omega)$ is suitably smooth, by the maximum principle, we have

$$|u_\varepsilon| \leq \|u_0\|_{L^\infty} \leq M. \tag{2.3}$$

Secondly, let us make the BV estimates on u_ε.

Theorem 2.3 *Let* u_ε *be a solution of* (1.15) *with initial boundary value conditions* (1.5), (1.6). *If the assumptions of Theorem* 1.2 *are true, then*

$$|\operatorname{grad} u_\varepsilon|_{L^1(\Omega)} \leq c,$$

where $|\operatorname{grad} u|^2 = \sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^2 + |\frac{\partial u}{\partial t}|^2$, *c is independent of* ε.

Proof Differentiate (1.15) with respect to x_s, $s = 1, 2, \ldots, N, N+1$, $x_{N+1} = t$, and sum up for s after multiplying the resulting relation by $u_{\varepsilon x_s} \frac{S_\eta(|\operatorname{grad} u_\varepsilon|)}{|\operatorname{grad} u_\varepsilon|}$. In what follows, we simply denote u_ε by u. Integrating over Ω yields

$$\begin{aligned} \int_\Omega \frac{\partial u_{x_s}}{\partial t} u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\, dx &= \int_\Omega \frac{\partial}{\partial t} \int_0^{|\operatorname{grad} u|} S_\eta(\tau)\, d\tau\, dx \\ &= \frac{d}{dt} \int_\Omega I_\eta\big(|\operatorname{grad} u|\big)\, dx, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial x_s} \left[\frac{\partial}{\partial x_i} \left(a^{ij}(u) \frac{\partial u}{\partial x_j} \right) \right] u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\, dx \\ &\quad = \int_\Omega \frac{\partial}{\partial x_i} \big[a_u^{ij}(u) u_{x_j} u_{x_s} + a^{ij}(u) u_{x_j x_s} \big] u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\, dx \\ &\quad = \int_\Omega \frac{\partial}{\partial x_i} \big(a_u^{ij}(u) u_{x_j} u_{x_s} \big) u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\, dx \\ &\qquad + \int_\Omega \frac{\partial}{\partial x_i} \big(a^{ij}(u) u_{x_j x_s} \big) u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\, dx; \end{aligned} \tag{2.5}$$

and, moreover,

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial x_i} \big(a_u^{ij}(u) u_{x_j} u_{x_s} \big) u_{x_s} \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\, dx \\ &\quad = \sum_{s=1}^{N+1} \int_\Omega \frac{\partial}{\partial x_i} \big(a_u^{ij}(u) u_{x_j} \big) u_{x_s}^2 \frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\, dx + \int_\Omega a_u^{ij}(u) u_{x_j} \frac{\partial}{\partial x_i} I_\eta\big(|\operatorname{grad} u|\big)\, dx \\ &\quad = \int_\Omega \frac{\partial}{\partial x_i} \big(a_u^{ij}(u) u_{x_j} \big) |\operatorname{grad} u| S_\eta\big(|\operatorname{grad} u|\big)\, dx - \int_\Sigma a_u^{ij}(u) u_{x_i} n_j I_\eta\big(|\operatorname{grad} u|\big)\, d\sigma \end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega I_\eta\big(|\operatorname{grad} u|\big)\frac{\partial}{\partial x_i}\big(a_u^{ij}(u)u_{x_j}\big)\,dx \\
&=\int_\Omega \frac{\partial}{\partial x_i}\big(a_u^{ij}(u)u_{x_j}\big)\big[|\operatorname{grad} u|S_\eta\big(|\operatorname{grad} u|\big)-I_\eta\big(|\operatorname{grad} u|\big)\big]\,dx \\
&\quad-\int_\Sigma a_u^{ij}(u)u_{x_i}n_jI_\eta\big(|\operatorname{grad} u|\big)\,d\sigma, \qquad (2.6)
\end{aligned}
$$

where $\{n_i\}_{i=1}^N$ is the inner normal vector of Ω as before.

$$
\begin{aligned}
&\int_\Omega \frac{\partial}{\partial x_i}\big(a^{ij}(u)u_{x_jx_s}\big)u_{x_s}\frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\,dx \\
&=\int_\Omega \frac{\partial}{\partial x_i}\big(a^{ij}(u)u_{x_jx_s}\big)\frac{\partial}{\partial \xi_s}I_\eta\big(|\operatorname{grad} u|\big)\,dx \\
&=-\int_\Sigma a^{ij}(u)u_{x_ix_s}n_j\frac{\partial}{\partial \xi_s}I_\eta\big(|\operatorname{grad} u|\big)\,d\sigma \\
&\quad-\int_\Omega a^{ij}(u)\frac{\partial^2 I_\eta(|\operatorname{grad} u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_j}\,dx, \qquad (2.7)
\end{aligned}
$$

where $\xi_s = u_{x_s}$.

$$
\begin{aligned}
\varepsilon\int_\Omega \triangle u_{x_s}u_{x_s}\frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\,dx &= -\varepsilon\int_\Sigma \frac{\partial I_\eta(|\operatorname{grad} u|)}{\partial x_i}n_i\,d\sigma \\
&\quad-\varepsilon\int_\Omega \frac{\partial^2 I_\eta(|\operatorname{grad} u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}\,dx. \qquad (2.8)
\end{aligned}
$$

At the same time,

$$
\begin{aligned}
&\int_\Omega \nabla\big(\vec{b}'(u)u_{x_s}\big)u_{x_s}\frac{S_\eta(|\operatorname{grad} u|)}{|\operatorname{grad} u|}\,dx \\
&=\sum_{i=1}^N\int_\Omega \frac{\partial}{\partial x_i}\big(b_i'(u)\big)|\operatorname{grad} u|S_\eta\big(|\operatorname{grad} u|\big)\,dx+\sum_{i=1}^N\int_\Omega b_i'(u)\frac{\partial I_\eta(|\operatorname{grad} u|)}{\partial x_i}\,dx \\
&=\sum_{i=1}^N\int_\Omega \frac{\partial}{\partial x_i}\big(b_i'(u)\big)\big[|\operatorname{grad} u|S_\eta\big(|\operatorname{grad} u|\big)-I_\eta\big(|\operatorname{grad} u|\big)\big]\,dx \\
&\quad-\int_\Sigma b_i'(u)I_\eta\big(|\operatorname{grad} u|\big)n_i\,d\sigma. \qquad (2.9)
\end{aligned}
$$

