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Existence results for some nonlinear elliptic equations with measure data in Orlicz-Sobolev spaces

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Abstract
We prove the existence results in the setting of Orlicz spaces for the following nonlinear elliptic equation:

$$A(u) + g(x, u, Du) = \mu,$$

where A is a Leray-Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$, while g is a nonlinear term having a growth condition with respect to Du, but does not satisfy any sign condition. The right-hand side μ is a bounded Radon measure data.

Keywords: Orlicz-Sobolev spaces; nonlinear elliptic equations; measure data

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. In the classical Sobolev space $W_0^{1,p}(\Omega)$, Porretta [1] studied the solution of the following problem:

$$-\operatorname{div} a(x, u, Du) = H(x, u, Du) + \mu, \tag{1}$$

where a is supposed to satisfy a polynomial growth condition with respect to u and Du, H has natural growth with respect to Du without any sign condition (*i.e.*, $H(x, s, \xi)s \geq 0$), that is, a and H satisfy

(a1) $|a(x, s, \xi)| \leq \beta(k(x) + |s|^{p-1} + |\xi|^{p-1})$, $k(x) \in L^{p'}(\Omega)$, $\beta > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$,

(H) $|H(x, s, \xi)| \leq \gamma(x) + g(s)|\xi|^p$, $\gamma(x) \in L^1(\Omega)$, and $g : \mathbb{R} \to \mathbb{R}^+$ is continuous, $g \geq 0$, $g \in L^1(\mathbb{R})$,

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$. The right-hand side μ is a nonnegative bounded Radon measure on Ω. The model example is the equation

$$-\Delta_p(u) + g(u)|Du|^p = \mu$$

in Ω coupled with a Dirichlet boundary condition.

Aharouch *et al.* [2] proved the existence results in the setting of Orlicz spaces for the unilateral problem associated to the following equation:

$$A(u) + g(x, u, Du) = f, \tag{2}$$

where $A(u) = -\operatorname{div} a(x,u,Du)$ is a Leray-Lions operator defined on $D(A) \subset W_0^1L_M(\Omega)$, a and g satisfy the following growth conditions:

(a2) $|a(x,s,\xi)| \leq c(x) + k_1\bar{P}^{-1}(M(k_2|s|)) + k_3\bar{M}^{-1}(M(k_4|\xi|))$, $k_1,k_2,k_3,k_4 \geq 0$, $c(x) \in E_{\bar{M}}(\Omega)$,

(g) $|g(x,s,\xi)| \leq \gamma(x) + \rho(s)M(|\xi|)$, $\gamma(x) \in L^1(\Omega)$, and $\rho : \mathbb{R} \to \mathbb{R}^+$ is continuous, $\rho \geq 0$, $\rho \in L^1(\mathbb{R})$,

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, where M and P are N-functions such that $P \ll M$. The right-hand side f belongs to $L^1(\Omega)$. The obstacle is a measurable function.

Youssfi *et al.* [3] proved the existence of bounded solutions of problem (2) whose principal part has a degenerate coercivity, where g does not satisfy the sign condition and f is an appropriate integrable source term.

Some elliptic equations in Orlicz spaces with variational structure of the form

$$\int_\Omega M(|Du|)\,dx$$

have been studied, where $u : \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^n$ is a bounded open set (see, *e.g.*, [4–7]). The associated Euler-Lagrange system is

$$-\operatorname{div}\left(M'(|Du|)\frac{Du}{|Du|}\right) = 0 \quad (\text{see}, \textit{e.g.}, [5]).$$

In this case methods from the calculus of variations can be used and regularity of solutions can be shown. However, the assumptions are strong. For example, it is needed that M satisfies Δ_2 condition in [4] and [6].

The purpose of this paper is to study the existence of a solution for the following nonlinear Dirichlet problem:

$$A(u) + g(x,u,Du) = \mu, \tag{3}$$

where $A(u) = -\operatorname{div} a(x,u,Du)$ is a Leray-Lions operator defined on $D(A) \subset W_0^1L_M(\Omega)$ having the following growth condition:

$$|a(x,s,\xi)| \leq \beta\left[c(x) + \bar{M}^{-1}(M(|s|)) + \bar{M}^{-1}(M(|\xi|))\right], \quad \beta > 0, c(x) \in E_{\bar{M}}(\Omega)$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, g is a nonlinear term having the growth condition (g) without any sign condition, and μ is a nonnegative bounded Radon measure on Ω. When trying to relax the restriction on a and H in Eq. (1), we are led to replace Sobolev spaces by Orlicz-Sobolev spaces without assuming any restriction on M (*i.e.*, without the Δ_2 condition). The choice $M(t) = t^p$, $p > 1$, $t > 0$ leads to [1]. A nonstandard example is $M(t) = t\ln(1+t)$, $t > 0$ (see, *e.g.*, [8, 9]). Taking $M(t) = e^t - 1$, $t > 0$, M does not satisfy Δ_2-condition. Moreover, the elimination of the term g in Eq. (3) can lead to [10]. A specific example to which our result applies includes the following:

$$-\operatorname{div}\left(a(u)\frac{M(|Du|)Du}{|Du|^2}\right) + a'(u)\int_0^{|Du|}\frac{M(t)}{t}\,dt = \delta,$$

where $a(s)$ is a smooth function, and δ is a Dirac measure.

This paper is organized as follows. In Section 2, we recall some preliminaries and some technical lemmas which will be needed in Section 3. In Section 3, we first prove that there exist solutions in $W_0^1E_M(\Omega)$ for approximate equations by using a linear functional analysis method; next, following [1–3, 10], we prove the existence results for problem (9)-(10) and show that solutions belong to Orlicz-Sobolev spaces $W_0^1L_B(\Omega)$ for any $B \in \mathcal{P}_M$, where $\mathcal{P}_M$ is a special class of N-functions (see Theorem 3.1 below).

For some classical results on equations, we refer to [11–18].

2 Preliminaries

2.1 *N*-function

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function; *i.e.*, M is continuous, convex with $M(u) > 0$ for $u > 0$, $M(u)/u \to 0$ as $u \to 0$, and $M(u)/u \to \infty$ as $u \to \infty$. Equivalently, M admits the representation $M(u) = \int_0^u \phi(t)\,dt$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing, right-continuous function with $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$, and $\phi(t) \to \infty$ as $t \to \infty$.

The conjugated N-function $\bar{M}$ of M is defined by $\bar{M}(v) = \int_0^v \psi(s)\,ds$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\psi(s) = \sup\{t : \phi(t) \le s\}$.

The N-function M is said to satisfy the Δ_2 condition if, for some $k > 0$,

$$M(2u) \le kM(u), \quad \forall u \ge 0.$$

The N-function M is said to satisfy the Δ_2 condition near infinity if, for some $k > 0$ and $u_0 > 0$, $M(2u) \le kM(u)$, $\forall u \ge u_0$ (see [19, 20]).

Moreover, one has the following Young inequality:

$$\forall u, v \ge 0, \quad uv \le M(u) + \bar{M}(v).$$

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P, Q be two N-functions, $P \ll Q$ means that P grows essentially less rapidly than Q; *i.e.*, for each $\varepsilon > 0$, $P(t)/Q(\varepsilon t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} Q^{-1}(t)/P^{-1}(t) = 0$ (see [19, 21]).