From (2.4)-(2.9), by the assumption $a^{ij}(0) = 0$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega I_\eta\big(|\operatorname{grad} u|\big)\,dx \\
&=\int_\Omega \frac{\partial}{\partial x_i}\big(a_u^{ij}(u)u_{x_j}\big)\big[|\operatorname{grad} u|S_\eta\big(|\operatorname{grad} u|\big)-I_\eta\big(|\operatorname{grad} u|\big)\big]\,dx \\
&\quad-\int_\Omega a^{ij}(u)\frac{\partial^2 I_\eta(|\operatorname{grad} u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_j}\,dx-\varepsilon\int_\Omega \frac{\partial^2 I_\eta(|\operatorname{grad} u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}\,dx \\
&\quad+\sum_{i=1}^N\int_\Omega \frac{\partial}{\partial x_i}\big(b_i'(u)\big)\big[|\operatorname{grad} u|S_\eta\big(|\operatorname{grad} u|\big)-I_\eta\big(|\operatorname{grad} u|\big)\big]\,dx
\end{aligned}
$$

$$
\begin{aligned}
&-\int_\Sigma a_u^{ij}(u)u_{x_i}n_jI_\eta\big(|\operatorname{grad}u|\big)\,d\sigma-\int_\Sigma b_i'(u)I_\eta\big(|\operatorname{grad}u|\big)n_i\,d\sigma\\
&-\varepsilon\int_\Sigma\frac{\partial I_\eta(|\operatorname{grad}u|)}{\partial x_i}n_i\,d\sigma.
\end{aligned}
\tag{2.10}
$$

We shall use the fact that on Σ,

$$
b_i'(u)\frac{\partial u}{\partial n}n_i=\varepsilon\triangle u+\frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_i}\right),\qquad u=0,
\tag{2.11}
$$

to calculate the surface integrals in (2.10). Formula (2.11) involves the derivatives on the boundary, let us give some explanation in the concept of local coordinates. Let $\delta_0>0$ be small enough such that

$$
E^{\delta_0}=\big\{x\in\overline{\Omega};\operatorname{dist}(x,\Sigma)\le\delta_0\big\}\subset\bigcup_{\tau=1}^n V_\tau,
$$

where V_τ is a region, on which one can introduce local coordinates

$$
y_k=F_\tau^k(x)\quad(k=1,2,\ldots,N),\qquad y_N|_\Sigma=0,
$$

with F_τ^k appropriately smooth and $F_\tau^N=F_l^N$, such that the y_N-axis coincides with the normal vector. Since the domain is bounded, there exists finite V_τ, $\tau=1,2,\ldots,n$, such that $\bigcup_{\tau=1}^n V_\tau\supset\Sigma$.

Using these local coordinates on V_τ, $\tau=1,2,\ldots,n$, by elementary computations (refer to [24]), we obtain on $\Sigma\cap V_\tau$

$$
u_{x_ix_j}=\sum_{k=1}^N u_{y_Ny_k}F_{x_i}^NF_{x_j}^k+\sum_{k=1}^{N-1}u_{y_Ny_k}F_{x_i}^NF_{x_j}^k+u_{y_m}F_{x_ix_j}^m.
\tag{2.12}
$$

By this formula, what (2.11) means is clear.

Moreover, by (2.11), the surface integrals in (2.10) can be rewritten as

$$
\begin{aligned}
S={}&-\left[\int_\Sigma b_i'(u)I_\eta\big(|\operatorname{grad}u|\big)n_i\,d\sigma+\varepsilon\int_\Sigma\frac{\partial I_\eta\big(|\operatorname{grad}u|\big)}{\partial x_i}n_i\,d\sigma\right.\\
&\left.+\int_\Sigma a_u^{ij}(u)u_{x_i}n_jI_\eta\big(|\operatorname{grad}u|\big)\,d\sigma\right]\\
={}&-\varepsilon\int_\Sigma\left[\frac{\partial I_\eta(|\operatorname{grad}u|)}{\partial x_i}n_i-\triangle u\frac{I_\eta(|\operatorname{grad}u|)}{\frac{\partial u}{\partial n}}\right]d\sigma\\
&+\int_\Sigma a^{ij}(u)\left[\frac{\partial I_\eta(|\operatorname{grad}u|)}{\partial x_i}n_j-u_{x_ix_j}\frac{I_\eta(|\operatorname{grad}u|)}{\frac{\partial u}{\partial n}}\right]d\sigma\\
={}&-\varepsilon\int_\Sigma\left[\frac{\partial I_\eta(|\operatorname{grad}u|)}{\partial x_i}n_i-\triangle u\frac{I_\eta(|\operatorname{grad}u|)}{\frac{\partial u}{\partial n}}\right]d\sigma.
\end{aligned}
$$

Since

$$
u_{x_{N+1}}|_\Sigma=u_t|_\Sigma=0,
$$

we have

$$\lim_{\eta\to 0} S = \varepsilon \int_{\Sigma} \operatorname{sgn}\left(\frac{\partial u}{\partial n}\right)(u_{x_s x_i} n_i n_s - \triangle u)\, d\sigma.$$

Noticing that

$$u_{x_i x_j} n_j n_i = \frac{\sum_{k=1}^{N} u_{y_N y_k} F_{x_i}^N F_{x_j}^k F_{x_j}^N F_{x_i}^N}{|\operatorname{grad} F^N|^2} + \sum_{k=1}^{N-1} u_{y_N y_k} F_{x_i}^k F_{x_j}^N + \frac{u_{y_m} F_{x_i x_j}^m F_{x_j}^N F_{x_i}^N}{|\operatorname{grad} F^N|^2},$$

in which $F^k = F_\tau^k$, by the fact that the normal vector is

$$\vec{n} = \left(\frac{\partial F^N}{\partial x_1}, \dots, \frac{\partial F^N}{\partial x_N}\right) = \operatorname{grad} F^N,$$

we have

$$u_{x_i x_j} n_j n_i - \triangle u = u_{y_m}\left(\frac{F_{x_i x_j}^m F_{x_j}^N F_{x_i}^N}{|\operatorname{grad} F^N|^2} - F_{x_i x_i}^m\right).$$

Using Lemma 2.2, we are able to deduce that $\lim_{\eta\to 0} S$ can be estimated by $|\operatorname{grad} u|_{L_1(\Omega)}$.