2.2 Orlicz spaces

Let Ω be an open subset of $\mathbb{R}^N$ and M be an N-function. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M\big(u(x)\big)\,dx < +\infty \quad \left(\text{resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right).$$

$L_M(\Omega)$ is a Banach space under the Luxemburg norm

$$\|u\|_{(M)} = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \le 1\right\},$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$ but not necessarily a linear space.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the

Δ_2 condition for all t or for t large according to whether Ω has infinite measure or not. The dual space of $E_M(\Omega)$ can be identified with $L_{\bar{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)\,dx$, and the dual norm of $L_{\bar{M}}(\Omega)$ is equivalent to $\|\cdot\|_{(\bar{M})}$.

2.3 Orlicz-Sobolev spaces

We now turn to the Orlicz-Sobolev spaces. The class $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) consists of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). The classes $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ of such functions may be given the norm

$$\|u\|_{\Omega,M} = \sum_{|\alpha|\leq 1} \left\|D^\alpha u\right\|_{(M)}.$$

These classes will be Banach spaces under this norm. We refer to spaces of the forms $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ as Orlicz-Sobolev spaces. Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ and $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. If M satisfies Δ_2 condition (near infinity only when Ω has finite measure), then $W^1L_M(\Omega) = W^1E_M(\Omega)$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We recall that a sequence u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega M\left(\frac{|D^\alpha u_n - D^\alpha u|}{\lambda}\right) dx \to 0 \quad \text{as } n \to \infty \text{ for all } |\alpha| \leq 1.$$

Let $W^{-1}L_{\bar{M}}(\Omega)$ (resp. $W^{-1}E_{\bar{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\bar{M}}(\Omega)$ (resp. $E_{\bar{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\bar{M}})$. Consequently, the action of a distribution in $W^{-1}L_{\bar{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined. The dual space of $W_0^1E_M(\Omega)$ is $W^{-1}L_{\bar{M}}(\Omega)$ and the dual space of $W^{-1}E_{\bar{M}}(\Omega)$ is $W_0^1L_M(\Omega)$ (see [21, 22]).

For the above results, the readers can also be referred to [8, 23–25].

We recall some lemmas which will be used later.

Lemma 2.1 (see [26]) *For all $u \in W_0^1L_M(\Omega)$, one has*

$$\int_\Omega M\bigl(|u|/\operatorname{diam}\Omega\bigr)\,dx \leq \int_\Omega M\bigl(|Du|\bigr)\,dx,$$

where diam Ω *is the diameter of* Ω.

Lemma 2.2 (see [22]) *If the open set Ω has the segment property, $u \in W_0^1L_M(\Omega)$, then there exists $\lambda > 0$ and a sequence $u_k \in \mathcal{D}(\Omega)$ such that for any $|\alpha| \leq 1$, $\rho_M(|D^\alpha u_k - D^\alpha u|/\lambda) \to 0$, $k \to \infty$.*

Definition 2.1 (see [27]) Let $V_m = \operatorname{span}\{\omega_1, \dots, \omega_m\}$; then $u_m \in V_m$ is called a Galerkin solution of $A(u) = f$ in V_m if and only if

$$\big(A(u_m), v\big) = (f, v) \quad \forall v \in V_m.$$

The proof of the following lemma can be found in Lemma 5.12.1 in [28].

Lemma 2.3 *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous mapping with*

$$\lim_{|x|\to\infty} \frac{\langle x, f(x)\rangle}{|x|} = a, \tag{4}$$

where a is a constant with $-\infty \le a < 0$ or $0 < a \le +\infty$, $|\cdot|$ is a norm in $\mathbb{R}^m$, $\langle \cdot, \cdot \rangle$ is an inner product defined as $\langle x, f(x)\rangle = \sum_{i=1}^m x_i f_i(x)$ with $x = (x_1, x_2, \dots, x_m)$ and $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$. Then the range of f is the whole of $\mathbb{R}^m$.

Proof Let $u_0 \in \mathbb{R}^m$ and define $f^*(x) = f(x) - u_0$. Then f^* satisfies (4). Consequently, it is sufficient to prove that the range of any map satisfying (4) contains the origin.

If $0 < a \le +\infty$, using (4) we see that we may choose r large enough so that

$$\frac{\langle x, f(x)\rangle}{|x|} > 0 \quad \text{for } |x| = r. \tag{5}$$

But from (5), it follows that the mapping

$$w(\xi) = -\frac{r f(\xi)}{|f(\xi)|}, \quad |\xi| \le r.$$

Then $w : B(0, r) \to B(0, r)$ is continuous where $B(0, r) = \{x \in \mathbb{R}^m, |x| \le r\}$. By the Brouwer fixed point theorem, f is continuous from $B(0, r) \subset \mathbb{R}^m$ into $B(0, r)$, and f has a fixed point, *i.e.*, there exists $x \in B(0, r)$ such that $x = w(x)$. Then

$$|x| = \big|w(x)\big| = \left| -\frac{r f(x)}{|f(x)|} \right| = r,$$

which implies that

$$\frac{\langle x, f(x)\rangle}{|x|} = \frac{\langle x, -\frac{x}{r}|f(x)|\rangle}{|x|} = -\frac{|f(x)|\langle x, x\rangle}{r|x|} = -\frac{|f(x)| \cdot |x|^2}{r|x|} = -\frac{|f(x)| \cdot |x|}{r} < 0.$$

It is a contradiction with (5). Therefore, f is surjective.

If $-\infty \le a < 0$, then let $g = -f$. Thanks to (4), we have

$$\lim_{|x|\to\infty} \frac{\langle x, g(x)\rangle}{|x|} = -a.$$

From this we deduce that g is surjective. Therefore $-g$ is surjective, too. Immediately, f is surjective, *i.e.*, the range of f is the whole of $\mathbb{R}^m$. □

Remark 2.1 Let V be a vector space of finite dimension and $A : V \to V^*$ be a continuous mapping with

$$\lim_{\|u\|_V \to \infty} \frac{(A(u), u)}{|x|} = a, \tag{6}$$

where a is the constant in Lemma 2.3 and V^* is the dual space of V, then A is surjective.

Clearly, condition (4) is weaker than the one of Lemma 5.12.1 in [28].

Remark 2.2 If condition (4) is replaced by

$$\lim_{|x| \to \infty} \frac{|\langle x, f(x)\rangle|}{|x|} = a,$$

then f is not surjective. For example, let $f(x) = |x|$, then $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$\frac{|\langle x, f(x)\rangle|}{|x|} = \frac{|x \cdot |x||}{|x|} = |x| \to +\infty \quad \text{as } |x| \to +\infty.$$

However, the range of f is $[0, +\infty)$. Therefore, Lemma 1 in Landes [27] should be without absolute.

Lemma 2.4 (see [20] and [21]) *If a sequence $u_n \in L_M(\Omega)$ converges a.e. to u and if u_n remains bounded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \rightharpoonup u$ for $\sigma(L_M, E_{\bar{M}})$.*

Lemma 2.5 (see [22]) *Let $u_k, u \in L_M(\Omega)$. If $u_k \to u$ with respect to the modular convergence, then $u_k \to u$ for $\sigma(L_M, L_{\bar{M}})$.*

For N-function M, $\mathcal{T}_0^{1,M}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ such that for all $k > 0$ the truncated functions $T_k(u) \in W_0^1 L_M(\Omega)$ with $T_k(s) = \max(-k, \min(k, s))$.

The following lemmas will be applied to the truncation operators.