Thus, letting $\eta \to 0$ in (2.10) and noticing that

$$\lim_{\eta\to 0}\left[|\operatorname{grad} u| S_\eta\big(|\operatorname{grad} u|\big) - I_\eta\big(|\operatorname{grad} u|\big)\right] = 0,$$

using the fact that $\lim_{\eta\to 0} S$ can be estimated by $|\operatorname{grad} u|_{L_1(\Omega)}$, we have

$$\frac{d}{dt}\int_{\Omega} |\operatorname{grad} u|\, dx \le c_1 + c_2 \int_{\Omega} |\operatorname{grad} u|\, dx.$$

By the well-known Gronwall lemma, we have

$$\int_{\Omega} |\operatorname{grad} u|\, dx\, dt \le c. \tag{2.13}$$

By (2.13), it is easy to show that

$$\iint_{Q_T} a^{ij}(u) u_{x_i} u_{x_j}\, dx\, dt \le c. \tag{2.14}$$

Thus there exist a subsequence $\{u_{\varepsilon_n}\}$ of u_ε and a function $u \in BV(Q_T) \cap L^\infty(Q_T)$ such that $u_{\varepsilon_n} \to u$ a.e. on Q_T; there exist functions $g^i \in L^2(Q_T)$ and a subsequence of $\{\varepsilon\}$ (we can simply denote this subsequence as $\{\varepsilon\}$ itself) such that when $\varepsilon \to 0$,

$$\widehat{\gamma}^{ij}\frac{\partial u_\varepsilon}{\partial x_j} \rightharpoonup g^i, \quad \text{in } L^2(Q_T).$$

□

Proof of Theorem 1.2 We now prove that u is a generalized solution of (1.1)-(1.5)-(1.19). Let $\varphi \in C^2(\overline{Q_T})$, $\varphi_1 \ge 0$, $\operatorname{supp}\varphi \subset \overline{\Omega} \times (0,T)$, $\nabla\varphi_1|_\Omega = 0$ and $\{n_i\}$ be the inner normal vector of Ω. Multiply (1.15) by $\varphi_1 S_\eta(u_\varepsilon - k)$ and integrate over Q_T to obtain

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi_1 S_\eta(u_\varepsilon - k)\,dx\,dt \\
&\quad= \iint_{Q_T} \frac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi_1 S_\eta(u_\varepsilon - k)\,dx\,dt \\
&\qquad+ \varepsilon\iint_{Q_T} \Delta u_\varepsilon \varphi_1 S_\eta(u_\varepsilon - k)\,dx\,dt + \sum_{i=1}^{N}\iint_{Q_T} \frac{\partial b_i(u_\varepsilon)}{\partial x_i}\varphi_1 S_\eta(u_\varepsilon - k)\,dx\,dt.
\end{aligned} \tag{2.15}$$

Let us calculate every term in (2.15) by the part integral method.

$$\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi_1 S_\eta(u_\varepsilon - k)\,dx\,dt = -\iint_{Q_T} I_\eta(u_\varepsilon - k)\varphi_{1t}\,dx\,dt, \tag{2.16}$$

$$\begin{aligned}
&\varepsilon\iint_{Q_T} \Delta u_\varepsilon \varphi_1 S_\eta(u_\epsilon - k)\,dx\,dt \\
&\quad= -\varepsilon\int_0^T\int_\Sigma \nabla u_\varepsilon\cdot\vec{n}\varphi_1 S_\eta(u_\epsilon - k)\,dt\,d\sigma \\
&\qquad- \varepsilon\iint_{Q_T} \nabla u_\varepsilon\big[S_\eta(u_\varepsilon - k)\nabla\varphi_1 + \varphi_1 S_\eta'(u_\varepsilon - k)\nabla u_\varepsilon\big]\,dx\,dt \\
&\quad= \varepsilon S_\eta(k)\int_0^T\int_\Sigma \nabla u_\varepsilon\cdot\vec{n}\varphi_1\,dt\,d\sigma - \varepsilon\iint_{Q_T} \nabla u_\varepsilon S_\eta(u_\varepsilon - k)\nabla\varphi_1\,dx\,dt \\
&\qquad- \varepsilon\iint_{Q_T} |\nabla u_\varepsilon|^2 S_\eta'(u_\varepsilon - k)\varphi_1\,dx\,dt,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}\right)\varphi_1 S_\eta(u_\varepsilon - k)\,dx\,dt \\
&\quad= S_\eta(k)\int_0^T\int_\Sigma a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}n_i\varphi_1\,dt\,d\sigma \\
&\qquad- \iint_{Q_T} a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}\big(S_\eta(u_\varepsilon - k)\varphi_{1x_i} + \varphi_1 S_\eta'(u_\varepsilon - k)u_{\varepsilon x_i}\big)\,dx\,dt \\
&\quad= S_\eta(k)\int_0^T\int_\Sigma a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}n_i\varphi_1\,dt\,d\sigma - \iint_{Q_T} a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}S_\eta(u_\varepsilon - k)\varphi_{1x_i}\,dx\,dt \\
&\qquad- \iint_{Q_T} a^{ij}(u_\varepsilon)u_{\varepsilon x_i}u_{\varepsilon x_j}S_\eta'(u_\varepsilon - k)\varphi_1\,dx\,dt,
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&-\iint_{Q_T} a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}S_\eta(u_\varepsilon - k)\varphi_{1x_i}\,dx\,dt \\
&\quad= \iint_{Q_T} A_\eta^{ij}(u_\varepsilon, k)\varphi_{1x_i x_j}\,dx\,dt + \int_0^T\int_\Sigma \nabla\varphi_{1x_i}n_j A_\eta^{ij}(u_\varepsilon, k)\,dt\,d\sigma,
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial b_i(u_\varepsilon)}{\partial x_i}\varphi_1 S_\eta(u_\varepsilon - k)\,dx\,dt \\
&\quad= -\int_0^T\int_\Sigma \big[b_i(u_\varepsilon) - b(k)\big]n_i\varphi_1 S_\eta(u_\varepsilon - k)\,dt\,d\sigma
\end{aligned}$$

$$-\iint_{Q_T}\big[b_i(u_\varepsilon)-b_i(k)\big]\left[\frac{\partial\varphi_1}{\partial x_i}S_\eta(u_\varepsilon-k)+\varphi_1 S'_\eta(u_\varepsilon-k)\frac{\partial u_\varepsilon}{\partial x_i}\right]dx\,dt$$
$$=S_\eta(k)\int_0^T\int_\Sigma\varphi_1\big[b_i(0)-b_i(k)\big]n_i\,d\sigma\,dt-\iint_{Q_T}B^i_\eta(u_\varepsilon,k)\varphi_{1x_i}\,dx\,dt. \tag{2.20}$$

From (2.15)-(2.20), we have

$$\begin{aligned}
&\iint_{Q_T}I_\eta(u_\varepsilon-k)\varphi_{1t}\,dx\,dt+\iint_{Q_T}A^{ij}_\eta(u_\varepsilon,k)\varphi_{1x_ix_j}\,dx\,dt-\iint_{Q_T}B^i_\eta(u_\varepsilon,k)\varphi_{1x_i}\,dx\,dt\\
&\quad-\varepsilon\iint_{Q_T}\nabla u_\varepsilon\cdot\nabla\varphi_1 S_\eta(u_\varepsilon-k)\,dx\,dt-\varepsilon\iint_{Q_T}|\nabla u_\varepsilon|^2S'_\eta(u_\varepsilon-k)\varphi_1\,dx\,dt\\
&\quad-\iint_{Q_T}a^{ij}(u_\varepsilon)u_{\varepsilon x_i}u_{\varepsilon x_i}S'_\eta(u_\varepsilon-k)\varphi_1\,dx\,dt+\varepsilon S_\eta(k)\int_0^T\int_\Sigma\nabla u_\varepsilon\cdot\vec{n}\varphi_1\,dt\,d\sigma\\
&\quad+S_\eta(k)\int_0^T\int_\Sigma\frac{\partial}{\partial x_i}\Big(a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}n_i\varphi_1\,dt\,d\sigma+S_\eta(k)\int_0^T\int_\Sigma\varphi_{1x_i}n_jA^{ij}_\eta(0,k)\,dt\,d\sigma\\
&\quad+S_\eta(k)\int_0^T\int_{\Sigma_{1\eta k}}\big(b_i(0)-b_i(k)\big)n_i\varphi_1\,dt\,d\sigma\\
&\quad+S_\eta(k)\int_0^T\int_{\Sigma_{2\eta k}}\big(b_i(0)-b_i(k)\big)n_i\varphi_1\,dt\,d\sigma=0.
\end{aligned}\tag{2.21}$$

Taking $\varphi_2\in C^2(\overline{Q_T})$, $\varphi_1|_{\partial\Omega\times[0,T]}=\varphi_2|_{\partial\Omega\times[0,T]}$, $\operatorname{supp}\varphi_2\subset\overline{\Omega}\times(0,T)$,

$$\begin{aligned}
&S_\eta(k)\int_0^T\int_\Sigma a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}n_i\varphi_1\,dt\,d\sigma+\varepsilon S_\eta(k)\int_0^T\int_\Sigma\nabla u_\varepsilon\cdot\vec{n}\varphi_1\,dt\,d\sigma\\
&\quad=S_\eta(k)\bigg\{-\varepsilon\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_2}{\partial x_i}\,dx\,dt-\iint_{Q_T}a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}\varphi_{2x_i}\,dx\,dt\\
&\quad-\iint_{Q_T}\big[b_i(u_\varepsilon)-b_i(0)\big]\frac{\partial\varphi_2}{\partial x_i}\,dx\,dt\\
&\quad+\iint_{Q_T}u_\varepsilon\frac{\partial\varphi_2}{\partial t}\,dx\,dt-\int_0^T\int_\Sigma\big[b_i(0)-b_i(0)\big]n_i\varphi_2\,dt\,d\sigma\bigg\},
\end{aligned}\tag{2.22}$$

$$\begin{aligned}
&\iint_{Q_T}a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}\varphi_{2x_i}\,dx\,dt\\
&\quad=-\int_0^T\int_\Sigma a^{ij}(0)\frac{\partial\varphi_2}{\partial x_i}n_j\,dt\,d\sigma-\iint_{Q_T}a^{ij}(u_\varepsilon)\varphi_{2x_ix_j}\,dx\,dt\\
&\quad=-\iint_{Q_T}a^{ij}(u_\varepsilon)\varphi_{2x_ix_j}\,dx\,dt.
\end{aligned}\tag{2.23}$$

For $\nabla\varphi_1|_\Sigma=0$ and $a^{ij}(0)=0$, from (2.21)-(2.23), we have

$$\begin{aligned}
&\iint_{Q_T}I_\eta(u_\varepsilon-k)\varphi_{1t}\,dx\,dt+\iint_{Q_T}A^{ij}_\eta(u_\varepsilon,k)\triangle\varphi_1\,dx\,dt-\iint_{Q_T}B^i_\eta(u_\varepsilon,k)\varphi_{1x_i}\,dx\,dt\\
&\quad+S_\eta(k)\bigg[-\varepsilon\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_2}{\partial x_i}\,dx\,dt+\iint_{Q_T}a^{ij}(u_\varepsilon)\varphi_{2x_ix_j}\,dx\,dt\\
&\quad-\iint_{Q_T}b_i(u_\varepsilon)\frac{\partial\varphi_2}{\partial x_i}\,dx\,dt+\iint_{Q_T}u_\varepsilon\frac{\partial\varphi_2}{\partial t}\,dx\,dt\bigg]
\end{aligned}$$

$$-\varepsilon\iint_{Q_T}\nabla u_\varepsilon\cdot\nabla\varphi_1 S_\eta(u_\varepsilon-k)\,dx\,dt-\iint_{Q_T}a^{ij}(u_\varepsilon)u_{\varepsilon x_i}u_{\varepsilon x_j}S'_\eta(u_\varepsilon-k)\varphi_1\,dx\,dt$$
$$+S_\eta(k)\int_0^T\int_{\Sigma_{1\eta k}}\big[(b_i(0)-b_i(k)\big]n_i\varphi_1\,dt\,d\sigma\ge 0. \tag{2.24}$$

By Lemma 2.1,

$$\liminf_{\varepsilon\to 0}\iint_{Q_T}S'_\eta(u_\varepsilon-k)a^{ij}(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}\varphi_1\,dx\,dt$$
$$\ge\iint_{Q_T}\sum_{i=1}^N\big|g^i\big|^2S'_\eta(u-k)\varphi_1\,dx\,dt. \tag{2.25}$$

Let $\varepsilon\to 0$ in (2.24). By (2.25), we get (1.21). At the same time, (1.22) is naturally concealed in the limiting process.

The proof of (1.23) is similar to that in [2, 12], we omit the details here. □

3 Proof of Theorem 1.3

Similar as the proof of Lemma 2 in [9], we can prove the following lemma.

Lemma 3.1 *Let u be a solution of* (1.1). *Then*

$$\int_{u^-}^{u^+}\gamma^{ij}(s)\,ds\cdot\nu_i=0,\quad a.e.\ (x,t)\ on\ \Gamma^u, j=1,2,\ldots,N,$$

is true in the sense of Hausdorff measure $H_N(\Gamma^u)$.