Lemma 2.6 (see [2, 23] and [24]) *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Let M be an N-function, and let $u \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Moreover, we have $\frac{\partial}{\partial x_i} F(u) = F'(u) \frac{\partial}{\partial x_i} u$, a.e. in $\{x \in \Omega | u(x) \notin D\}$, and $\frac{\partial}{\partial x_i} F(u) = 0$, a.e. in $\{x \in \Omega | u(x) \in D\}$, where D is the set of discontinuity points of F'.*

Lemma 2.7 (see [29]) *If $u \in W^1 L_M(\Omega)$, then $u^+, u^- \in W^1 L_M(\Omega)$ and*

$$Du^+ = \begin{cases} Du, & \text{if } u > 0, \\ 0, & \text{if } u \le 0, \end{cases} \quad \text{and} \quad Du^- = \begin{cases} 0, & \text{if } u \ge 0, \\ -Du, & \text{if } u < 0. \end{cases} \tag{7}$$

Lemma 2.8 (see [2]) *For every $u \in \mathcal{T}_0^{1,M}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}$ such that $DT_k(u) = v\chi_{\{|u|<k\}}$ almost everywhere in Ω for every $k > 0$. Define the gradient of u as the function v, and denote it by $v = Du$.*

3 Existence theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the segment property, $N \geq 2$, M be an N-function, $\bar{M}$ be a complementary function of M. Assume that M is twice continuously differentiable. Denote by $\mathcal{P}_M$ the following subset of N-functions defined as:

$$\mathcal{P}_M = \biggl\{ B : \mathbb{R}^+ \to \mathbb{R}^+ : N\text{-function} : B \text{ is twice continuously differentiable,} \\ B''/B' \leq M''/M'; \int_0^1 B \circ H^{-1}\bigl(1/t^{1-1/N}\bigr)\,dt < \infty \biggr\},$$

where $H(r) = M(r)/r$. Assume that there exists $Q \in \mathcal{P}_M$ such that

$$Q \circ H^{-1} \text{ is an } N\text{-function.} \tag{8}$$

Let μ be a bounded nonnegative Radon measure on Ω. We consider the following Dirichlet problem:

$$A(u) + g(x,u,Du) = \mu \quad \text{in } \Omega, \tag{9}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{10}$$

where $A : D(A) \subset W_0^1 L_M(\Omega) \to W^{-1}L_{\bar{M}}(\Omega)$ is a mapping given by $A(u) = -\operatorname{div} a(x,u,Du)$. $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$:

$$\bigl|a(x,s,\xi)\bigr| \leq \beta\bigl[c(x) + \bar{M}^{-1}\bigl(M\bigl(|s|\bigr)\bigr) + \bar{M}^{-1}\bigl(M\bigl(|\xi|\bigr)\bigr)\bigr], \tag{11}$$

$$\bigl[a(x,s,\xi) - a(x,s,\eta)\bigr][\xi - \eta] > 0, \tag{12}$$

$$a(x,s,\xi)\xi \geq \alpha M\bigl(|\xi|\bigr), \tag{13}$$

where $\alpha, \beta > 0$, $k_1, k_2 \geq 0$, $c(x) \in E_{\bar{M}}(\Omega)$.

$g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$:

$$\bigl|g(x,s,\xi)\bigr| \leq \gamma(x) + \rho(s)M\bigl(|\xi|\bigr), \tag{14}$$

where $\rho : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\mathbb{R})$ and $\gamma(x)$ belongs to $L^1(\Omega)$. For example, $g(x,u,Du) = \gamma(x) + |\sin u| e^{-u} M(|Du|)$ (see [2]).

We have the following theorem.

Theorem 3.1 *Assume that* (8)-(14) *hold. Then there exists at least one solution of the following problem*:

$$\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega) \cap W_0^1 L_B(\Omega), & \forall B \in \mathcal{P}_M, \\ \langle A(u), \phi\rangle + \int_\Omega g(x,u,Du)\phi\,dx = \langle \mu, \phi\rangle, & \forall \phi \in \mathcal{D}(\Omega). \end{cases} \tag{15}$$

Remark 3.1 It is well known that there exists a sequence $\mu_n \in \mathcal{D}(\Omega)$ such that μ_n converges to μ in the distributional sense with $\|\mu_n\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)}$ and μ_n is nonnegative if μ is nonnegative.

Remark 3.2 (1) Benkirane and Bennouna [30, Remark 2.2] give some examples of N-functions M for which the set $\mathcal{P}_M$ is not empty. For example, assume that the N-function M is defined only at infinity, and let $M(t) = t^2 \log t$ and $B(t) = t \log t$, then $H(t) = t \log t$ and $H^{-1}(t) = t(\log t)^{-1}$ at infinity (see, *e.g.*, [30] or [20]). Hence, the N-function B belongs to $\mathcal{P}_M$.

(2) Let $M(t) = |t|^p$ and $B(t) = |t|^q$, then $B \in \mathcal{P}_M \Leftrightarrow 1 < q < \tilde{p} = \frac{N(p-1)}{N-1}$ and $p > 2 - \frac{1}{N}$. So that we find the same result given in [1]. Our theorem gives a refinement of the regularity result. For example, take $B_1(t) = \frac{t^{\tilde{p}}}{\log^\alpha(e+t)}$ with $\alpha > 1$.

We have the following proposition.

Proposition 3.1 *Assume that* (8)-(14) *hold. Then, for any $n \in \mathbb{N}$, there exists at least one solution $u_n \in W_0^1 E_M(\Omega)$ of the following approximate equation*:

$$\int_\Omega \bigl[a(x,u,Du)Dv + g_n(x,u,Du)v\bigr]\,dx = \int_\Omega \mu_n v\,dx, \quad \forall v \in W_0^1 L_M(\Omega), \tag{16}$$

where $g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$.

Proof Denote $V = W_0^1 E_M(\Omega)$. Define $A_n : V \to V^*$,

$$(A_n u, w) := \int_\Omega \bigl[a(x,u,Du)Dw(x) + g_n(x,u,Du)w(x)\bigr]\,dx, \quad \forall w \in V.$$

Then A_n is well defined. Indeed, from (11) we have

$$\int_\Omega \bar{M}\biggl(\frac{1}{3\beta}\bigl|a(x,u,Du)\bigr|\biggr)\,dx \leq \int_\Omega \frac{1}{3}\bigl[\bar{M}\bigl(c(x)\bigr) + M\bigl(|u|\bigr) + M\bigl(|Du|\bigr)\bigr]\,dx < \infty.$$

Therefore, $a(x,u,Du) \in (L_{\bar{M}}(\Omega))^N$. On the other hand, for every fixed n, $\int_\Omega \bar{M}(|g_n(x,u,Du)|)\,dx \leq \bar{M}(n)\operatorname{meas}(\Omega) < \infty$. Thus $g_n(x,u,Du) \in L_{\bar{M}}(\Omega)$.

There exists a sequence $\{w_j\}_{n=1}^\infty \subset \mathcal{D}(\Omega)$ such that $\{w_j\}_{n=1}^\infty$ dense in V. Let $V_m = \operatorname{span}\{w_1,\ldots,w_m\}$ and consider $A_n|_{V_m}$. $\int_\Omega |Du|\,dx$ and $\|Du\|_{(M)}$ to be two norms of V_m equivalent to the usual norm of finite dimensional vector spaces.

Claim: the mapping $u \to A_n|_{V_m} u : V_m \to V_m^*$ is continuous. Indeed, if $u_j \to u$ in V_m and there exists $\varepsilon_0 > 0$ such that

$$\|A_n|_{V_m} u_j - A_n|_{V_m} u\|_{V_m^*} \geq \varepsilon_0, \tag{17}$$

and since $u_j \to u$ strongly in V_m,

$$\int_\Omega M\bigl(2|u_j - u|\bigr)\,dx \to 0 \quad \text{and} \quad \int_\Omega M\bigl(2|Du_j - Du|\bigr)\,dx \to 0,$$

then there exists a subsequence of $\{u_j\}$ still denoted by $\{u_j\}$ and $f_1, f_2 \in L^1(\Omega)$ such that $M(2|u_j - u|) \le f_1$ and $M(2|Du_j - Du|) \le f_2$. By the convexity of M, we deduce that

$$M(|u_j|) \le \frac{1}{2}M(2|u_j - u|) + \frac{1}{2}M(2|u|) \le \frac{1}{2}f_1 + \frac{1}{2}M(2|u|). \tag{18}$$

Similarly,

$$M(|Du_j|) \le \frac{1}{2}f_2 + \frac{1}{2}M(2|Du|). \tag{19}$$

For $\forall w \in V_m$, by (11), (18), (19) and Young inequality, one has

$$\begin{aligned}
&\bigl|a(x,u_j,Du_j)Dw(x) + g_n(x,u_j,Du_j)w(x)\bigr| \\
&\quad\le \beta\bigl[c(x) + \bar{M}^{-1}\bigl(M(|u_j|)\bigr) + \bar{M}^{-1}M\bigl(|Du_j|\bigr)\bigr]|Dw| + n|w| \\
&\quad\le \beta\bigl[\bar{M}\bigl(c(x)\bigr) + 3M\bigl(|Dw|\bigr) + M\bigl(|u_j|\bigr) + M\bigl(|Du_j|\bigr)\bigr] + \bigl[\bar{M}(n) + M\bigl(|w|\bigr)\bigr] \\
&\quad\le \beta\biggl[\bar{M}\bigl(c(x)\bigr) + 3M\bigl(|Dw|\bigr) + \frac{1}{2}f_1 + \frac{1}{2}M\bigl(2|u|\bigr) + \frac{1}{2}f_2 + \frac{1}{2}M\bigl(2|Du|\bigr)\biggr] \\
&\qquad + \bar{M}(n) + M\bigl(|w|\bigr).
\end{aligned} \tag{20}$$

Hence $(A_n|_{V_m}u_j, w) < \infty$ for all $w \in V_m$. By the Banach-Steinhaus theorem $\{\|A_n|_{V_m}u_j\|_{V_m^*}\}_j$ is bounded. Hence $\{A_n|_{V_m}u_j\}_j$ is relatively sequently compact in V_m^*. Passing to a subsequence if necessary, there exists $\eta_n \in V_m^*$ such that

$$\|A_n|_{V_m}u_j - \eta_n\|_{V_m^*} \to 0.$$

On the other hand, passing to a subsequence if necessary,

$$u_j(x) \to u(x) \quad \text{a.e. in } \Omega \quad \text{and} \quad Du_j(x) \to Du(x) \quad \text{a.e. in } \Omega.$$

By the Lebesgue theorem, we know that for each $w \in V_m$,

$$\lim_{j\to\infty}(A_n|_{V_m}u_j, w) = (A_n|_{V_m}u, w).$$

Hence $A_n|_{V_m}u = \eta_n$, it is a contradiction with (17).

Thanks to (13) and Lemma 2.1, for all $u \in V_m$,

$$\begin{aligned}
(A_n u, u) &= \int_\Omega \bigl[a(x,u,Du)Du + g_n(x,u,Du)u\bigr]\,dx \\
&\ge \int_\Omega \bigl[\alpha M\bigl(|Du|\bigr) - n|u|\bigr]\,dx \\
&\ge \alpha \int_\Omega M\bigl(|Du|\bigr)\,dx - \int_\Omega \biggl[\bar{M}\biggl(\frac{1}{\alpha_0}(n \operatorname{diam}\Omega)\biggr) + M\biggl(\alpha_0 \frac{|u|}{\operatorname{diam}\Omega}\biggr)\biggr]\,dx \\
&\ge \alpha \int_\Omega M\bigl(|Du|\bigr)\,dx - \bar{M}\biggl(\frac{1}{\alpha_0}(n \operatorname{diam}\Omega)\biggr)\operatorname{meas}\Omega - \int_\Omega \alpha_0 M\bigl(|Du|\bigr)\,dx \\
&= (\alpha - \alpha_0)\int_\Omega M\bigl(|Du|\bigr)\,dx - \bar{M}\biggl(\frac{1}{\alpha_0}(n \operatorname{diam}\Omega)\biggr)\operatorname{meas}\Omega,
\end{aligned} \tag{21}$$

where $\alpha_0 = \min\{\frac{\alpha}{2}, 1\}$. By Lemma 2.1, one has $\|u\|_{(M)} \le \operatorname{diam}\Omega \|Du\|_{(M)}$. It follows that $\|u\|_{\Omega,M} \le (1 + \operatorname{diam}\Omega)\|Du\|_{(M)}$. We have

$$\frac{\int_\Omega M(|Du|)\,dx}{\|u\|_{\Omega,M}} \ge \frac{1}{1 + \operatorname{diam}\Omega} \frac{\int_\Omega M(|Du|)\,dx}{\|Du\|_{(M)}} \ge \frac{1}{1 + \operatorname{diam}\Omega} \tag{22}$$

since $\int_\Omega M(u)\,dx > \|u\|_{(M)}$ whenever $\|u\|_{(M)} > 1$. Combining (21) and (22), one has

$$\frac{(A_n u, u)}{\|u\|_{\Omega,M}} \ge \frac{1}{1 + \operatorname{diam}\Omega}. \tag{23}$$

By Remark 2.1, A_n is surjective, *i.e.*, there exists a Galerkin solution $u_m \in V_m$ for every m such that

$$(A_n u_m, v) = (\mu_n, v), \quad \forall v \in V_m. \tag{24}$$

We will show that the sequence $\{u_m\}$ is bounded in V.

In fact, for every $u_m \in V$, if $\|u_m\|_{\Omega,M} \to \infty$, then by (23), $(A_n u_m, u_m) \to \infty$. It is a contradiction with (24). Therefore $\{u_m\}$ is bounded in V.

It follows from (20) that we can deduce $\{\|A_n|_{V_m} u_m\|_{V^*}\}_m$ is bounded. So we can extract a subsequence $\{u_k\}_{k=1}^\infty$ of $\{u_m\}_{m=1}^\infty$ such that

$$u_k \rightharpoonup u_0 \quad \text{in } V \text{ for } \sigma(\Pi L_M, \Pi E_{\bar M}), \qquad A_n u_k \rightharpoonup \xi_n \quad \text{in } V^* \text{ for } \sigma(\Pi L_{\bar M}, \Pi E_M), \tag{25}$$

as $k \to \infty$ and $(\xi_n, w) = (\mu_n, w)$ for all $w \in \bigcup_{m=1}^\infty \{w_m\}$. By the density of $\{w_m\}$, we get

$$(\xi_n, w) = (\mu_n, w), \quad \forall w \in V.$$

By the imbedding theorem (see, *e.g.*, [31]) we have

$$u_k \to u_0 \quad \text{strongly in } L_M(\Omega) \text{ as } k \to \infty. \tag{26}$$

Hence, passing to a subsequence if necessary

$$u_k(x) \to u_0(x) \quad \text{a.e. } x \in \Omega \text{ as } k \to \infty. \tag{27}$$

On the other hand, thanks to (26), we have

$$\int_\Omega g_n(x, u_k, Du_k)(u_k - u_0)\,dx \to 0 \quad \text{and} \quad \int_\Omega \mu_n(u_k - u_0)\,dx \to 0$$

as $k \to \infty$. Thus we obtain that

$$\begin{aligned} &\int_\Omega a(x, u_k, Du_k)(Du_k - Du_0)\,dx \\ &\quad = \int_\Omega \mu_n(u_k - u_0)\,dx - \int_\Omega g_n(x, u_k, Du_k)(u_k - u_0)\,dx \to 0. \end{aligned} \tag{28}$$

Fix a positive real number r and define $\Omega_r = \{x \in \Omega : |Du_0(x)| \leq r\}$ and denote by χ_r the characteristic function of Ω_r.