Proof of Theorem 1.3 Let u, v be two entropy solutions of (1.1) with different initial values

$$u(x,0)=u_0(x),\qquad v(x,0)=v_0(x) \tag{3.1}$$

and with the same homogeneous boundary value $\gamma u(x,t)=\gamma v(x,t)=0$, $(x,t)\in\Sigma_1$.

By Definition 1.1, for any $\varphi_1,\varphi_2\in C^2(\overline{Q_T})$, $\varphi_1\ge 0$, $\varphi_1|_{\partial\Omega\times[0,T]}=\varphi_2|_{\partial\Omega\times[0,T]}$, $\operatorname{supp}\varphi_2$, $\operatorname{supp}\varphi_1\subset\overline{\Omega}\times(0,T)$, $\eta>0$, $k,l\in\mathbb{R}$, we have

$$\iint_{Q_T}\left[I_\eta(u-k)\varphi_{1t}-B^i_\eta(u,k)\varphi_{1x_i}+A^{ij}_\eta(u,k)\varphi_{1x_ix_j}-S'_\eta(u-k)\sum_{i=1}^N\big|g^i(u)\big|^2\varphi_1\right]dx\,dt$$
$$+S_\eta(k)\big[b_i(0)-b_i(k)\big]\int_0^T\int_{\Sigma_{1\eta k}}\varphi_1 n_i\,dt\,d\sigma$$
$$+S_\eta(k)\iint_{Q_T}\big[u\varphi_{2t}-\big(b_i(u)-b_i(0)\big)\varphi_{2x_i}+A^{ij}(u)\varphi_{2x_ix_j}\big]\,dx\,dt\ge 0, \tag{3.2}$$

$$\iint_{Q_T}\left[I_\eta(v-l)\varphi_{1\tau}-B^i_\eta(v,l)\varphi_{1y_i}+A^{ij}_\eta(v,l)\varphi_{1y_iy_j}-S'_\eta(v-l)\sum_{i=1}^N\big|g^i(v)\big|^2\varphi_1\right]dy\,d\tau$$
$$+S_\eta(l)\big[(b_i(0)-b_i(l)\big]\int_0^T\int_{\Sigma_{1\eta k}}\varphi_1 n_i\,d\tau\,d\sigma$$
$$+S_\eta(l)\iint_{Q_T}\big[v\varphi_{2\tau}-\big(b_i(v)-b_i(0)\big)\varphi_{2y_i}+A^{ij}(v)\varphi_{2y_iy_j}\big]\,dy\,d\tau\ge 0. \tag{3.3}$$

Especially, if $\varphi_1 \in C_0^2(Q_T)$, $\varphi_2 \equiv 0$, we have

$$\iint_{Q_T} \Bigg[I_\eta(u-k)\varphi_{1t} - B_\eta^i(u,k)\varphi_{1x_i} + A_\eta^{ij}(u,k)\varphi_{1x_ix_j} - S'_\eta(u-k)\sum_{i=1}^N \big|g^i(u)\big|^2\varphi_1 \Bigg]\,dx\,dt \geq 0, \tag{3.4}$$

$$\iint_{Q_T} \Bigg[I_\eta(v-l)\varphi_{1\tau} - B_\eta^i(v,l)\varphi_{1y_i} + A_\eta^{ij}(v,l)\varphi_{1y_iy_j} - S'_\eta(v-l)\sum_{i=1}^N \big|g^i\big|^2(v)\varphi_1 \Bigg]\,dy\,d\tau \geq 0. \tag{3.5}$$

Let $\psi(x,t,y,\tau) = \phi(x,t)j_h(x-y,t-\tau)$. Here $\phi(x,t) \geq 0$, $\phi(x,t) \in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau) = \omega_h(t-\tau)\prod_{i=1}^N \omega_h(x_i-y_i), \tag{3.6}$$

$$\omega_h(s) = \frac{1}{h}\omega\left(\frac{s}{h}\right), \qquad \omega(s) \in C_0^\infty(R), \qquad \omega(s) \geq 0, \qquad \omega(s) = 0 \quad \text{if } |s| > 1, \int_{-\infty}^{\infty} \omega(s)\,ds = 1. \tag{3.7}$$

Then we choose $k = v(y,\tau)$, $l = u(x,t)$, $\varphi_1 = \psi(x,t,y,\tau)$ in (3.4), (3.5), integrate over Q_T respectively, plus them together and get the following inequality

$$\iint_{Q_T}\iint_{Q_T} \big[I_\eta(u-v)(\psi_t+\psi_\tau) - \big(B_\eta^i(u,v)\psi_{x_i} + B_\eta^i(v,u)\psi_{y_i}\big) + A_\eta^{ij}(u,v)\psi_{x_ix_j} + A_\eta^{ij}(v,u)\psi_{y_iy_j} \big] - S'_\eta(u-v)\left(\sum_{i=1}^N \big|g^i(u)\big|^2 + \sum_{i=1}^N \big|g^i(v)\big|^2\right)\psi\,dx\,dt\,dy\,d\tau \geq 0. \tag{3.8}$$

Clearly,

$$\frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = 0, \qquad \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0, \quad i = 1,\ldots,N;$$
$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \phi}{\partial t}j_h, \qquad \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i}j_h.$$

Noticing that

$$\lim_{\eta\to 0} B_\eta^i(u,v) = \lim_{\eta\to 0} B_\eta^i(v,u) = \operatorname{sgn}(u-v)\big(b_i(u) - b_i(v)\big),$$

as $\eta \to 0$, we have

$$\iint_{Q_T}\iint_{Q_T} \big[B_\eta^i(u,v)\psi_{x_i} + B_\eta^i(v,u)\psi_{y_i}\big]\,dx\,dt\,dy\,d\tau \to \iint_{Q_T}\iint_{Q_T} \operatorname{sgn}(u-v)\big[b_i(u) - b_i(v)\big]\phi_{x_i}j_h\,dx\,dt\,dy\,d\tau,$$

as $h \to 0$, we have

$$\iint_{Q_T} \iint_{Q_T} \operatorname{sgn}(u-v)\bigl[b_i(u)-b_i(v)\bigr]\phi_{x_i} j_h \,dx\,dt\,dy\,d\tau \\ \to \iint_{Q_T} \operatorname{sgn}(u-v)\bigl[b_i(u)-b_i(v)\bigr]\phi_{x_i}\,dx\,dt. \tag{3.9}$$