Taking $s \geq r$, one has

$$\begin{aligned} 0 &\leq \int_{\Omega_r} \big[a(x,u_k,Du_k) - a(x,u_k,Du_0)\big](Du_k - Du_0)\,dx \\ &\leq \int_{\Omega_s} \big[a(x,u_k,Du_k) - a(x,u_k,Du_0)\big](Du_k - Du_0)\,dx \\ &= \int_{\Omega_s} \big[a(x,u_k,Du_k) - a(x,u_k,Du_0\chi_s)\big](Du_k - Du_0\chi_s)\,dx \\ &\leq \int_{\Omega} \big[a(x,u_k,Du_k) - a(x,u_k,Du_0\chi_s)\big](Du_k - Du_0\chi_s)\,dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_{\Omega} a(x,u_k,Du_k)(Du_k - Du_0)\,dx \\ &\quad = \int_{\Omega} \big[a(x,u_k,Du_k) - a(x,u_k,Du_0\chi_s)\big](Du_k - Du_0\chi_s)\,dx \\ &\qquad - \int_{\Omega} a(x,u_k,Du_k)Du_0\chi_{\Omega\setminus\Omega_s}\,dx + \int_{\Omega} a(x,u_k,Du_0\chi_s)(Du_k - Du_0\chi_s)\,dx. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_{\Omega} \big[a(x,u_k,Du_k) - a(x,u_k,Du_0\chi_s)\big](Du_k - Du_0\chi_s)\,dx \\ &\quad = \int_{\Omega} a(x,u_k,Du_k)(Du_k - Du_0)\,dx \\ &\qquad + \int_{\Omega} a(x,u_k,Du_k)Du_0\chi_{\Omega\setminus\Omega_s}\,dx - \int_{\Omega} a(x,u_k,Du_0\chi_s)(Du_k - Du_0\chi_s)\,dx. \end{aligned} \tag{29}$$

In view of (28) the first term of the right-hand side of (29) tends to 0 as $k \to \infty$.

$\{a(x,u_k,Du_k)\}_k$ is bounded in $(L_{\bar{M}}(\Omega))^N$. Indeed, for every $w \in (E_M(\Omega))^N$,

$$\begin{aligned} &\int_{\Omega} a(x,u_k,Du_k)w\,dx \\ &\quad = \int_{\Omega} \mu_n w\,dx - \int_{\Omega} g_n(x,u_k,Du_k)w\,dx \\ &\quad \leq \|\mu_n\|_{\bar{M}} \cdot \|w\|_{(M)} + \|n\|_{\bar{M}} \cdot \|w\|_{(M)} = \big(\|\mu_n\|_{\bar{M}} + \|n\|_{\bar{M}}\big)\|w\|_{(M)} < +\infty. \end{aligned}$$

By the Banach-Steinhaus theorem, $\{\|a(x,u_k,Du_k)\|_{\bar{M}}\}_k$ is bounded.

Thus, there exists $h \in (L_{\bar{M}}(\Omega))^N$ such that (for a subsequence still denoted by $\{u_k\}$)

$$a(x,u_k,Du_k) \rightharpoonup h \quad \text{in } \big(L_{\bar{M}}(\Omega)\big)^N \text{ for } \sigma(\Pi L_{\bar{M}}, \Pi E_M).$$

It follows that the second term of the right-hand side of (29) tends to $\int_{\Omega\setminus\Omega_s} hDu_0\,dx$ as $k \to \infty$.

Since $a(x,u_k,Du_0\chi_s) \rightharpoonup a(x,u_0,Du_0\chi_s)$ strongly in $(E_{\bar{M}}(\Omega))^N$, while by (25) $Du_k - Du_0\chi_s \rightharpoonup Du_0 - Du_0\chi_s$ tends weakly in $(E_M(\Omega))^N$ for $\sigma((L_M(\Omega))^N,(E_{\bar{M}}(\Omega))^N)$, the third term of the right-hand side of (29) tends to $-\int_\Omega a(x,u_0,Du_0\chi_s)(Du_0 - Du_0\chi_s)\,dx = -\int_{\Omega\setminus\Omega_s} a(x,u_0,0)Du_0\,dx$.

Therefore,

$$\begin{aligned}&\int_\Omega \bigl[a(x,u_k,Du_k) - a(x,u_k,Du_0\chi_s)\bigr](Du_k - Du_0\chi_s)\,dx\\&\quad = \int_{\Omega\setminus\Omega_s} \bigl[h - a(x,u_0,0)\bigr]Du_0\,dx + \varepsilon(k).\end{aligned}$$

We have then proved that

$$\begin{aligned}&0 \le \limsup_{k\to\infty} \int_{\Omega_r} \bigl[a(x,u_k,Du_k) - a(x,u_k,Du_0)\bigr](Du_k - Du_0)\,dx\\&\quad = \int_{\Omega\setminus\Omega_s} \bigl[h - a(x,u_0,0)\bigr]Du_0\,dx.\end{aligned}$$

Using the fact that $[h - a(x,u_0,0)]Du_0 \in L^1(\Omega)$ and letting $s \to \infty$, we get, since $\operatorname{meas}(\Omega\setminus\Omega_s) \to 0$,

$$\int_{\Omega_r} \bigl[a(x,u_k,Du_k) - a(x,u_k,Du_0)\bigr](Du_k - Du_0)\,dx \to 0 \quad \text{as } k \to \infty,$$

which gives

$$\bigl[a(x,u_k,Du_k) - a(x,u_k,Du_0)\bigr](Du_k - Du_0)\,dx \to 0 \quad \text{a.e. in } \Omega_r \tag{30}$$

(for a subsequence still denoted by $\{u_k\}$), say, for each $x \in \Omega_r\setminus Z$ with $\operatorname{meas}(Z) = 0$. As the proof of Eq. (3.23) in [32], we can construct a subsequence such that

$$Du_k(x) \to Du_0(x) \quad \text{a.e. in } \Omega. \tag{31}$$

Consequently, we get

$$a(x,u_k,Du_k) \to a(x,u_0,Du_0) \quad \text{a.e. in } \Omega,$$

and

$$g_n(x,u_k,Du_k) \to g_n(x,u_0,Du_0) \quad \text{a.e. in } \Omega.$$

By Lemma 2.4, we get

$$a(x,u_k,Du_k) \rightharpoonup a(x,u_0,Du_0) \quad \text{in } \bigl(L_{\bar{M}}(\Omega)\bigr)^N \text{ for } \sigma\bigl(\bigl(L_{\bar{M}}(\Omega)\bigr)^N,\bigl(E_M(\Omega)\bigr)^N\bigr),$$

and

$$g_n(x,u_k,Du_k) \rightharpoonup g_n(x,u_0,Du_0) \quad \text{in } \bigl(L_{\bar{M}}(\Omega)\bigr)^N \text{ for } \sigma\bigl(\bigl(L_{\bar{M}}(\Omega)\bigr)^N,\bigl(E_M(\Omega)\bigr)^N\bigr).$$

Then

$$\int_\Omega \big[a(x,u_k,Du_k)Dw + g_n(x,u_k,Du_k)w\big]\,dx \to \int_\Omega \big[a(x,u_0,Du_0)Dw + g_n(x,u_0,Du_0)w\big]\,dx$$

for every $w \in V$. Thus, we get $(A_n u_k, w) \to (A_n u_0, w)$ for every $w \in V$. It follows that $A_n u_0 = \xi_n$. Therefore,

$$(A_n u_0, w) = (\mu_n, w), \quad \forall w \in W_0^1 E_M(\Omega).$$

Furthermore, by Lemmas 2.2 and 2.5, we have

$$(A_n u_0, v) = (\mu_n, v), \quad \forall v \in W_0^1 L_M(\Omega).$$

Hence, for every n, there exists at least one solution u_n of (16) with $u_n \in W_0^1 E_M(\Omega)$. □

Remark 3.3 From Proposition 3.1, we have the following approximate equations:

$$\int_\Omega \big[a(x,u_n,Du_n)Dv + g_n(x,u_n,Du_n)v\big]\,dx = \int_\Omega \mu_n v\,dx, \quad \forall v \in W_0^1 L_M(\Omega), \tag{32}$$

where $u_n \in W_0^1 E_M(\Omega)$.

Remark 3.4 Clearly, condition (11) is weaker than

$$\big|a(x,s,\xi)\big| \le \beta\big[c(x) + \bar{P}^{-1}\big(M\big(|s|\big)\big) + \bar{M}^{-1}\big(M\big(|\xi|\big)\big)\big], \tag{33}$$

whenever $P \ll M$. If condition (11) is replaced by (33) in Proposition 3.1, the approximate equations (16) has at least one solution $u_n \in W_0^1 L_M(\Omega)$ by the classical result of [33].

The proof of the following proposition is similar to the proof of Lemma 2.2 in [34].

Proposition 3.2 *Assume that* (8)-(14) *hold true, and let* $\{u_n\}_n$ *be a solution of the approximate problem* (16). *Let* $\varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ *with* $\varphi \ge 0$. *Then*

(1) $\exp(G(T_k(u_n)))\varphi$ *can be taken as a test function in* (32) *and*

$$\begin{aligned}&\int_\Omega a(x,u_n,Du_n)\exp\big(G(u_n)\big)D\varphi\,dx\\&\quad\le \int_\Omega \mu_n \exp\big(G(u_n)\big)\varphi\,dx + \int_\Omega \gamma(x)\exp\big(G(u_n)\big)\varphi\,dx;\end{aligned} \tag{34}$$

(2) $\exp(-G(T_k(u_n)))\varphi$ *can be taken as a test function in* (32) *and*

$$\begin{aligned}&\int_\Omega a(x,u_n,Du_n)\exp\big(-G(u_n)\big)D\varphi\,dx + \int_\Omega \gamma(x)\exp\big(-G(u_n)\big)\varphi\,dx\\&\quad\ge \int_\Omega \mu_n \exp\big(-G(u_n)\big)\varphi\,dx.\end{aligned} \tag{35}$$

Proof (1) Choosing $\exp(G(T_k(u_n)))\varphi$ as a test function in (32), we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\frac{\rho(T_k(u_n))}{\alpha}DT_k(u_n)\varphi\,dx \\
&\quad+\int_\Omega a(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)D\varphi\,dx \\
&\quad+\int_\Omega g_n(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\varphi\,dx \\
&=\int_\Omega \mu_n\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\varphi\,dx,
\end{aligned} \tag{36}$$

which implies by (13)

$$\begin{aligned}
&\int_\Omega \alpha M\bigl(\bigl|DT_k(u_n)\bigr|\bigr)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\frac{\rho(T_k(u_n))}{\alpha}\varphi\,dx \\
&\quad\le\int_\Omega a\bigl(x,T_k(u_n),DT_k(u_n)\bigr)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\frac{\rho(T_k(u_n))}{\alpha}DT_k(u_n)\varphi\,dx \\
&\quad=\int_\Omega a(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\frac{\rho(T_k(u_n))}{\alpha}DT_k(u_n)\varphi\,dx.
\end{aligned}$$

Since $T_k(u_n)\to u_n$ and $DT_k(u_n)\to Du_n$ a.e. in Ω as $k\to\infty$, by the Fatou lemma, we get

$$\begin{aligned}
&\int_\Omega \alpha M\bigl(|Du_n|\bigr)\exp\bigl(G(u_n)\bigr)\frac{\rho(u_n)}{\alpha}\varphi\,dx \\
&\quad\le\liminf_{k\to\infty}\int_\Omega \alpha M\bigl(\bigl|DT_k(u_n)\bigr|\bigr)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\frac{\rho(T_k(u_n))}{\alpha}\varphi\,dx \\
&\quad\le\liminf_{k\to\infty}\int_\Omega a(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\frac{\rho(T_k(u_n))}{\alpha}DT_k(u_n)\varphi\,dx.
\end{aligned}$$

On the other hand, the functions $a(x,u_n,Du_n)D\varphi$, $g_n(x,u_n,Du_n)\varphi$, and $\mu_n\varphi$ are summable, and the functions $\exp(G(T_k(u_n)))$ are bounded in $L^\infty(\Omega)$; so Lebesgue's dominated convergence theorem may be applied in the remaining integrals. Indeed, thanks to (11) and Young inequality, one has

$$\begin{aligned}
&\bigl|a(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)D\varphi\bigr| \\
&\quad\le e^{\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}}\beta\bigl[c(x)+\bar{M}^{-1}\bigl(M\bigl(|u_n|\bigr)\bigr)+\bar{M}^{-1}\bigl(M\bigl(|Du_n|\bigr)\bigr)\bigr]|D\varphi| \\
&\quad\le e^{\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}}\beta\bigl[\bar{M}\bigl(c(x)\bigr)+M\bigl(|u_n|\bigr)+M\bigl(|Du_n|\bigr)+3M\bigl(|D\varphi|\bigr)\bigr].
\end{aligned}$$

Since $a(x,u_n,Du_n)\exp(G(T_k(u_n)))D\varphi\to a(x,u_n,Du_n)\exp(G(u_n))D\varphi$ a.e. in Ω as $k\to\infty$, and by Lebesgue's dominated convergence theorem, we deduce that

$$\int_\Omega a(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)D\varphi\,dx\to\int_\Omega a(x,u_n,Du_n)\exp\bigl(G(u_n)\bigr)D\varphi\,dx$$

as $k \to \infty$. Since $g_n(x,u_n,Du_n)\exp(G(T_k(u_n)))\varphi \to g_n(x,u_n,Du_n)\exp(G(u_n))\varphi$ a.e. in Ω as $k \to \infty$, and

$$\left|g_n(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\varphi\right| \le ne^{\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}}\|\varphi\|_\infty,$$

by Lebesgue's dominated convergence theorem one has

$$\int_\Omega g_n(x,u_n,Du_n)\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\varphi\,dx \to \int_\Omega g_n(x,u_n,Du_n)\exp\bigl(G(u_n)\bigr)\varphi\,dx$$

as $k \to \infty$. Since $\mu_n\exp(G(T_k(u_n)))\varphi \to \mu_n\exp(G(u_n))\varphi$ a.e. in Ω as $k \to \infty$, and

$$\left|\mu_n\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\varphi\right| \le e^{\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}}\mu_n\|\varphi\|_\infty,$$

we have

$$\int_\Omega \mu_n\exp\bigl(G\bigl(T_k(u_n)\bigr)\bigr)\varphi\,dx \to \int_\Omega \mu_n\exp\bigl(G(u_n)\bigr)\varphi\,dx$$

as $k \to \infty$.