For the third term and the fourth term on the left-hand side of (3.8), we have

$$\begin{aligned} &\iint_{Q_T} \bigl[A_\eta^{ij}(u,v)\psi_{x_i x_j} + A_\eta^{ij}(v,u)\psi_{y_i y_j}\bigr]\,dx\,dt\,dy\,d\tau \\ &= \iint_{Q_T}\iint_{Q_T} \bigl\{A_\eta^{ij}(u,v)(\phi_{x_i x_j} j_h + 2\phi_{x_i} j_{hx_j} + \phi j_{hx_i x_j}) + A_\eta^{ij}(v,u)\phi j_{hy_i y_j}\bigr\}\,dx\,dt\,dy\,d\tau \\ &= \iint_{Q_T}\iint_{Q_T} \bigl\{A_\eta^{ij}(u,v)\phi j_{hx_i x_j} + A_\eta^{ij}(u,v)\phi_{x_i} j_{hx_j} + A_\eta^{ij}(v,u)\phi_{x_i} j_{hy_j}\bigr\}\,dx\,dt\,dy\,d\tau \\ &\quad - \iint_{Q_T}\iint_{Q_T} \biggl\{\widehat{a^{ij}(u) S_\eta(u} - v)\frac{\partial u}{\partial x_i} \\ &\quad - \int_u^v \widehat{a^{ij}(s) S'_\eta(s} - v)\,ds \frac{\partial u}{\partial x_i}\phi j_{hx_j}\biggr\}\,dx\,dt\,dy\,d\tau, \end{aligned} \tag{3.10}$$

where Definition 1.1 and formula (2.2) are used, *i.e.*,

$$\widehat{a^{ij}(u) S_\eta(u} - v) = \int_0^1 a^{ij}\bigl(su^+ + (1-s)u^-\bigr) S_\eta\bigl(su^+ + (1-s)u^- - v\bigr)\,ds,$$

$$\int_u^v \widehat{a^{ij}(s) S'_\eta(s} - v)\,ds = \int_0^1 \int_{su^+ + (1-s)u^-}^v a^{ij}(\sigma) S_\eta\bigl(\sigma - su^+ - (1-s)u^-\bigr)\,d\sigma\,ds.$$

For the fifth term on the left-hand side of (3.8), we have

$$\begin{aligned} &\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\Biggl(\sum_{i=1}^N \bigl|g^i(u)\bigr|^2 + \sum_{i=1}^N \bigl|g^i(v)\bigr|^2\Biggr)\psi\,dx\,dt\,dy\,d\tau \\ &= \sum_{i=1}^N \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\bigl(\bigl|g^i(u)\bigr| - \bigl|g^i(v)\bigr|\bigr)^2 \psi\,dx\,dt\,dy\,d\tau \\ &\quad + 2\sum_{i=1}^N \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v) g^i(u) g^i(v)\psi\,dx\,dt\,dy\,d\tau \\ &= \sum_{i=1}^N \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\bigl(\bigl|g^i(u)\bigr| - \bigl|g^i(v)\bigr|\bigr)^2 \psi\,dx\,dt\,dy\,d\tau \\ &\quad + 2\sum_{n=1}^N \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\partial_{x_j}\int_v^u \gamma^{ni}(s)\,ds\,\partial_{y_j}\int_u^v \gamma^{nj}(s)\psi\,dx\,dy\,dt\,d\tau. \end{aligned} \tag{3.11}$$

Now, by the properties of BV function,

$$\begin{aligned} &\int_{Q_T}\int_{Q_T} \partial_{x_i}\partial_{y_j}\int_v^u r^{ni}(\delta)\int_\delta^v r^{nj}(\sigma) S'_\eta(\sigma-\delta)\,d\sigma\,d\delta\psi\,dx\,dt\,dy\,d\tau \\ &= \int_{Q_T}\int_{Q_T} \psi\,\partial_{y_j}\int_0^1 r^{in}\bigl(su^+ + (1-s)u^-\bigr) \end{aligned}$$

$$\times \int_{su^+ +(1-s)u^-}^{v} r^{nj}(\sigma) S'_\eta\big(\sigma - su^+ - (1-s)u^-\big)\, d\sigma\, ds \frac{\partial u}{\partial x_i}\, dx\, dt\, dy\, d\tau$$

$$= \int_{Q_T}\int_{Q_T} \psi\, \partial_{x_i} \int_0^u r^{in}(\delta)\, d\delta \cdot \partial_{y_j} \int_0^v r^{nj}(\delta)\, d\delta S'_\eta(v-u)\, dx\, dt\, dy\, d\tau$$

$$\int_{Q_T}\int_{Q_T} \partial_{x_i}\partial_{y_j} \int_v^u r^{in}(\delta) \int_\delta^v r^{nj}(\sigma) S'_\eta(\sigma-\delta)\, d\sigma\, d\delta \psi\, dx\, dt\, dy\, d\tau$$

$$= \int_{Q_T}\int_{Q_T} \psi\, \partial_{y_j} \int_0^1 r^{in}\big(su^+ + (1-s)u^-\big)$$

$$\times \int_{su^+ +(1-s)u^-}^{v} r^{nj}(\sigma) S'_\eta\big(\sigma - su^+ - (1-s)u^-\big) \frac{\partial u}{\partial x_i}\, d\sigma\, ds$$

$$= \int_{Q_T}\int_{Q_T} \phi j_{hx_j} \int_0^1 r^{in}\big(su^+ + (1-s)u^-\big)$$

$$\times \int_{su^+ +(1-s)u^-}^{v} r^{nj}(\sigma) S'_\eta\big(\sigma - su^+ - (1-s)u^-\big)\, d\sigma\, ds \frac{\partial u}{\partial x_i}\, dx\, dt\, dy\, d\tau,$$

we have

$$-\int_{Q_T}\int_{Q_T} \left(\widehat{a^{ij}(u) S_\eta(u} - v) \frac{\partial u}{\partial x_i} + \int_u^v \widehat{a^{ij}(s) S'_\eta(s} - u)\, ds \frac{\partial u}{\partial x_i} \right) j_{hx_j} \phi\, dx\, dt\, dy\, d\tau$$

$$- 2 \int_{Q_T}\int_{Q_T} S'_\eta(u-v) \partial x_i \int_0^u r^{in}(s)\, ds \cdot \partial y_j \int_0^v r^{nj}(s)\, ds \psi\, dx\, dt\, dy\, d\tau$$

$$= -\int_{Q_T}\int_{Q_T} \left\{ \int_0^1 a^{ij}\big(su^+ + (1-s)u^-\big) S_\eta\big(su^+ + (1-s)u^- - v\big)\, ds \right.$$