Thus, letting k tend to ∞ in (36), we obtain

$$\begin{aligned}&\int_\Omega M\bigl(|Du_n|\bigr)\exp\bigl(G(u_n)\bigr)\rho(u_n)Du_n\varphi\,dx\\&\quad+\int_\Omega a(x,u_n,Du_n)\exp\bigl(G(u_n)\bigr)D\varphi\,dx+\int_\Omega g_n(x,u_n,Du_n)\exp\bigl(G(u_n)\bigr)\varphi\,dx\\&\le\int_\Omega \mu_n\exp\bigl(G(u_n)\bigr)\varphi\,dx.\end{aligned}\tag{37}$$

By (14), (37) is reduced to (34).

(2) Similarly, taking $\exp(-G(T_k(u_n)))\varphi$ as a test function in (32), we obtain (35). □

Proposition 3.3 *Assume that* (8)-(14) *hold true, and let* $\{u_n\}_n$ *be a solution of the approximate problem* (16). *Then, for all* $k > 0$, *there exists a constant* C (*which does not depend on the* n *and* k) *such that*

$$\int_\Omega M\bigl(\bigl|DT_k(u_n)\bigr|\bigr)\,dx \le Ck.\tag{38}$$

Proof Let $\varphi = T_k(u_n)^+$ in (34). Also let $G(\pm\infty) = \frac{1}{\alpha}\int_0^{\pm\infty}\rho(s)\,ds$ which are well defined since $\rho \in L^1(\mathbb{R})$, then $G(-\infty) \le G(s) \le G(+\infty)$ and $|G(\pm\infty)| \le \|\rho\|_{L^1(\mathbb{R})}/\alpha$. We have

$$\int_\Omega a(x,u_n,Du_n)\exp\bigl(G(u_n)\bigr)DT_k(u_n)^+\,dx \le e^{\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}}k\bigl[\|\mu\|_{\mathcal{M}_b(\Omega)}+\bigl\|\gamma(x)\bigr\|_{L^1(\Omega)}\bigr].$$

Immediately, by (13) we get

$$\int_\Omega M\bigl(\bigl|DT_k(u_n)^+\bigr|\bigr)\,dx \le Ck.\tag{39}$$

Similarly, let $\varphi = T_k(u_n)^-$ in (35). We obtain

$$\int_\Omega M\bigl(\bigl|DT_k(u_n)^-\bigr|\bigr)\,dx \le Ck. \tag{40}$$

Combing (39) and (40), we deduce (38). □

Proposition 3.4 *Assume that* (8)-(14) *hold true, and let* $\{u_n\}_n$ *be a solution of the approximate problem* (16). *Then there exists a measurable function* u *such that for all* $k > 0$ *we have* (*for a subsequence still denoted by* $\{u_n\}_n$),

(1) $u_n \to u$ *a.e. in* Ω;

(2) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^1E_M(\Omega)$ *for* $\sigma(\Pi L_M, \Pi E_{\bar{M}})$;

(3) $T_k(u_n) \to T_k(u)$ *strongly in* $E_M(\Omega)$ *and a.e. in* Ω.

Proof Since

$$\begin{aligned} &M\biggl(\frac{k}{\operatorname{diam}\Omega}\biggr)\operatorname{meas}\bigl\{\bigl|T_k(u_n)\bigr| = k\bigr\} \\ &\quad= \int_{\{|T_k(u_n)|=k\}} M\biggl(\frac{|T_k(u_n)|}{\operatorname{diam}\Omega}\biggr)\,dx \le \int_\Omega M\bigl(\bigl|DT_k(u_n)\bigr|\bigr)\,dx \le Ck \end{aligned}$$

and $\{|u_n| > k\} \subset \{|T_k(u_n)| = k\}$, we get

$$\operatorname{meas}\bigl\{|u_n| > k\bigr\} \le \operatorname{meas}\bigl\{\bigl|T_k(u_n)\bigr| = k\bigr\} \le \frac{Ck}{M(\frac{k}{\operatorname{diam}\Omega})}$$

for all n and for all k. Similar to the proof of Proposition 4.3 in [2], assertions (1)-(3) hold. □

Proposition 3.5 *Assume that* (8)-(14) *hold true, and let* $\{u_n\}_n$ *be a solution of the approximate problem* (16). *Then, for all* $k > 0$,

(1) $\{a(x, T_k(u_n), DT_k(u_n))\}_n$ *is bounded in* $L_{\bar{M}}(\Omega)^N$;

(2) $Du_n \to Du$ *a.e. in* Ω (*for a subsequence*) *as* $n \to \infty$.

Proof (1) Let $w \in (E_M(\Omega))^N$ be arbitrary. By condition (11) and Young inequality, we have

$$\begin{aligned} &\int_\Omega a\bigl(x, T_k(u_n), DT_k(u_n)\bigr)w\,dx \\ &\quad\le \beta\int_\Omega \bigl[\bar{M}\bigl(c(x)\bigr) + M(k) + M\bigl(\bigl|DT_k(u_n)\bigr|\bigr) + 3M\bigl(|w|\bigr)\bigr]\,dx \\ &\quad\le \beta\biggl[\int_\Omega \bar{M}\bigl(c(x)\bigr)\,dx + M(k)\operatorname{meas}\Omega + \int_\Omega M\bigl(\bigl|DT_k(u_n)\bigr|\bigr)\,dx + 3\int_\Omega M\bigl(|w|\bigr)\,dx\biggr] \\ &\quad\le \beta\biggl[\int_\Omega \bar{M}\bigl(c(x)\bigr)\,dx + M(k)\operatorname{meas}\Omega + Ck + 3\int_\Omega M\bigl(|w|\bigr)\,dx\biggr] = C(k) < +\infty, \end{aligned}$$

where $C(k)$ is a constant independent of n.

By the Banach-Steinhaus theorem $\{\|a(x, T_k(u_n), DT_k(u_n))\|_{\bar{M}}\}_n$ is bounded; this completes the proof of assertion (1).

(2) Let $\Omega_s = \{x \in \Omega \mid |DT_k(u_n)| < s\}$ and denote by χ_s the characteristic function of Ω_s. Clearly, $\Omega_s \subset \Omega_{s+1}$ and $\operatorname{meas}(\Omega\backslash\Omega_s) \to 0$ as $s \to \infty$.

Step (i). We shall show the following assertion:

$$\lim_{j\to\infty}\limsup_{n\to\infty}\int_{\{-(j+1)\le u_n\le -j\}} a(x,u_n,Du_n)Du_n\,dx = 0. \tag{41}$$

Indeed, the term in (35) with μ_n can be neglected since it is nonnegative. Hence

$$-\int_\Omega a(x,u_n,Du_n)\exp\bigl(-G(u_n)\bigr)D\varphi\,dx \le \int_\Omega \gamma(x)\exp\bigl(-G(u_n)\bigr)\varphi\,dx. \tag{42}$$

Taking $\varphi = T_1(u_n - T_j(u_n))^-$ in (42), we obtain

$$\begin{aligned}&\int_{\{-(j+1)\le u_n\le -j\}} a(x,u_n,Du_n)Du_n\exp\bigl(-G(u_n)\bigr)\,dx\\ &\quad\le \int_\Omega \gamma(x)\exp\bigl(-G(u_n)\bigr)T_1\bigl(u_n - T_j(u_n)\bigr)^-\,dx.\end{aligned}$$

Since $|\gamma(x)\exp(-G(u_n))T_1(u_n - T_j(u_n))^-| \le e^{\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}}|\gamma(x)|$, we deduce

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega \gamma(x)\exp\bigl(-G(u_n)\bigr)T_1\bigl(u_n - T_j(u_n)\bigr)^-\,dx = 0,$$

by Lebesgue's dominate convergence theorem, which implies (41).