$$+ \int_0^1 \int_{su^+ +(1-s)u^-}^{v} a^{ij}(\sigma) S'_\eta\big(\sigma - su^+ - (1-s)u^-\big)\, d\sigma\, ds$$

$$- 2 \int_0^1 r^{in}\big(su^+ + (1-s)u^-\big) \int_{su^+ +(1-s)u^-}^{v} r^{nj}(\sigma)$$

$$\left. \times S'_\eta\big(\sigma - su^+ - (1-s)u^-\big)\, d\sigma\, ds \right\} \frac{\partial u}{\partial x_i} j_{hx_j} \phi\, dx\, dt\, dy\, d\tau$$

$$= \int_{Q_T}\int_{Q_T} \left\{ \int_0^1 \int_{su^+ +(1-s)u^-}^{v} r^{nj}\big(su^+ + (1-s)u^-\big) \right.$$

$$\times r^{in}\big(su^+ + (1-s)u^-\big) S'_\eta\big(\sigma - su^+ - (1-s)u^-\big)\, d\sigma\, ds$$

$$+ \int_0^1 \int_{su^+ +(1-s)u^-}^{v} r^{in}(\sigma) r^{nj}(\sigma) S'_\eta\big(\sigma - su^+ - (1-s)u^-\big)\, d\sigma\, ds$$

$$- 2 \int_0^1 r^{in}\big(su^+ + (1-s)u^-\big) \int_{su^+ +(1-s)u^-}^{v} r^{nj}(\sigma)$$

$$\left. \times S'_\eta\big(\sigma - su^+ - (1-s)u^-\big)\, d\sigma\, ds \right\} \frac{\partial u}{\partial x_i} j_{hx_j} \phi\, dx\, dt\, dy\, d\tau$$

$$= \int_{Q_T}\int_{Q_T} \left\{ \int_0^1 \int_{su^+ +(1-s)u^-}^{v} \Big\{ r^{in}\big(su^+ + (1-s)u^-\big) \big(r^{nj}\big(su^+ + (1-s)u^-\big) - r^{nj}(\sigma) \big) \right.$$

$$- r^{nj}(\sigma) \big(r^{in}\big(su^+ + (1-s)u^-\big) - r^{in}(\sigma) \big) \Big\}$$

$$\left. \times S'_\eta\big(\sigma - su^+ - (1-s)u^-\big)\, d\sigma\, ds \right\} \frac{\partial u}{\partial x_i} j_{hx_j} \phi\, dx\, dt\, dy\, d\tau.$$

Since

$$\int_{su^+ +(1-s)u^-}^{v} \{r^{in}(su^+ + (1-s)u^-)(r^{nj}(su^+ + (1-s)u^-) - r^{nj}(\sigma)) \\ - r^{nj}(\sigma)(r^{in}(su^+ + (1-s)u^-) - r^{in}(\sigma))\} S'_\eta(\sigma - su^+ - (1-s)u^-)\, d\sigma$$

is uniformly bounded and leads to 0 as $\eta \to 0$, we have, when $\eta \to 0$,

$$-\int_{Q_T}\int_{Q_T}\left(a^{ij}(u)\widehat{S_\eta(u-v)}\frac{\partial u}{\partial x_i} - \int_u^v a^{ij}(s)\widehat{S'_\eta(s-u)}\, ds \frac{\partial u}{\partial x_i}\right) j_{hx_j}\phi\, dx\, dt\, dy\, d\tau \\ - 2\int_{Q_T}\int_{Q_T} S'_\eta(u-v)\partial x_i \int_0^u r^{in}(s)\, ds \cdot \partial y_j \int_0^v r^{nj}(s)\, ds \psi\, dx\, dt\, dy\, d\tau \to 0. \tag{3.12}$$

At the same time, noticing that $\lim_{\eta\to 0} A_\eta^{ij}(u,v) = \lim_{\eta\to 0} A_\eta^{ij}(v,u) = \operatorname{sgn}(u-v)(A^{ij}(u) - A^{ij}(v))$, we have

$$\lim_{\eta\to 0}\left(A_\eta^{ij}(u,v)\phi_{x_i} j_{hx_i} + A_\eta^{ij}(v,u)\phi_{x_i} j_{hy_i}\right) = 0. \tag{3.13}$$

Combining (3.8)-(3.13), and letting $\eta \to 0$, $h \to 0$, we get

$$\int_{Q_T}\{|u(x,t) - v(x,t)|\phi_t + \operatorname{sgn}(u-v)(A^{ij}(u) - A^{ij}(v))\phi_{x_i x_j} \\ - \operatorname{sgn}(u-v)(b_i(u) - b_i(v))\phi_{x_i}\}\, dx\, dt \geq 0. \tag{3.14}$$

Let δ_ε be the mollifier as usual. If $y = (x_1, \ldots, x_N)$, then

$$\delta(y) = \begin{cases} \frac{1}{A} e^{\frac{1}{|y|^2-1}} & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geq 1, \end{cases}$$

where

$$A = \int_{B_1(0)} e^{\frac{1}{|y|^2-1}}\, dx.$$

For any given $\varepsilon > 0$, $\delta_\varepsilon(y)$ is defined as

$$\delta_\varepsilon(y) = \frac{1}{\varepsilon^N}\delta\left(\frac{y}{\varepsilon}\right).$$

Especially, we can choose ϕ in (3.14) by

$$\phi(x,t) = \omega_{\lambda\varepsilon}(x)\eta(t),$$

where $\eta(t) \in C_0^\infty(0,T)$, $\omega_{\lambda\varepsilon}(x)$ is defined as follows. Let $\omega_\lambda(x) \in C_0^2(\Omega)$ be defined as follows: for any given small enough $0 < \lambda$, $0 \leq \omega_\lambda \leq 1$, $\omega|_{\partial\Omega} = 0$ and

$$\omega_\lambda(x) = 1, \quad \text{if } d(x) = \operatorname{dist}(x, \partial\Omega) \geq \lambda.$$

When $0 \leq d(x) \leq \lambda$,

$$\omega_\lambda\big(d(x)\big) = 1 - \frac{(d(x)-\lambda)^2}{\lambda^2}.$$

Then

$$\omega_{\lambda\varepsilon} = \omega_\lambda * \delta_\varepsilon(d),$$

$$\omega'_{\lambda\varepsilon}(d) = \int_{\{|s|<\varepsilon\}\cap\{0<d-s<\lambda\}} \omega'_\lambda(d-s)\delta_\varepsilon(s)\,ds$$

$$= -\int_{\{|s|<\varepsilon\}\cap\{0<d-s<\lambda\}} \frac{2(d-s-\lambda)}{\lambda^2}\delta_\varepsilon(s)\,ds,$$

$$\big|\omega'_{\lambda\varepsilon}(d)\big| \leq \frac{c}{\lambda}, \tag{3.15}$$

$$\omega''_{\lambda\varepsilon}(d) = \omega''_\lambda * \delta_\varepsilon(d) = -\frac{2}{\lambda^2}\int_{\{|s|<\varepsilon\}\cap\{0<d-s<\lambda\}} \delta_\varepsilon(s)\,ds. \tag{3.16}$$

Now,

$$\phi_{x_ix_j} = \eta(t)\big(\omega_{\lambda\varepsilon}\big(d(x)\big)\big)_{x_ix_j}$$

$$= \eta(t)\big(\omega'_{\lambda\varepsilon}(d)d_{x_i}\big)_{x_j}$$

$$= \eta(t)\big[\omega''_{\lambda\varepsilon}(d)d_{x_i}d_{x_j} + \omega'_{\lambda\varepsilon}(d)d_{x_ix_j}\big]$$

$$= \eta(t)\left[-\frac{2}{\lambda^2}d_{x_i}d_{x_j}\int_{\{|s|<\varepsilon\}\cap\{0<d-s<\lambda\}} \delta_\varepsilon(s)\,ds + \omega'_{\lambda\varepsilon}(d)d_{x_ix_j}\right]$$

using the conditions $|d_{x_ix_j}| \leq c$, and using the fact that $|\nabla d| = 1$, noticing that

$$\operatorname{sgn}(u-v)\big(A^{ij}(u) - A^{ij}(v)\big)d_{x_i}d_{x_j} = |u-v|a^{ij}(\zeta)d_{x_i}d_{x_j} \geq 0,$$

where $\zeta \in (v,u)$. Then by (1.26), from (3.14), we have

$$\int_{Q_T}\big|u(x,t)-v(x,t)\big|\phi_t\,dx\,dt + c\int_0^T\int_{\Omega\setminus\Omega_\lambda}\eta(t)\big|\omega'_{\lambda\varepsilon}(d)\big||u-v|\,dx\,dt \geq 0, \tag{3.17}$$

where $\Omega_\lambda = \{x \in \Omega : d(x,\partial\Omega) < \lambda\}$.

According to the definition of trace (3.3), let $\lambda \to 0$ in (3.17). By (3.15)-(3.16), we have

$$c \operatorname*{ess\,sup}_{\Sigma_2\times(0,T)}\big|f(x,t)-g(x,t)\big| + \int_{Q_T}\big|u(x,t)-v(x,t)\big|\eta'_t\,dx\,dt \geq 0. \tag{3.18}$$

Let $0 < s < \tau < T$, and

$$\eta(t) = \int_{\tau-t}^{s-t}\alpha_\varepsilon(\sigma)\,d\sigma, \quad \varepsilon < \min\{\tau, T-s\}.$$

Here $\alpha_\varepsilon(t)$ is the kernel of the mollifier with $\alpha_\varepsilon(t) = 0$ for $t \notin (-\varepsilon,\varepsilon)$. Then

$$c \operatorname*{ess\,sup}_{\Sigma_2\times(0,T)}\big|f(x,t)-g(x,t)\big| + \int_0^T\big[\alpha_\varepsilon(t-s)-\alpha_\varepsilon(t-\tau)\big]|u-v|_{L^1(\Omega)}\,dt \geq 0.$$

Let $\varepsilon \to 0$. Then

$$\left|u(x,\tau) - v(x,\tau)\right|_{L^1(\Omega)} \le \left|u(x,s) - v(x,s)\right|_{L^1(\Omega)} + c \operatorname*{ess\,sup}_{\Sigma_2 \times (0,T)} \left|f(x,t) - g(x,t)\right|$$

and the desired result follows by letting $s \to 0$. □

4 The case of $\Sigma_1 = \emptyset$

If $\Sigma_1 = \emptyset$, the solution of equation (1.1) is completely controlled by the initial value condition. Now, we should give the following definition.

Definition 4.1 A function u is said to be the entropy solution of equation (1.1) with initial value (1.5) if

1. u satisfies

$$u \in BV(Q_T) \cap L^\infty(Q_T), \qquad \frac{\partial}{\partial x_i} \int_0^u \sqrt{a(s)}\, ds \in L^2(Q_T). \tag{4.1}$$

2. For any $\varphi \in C_0^2(Q_T)$, $\varphi \ge 0$, for any $k \in \mathbb{R}$, for any small $\eta > 0$, u satisfies

$$\iint_{Q_T} \Bigg[I_\eta(u-k)\varphi_t - B_\eta^i(u,k)\varphi_{x_i} + A_\eta^{ij}(u,k) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} - S_\eta'(u-k) \sum_{j=1}^N \left|g^j\right|^2 \varphi \Bigg]\, dx\, dt \ge 0. \tag{4.2}$$

3. The initial value is true in the sense of

$$\lim_{t \to 0} \int_\Omega \left|u(x,t) - u_0(x)\right| dx = 0, \quad \text{a.e. } x \in \Omega. \tag{4.3}$$

Similarly as in the proofs of Theorems 1.2 and 1.3, we can prove the following theorems, and we omit the details here.

Theorem 4.2 *Suppose that* $A^{ij}(s)$ *is* C^3, $b_i(s)$ *is* C^2, $u_0(x) \in L^\infty(\Omega)$, *and suppose that*

$$a^{ij}(0) = 0. \tag{4.4}$$

Then equation (1.1) *with initial value condition* (1.5) *has an entropy solution in the sense of Definition* 4.1.

Theorem 4.3 *Suppose that* $A^{ij}(s)$, $b_i(s)$ *is* C^1. *Let* u, v *be solutions of equation* (1.1) *with different initial values* $u_0(x), v_0(x) \in L^\infty(\Omega)$, *respectively. Suppose that the distance function* $d(x) = \operatorname{dist}(x, \Sigma)$ *satisfies* (2.17), *and that*

$$\gamma u(x,t) = f(x,t), \qquad \gamma v = g(x,t), \quad (x,t) \in \Sigma \times (0,T). \tag{4.5}$$

Then

$$\int_\Omega \left|u(x,t) - v(x,t)\right| dx \le \int_\Omega |u_0 - v_0|\, dx + \operatorname*{ess\,sup}_{(x,t) \in \Sigma \times (0,T)} \left|f(x,t) - g(x,t)\right|. \tag{4.6}$$

Competing interests
The author declares that they have no competing interests.

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