Step (ii). Taking $\varphi = (T_k(u_n) - T_k(v_i))^-[1 - |T_1(u_n - T_j(u_n))|]$ and $\varphi = (T_k(v_i) - T_k(u_n))^-[1 - |T_1(u_n - T_j(u_n))|]$ in (42) with $j > k$, as in [2], we can deduce that, by passing to a subsequence if necessary,

$$DT_k(u_n) \to DT_k(u) \quad \text{a.e. in } \Omega, \tag{43}$$

and

$$Du_n \to Du \quad \text{a.e. in } \Omega. \tag{44}$$

□

Proof of Theorem 3.1 (1) We are going to show that as $n \to \infty$,

$$g_n(x,u_n,Du_n) \to g(x,u,Du) \quad \text{in } L^1(\Omega). \tag{45}$$

Indeed, taking $v = \exp(-G(T_k(u_n)))\int_{T_k(u_n)}^0 \rho(s)\chi_{\{s<-h\}}\,ds$ as a test function in (32), we have

$$\begin{aligned}&\int_\Omega a(x,u_n,Du_n)DT_k(u_n)\frac{\rho(T_k(u_n))}{\alpha}\exp\bigl(-G\bigl(T_k(u_n)\bigr)\bigr)\int_{T_k(u_n)}^0 \rho(s)\chi_{\{s<-h\}}\,ds\,dx\\ &\qquad+\int_\Omega a(x,u_n,Du_n)DT_k(u_n)\exp\bigl(-G\bigl(T_k(u_n)\bigr)\bigr)\rho\bigl(T_k(u_n)\bigr)\chi_{\{T_k(u_n)<-h\}}\,dx\\ &\quad=\int_\Omega g_n(x,u_n,Du_n)\exp\bigl(-G\bigl(T_k(u_n)\bigr)\bigr)\int_{T_k(u_n)}^0 \rho(s)\chi_{\{s<-h\}}\,ds\,dx\\ &\qquad-\int_\Omega \mu_n\exp\bigl(-G\bigl(T_k(u_n)\bigr)\bigr)\int_{T_k(u_n)}^0 \rho(s)\chi_{\{s<-h\}}\,ds\,dx.\end{aligned}$$

Using (13) and by Fatou's lemma and Lebesgue's theorem, we can deduce that

$$\begin{aligned}
&\int_\Omega \alpha M\big(|Du_n|\big)\exp\big(-G(u_n)\big)\frac{\rho(u_n)}{\alpha}\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds\,dx \\
&\qquad+\int_\Omega \alpha M\big(|Du_n|\big)\exp\big(-G(u_n)\big)\rho(u_n)\chi_{\{u_n<-h\}}\,dx \\
&\quad\le \int_\Omega g_n(x,u_n,Du_n)\exp\big(-G(u_n)\big)\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds\,dx \\
&\qquad-\int_\Omega \mu_n\exp\big(-G(u_n)\big)\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds\,dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_\Omega \alpha M\big(|Du_n|\big)\exp\big(-G(u_n)\big)\rho(u_n)\chi_{\{u_n<-h\}}\,dx \\
&\quad\le \int_\Omega \gamma(x)\exp\big(-G(u_n)\big)\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds\,dx \\
&\qquad-\int_\Omega \mu_n\exp\big(-G(u_n)\big)\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds\,dx.
\end{aligned}$$

Since $\rho \geq 0$, we get

$$\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}\,ds \le \int_{-\infty}^{-h}\rho(s)\,ds.$$

Hence we have

$$\begin{aligned}
&\int_\Omega M\big(|Du_n|\big)\exp\big(-G(u_n)\big)\rho(u_n)\chi_{\{u_n<-h\}}\,dx \\
&\quad\le \frac{1}{\alpha}e^{\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}}\int_{-\infty}^{-h}\rho(s)\,ds\big(\|\gamma\|_{L^1(\Omega)}+\|\mu\|_{\mathcal{M}_b(\Omega)}\big)=C\int_{-\infty}^{-h}\rho(s)\,ds.
\end{aligned}$$

Consequently, one has

$$\int_\Omega M\big(|Du_n|\big)\rho(u_n)\chi_{\{u_n<-h\}}\,dx \le C\int_{-\infty}^{-h}\rho(s)\,ds.$$

Letting $h \to +\infty$, one has

$$\int_{-\infty}^{-h}\rho(s)\,ds \to 0.$$

Therefore,

$$\lim_{h\to+\infty}\sup_{n\in N}\int_{\{u_n<-h\}} M\big(|Du_n|\big)\rho(u_n)\,dx = 0.$$

Taking $v = \exp(G(T_k(u_n))) \int_0^{T_k(u_n)} \rho(s)\chi_{\{s>h\}}\,ds$ as a test function in (32), similarly we obtain that

$$\lim_{h\to+\infty} \sup_{n\in N} \int_{\{u_n>h\}} M(|Du_n|)\rho(u_n)\,dx = 0.$$

Hence,

$$\lim_{h\to+\infty} \sup_{n\in N} \int_{\{|u_n|>h\}} M(|Du_n|)\rho(u_n)\,dx = 0. \tag{46}$$

Following the proof of step 1 in Theorem 3.1 of [2], we can deduce (45).

(2) We will prove that $a(x,u_n,Du_n) \rightharpoonup a(x,u,Du)$ weakly for $\sigma(\Pi L_{Q\circ H^{-1}}, \Pi E_{\overline{Q\circ H^{-1}}})$.

By (44), we have

$$a(x,u_n,Du_n) \to a(x,u,Du) \quad \text{a.e. in } \Omega. \tag{47}$$

By $Q \in \mathcal{P}_M$ and (8), one has $Q''/Q' \le M''/M'$. Then

$$\int \frac{Q''(t)}{Q'(t)}\,dt \le \int \frac{M''(t)}{M'(t)}\,dt.$$

Thus, there exists a constant C such that $\ln|Q'(t)| \le \ln|M'(t)| + C$. Therefore,

$$Q'(t) \le CM'(t).$$

It implies that

$$Q(r) = \int_0^r Q'(t)\,dt \le C\int_0^r M'(t)\,dt = CM(r). \tag{48}$$

Let $s = H(r)$, then $s = \frac{M\circ H^{-1}(s)}{H^{-1}(s)}$. By Young inequality we have

$$M\circ H^{-1}\left(\frac{s}{2}\right) = \frac{s}{2}\cdot H^{-1}\left(\frac{s}{2}\right) \le \frac{1}{2}\bar{M}(s) + \frac{1}{2}M\circ H^{-1}\left(\frac{s}{2}\right).$$

Hence

$$M\circ H^{-1}\left(\frac{s}{2}\right) \le \bar{M}(s). \tag{49}$$

In view of (48) and (49), we get

$$\int_\Omega Q\circ H^{-1}\left(\frac{1}{2}c(x)\right)dx \le C\int_\Omega M\circ H^{-1}\left(\frac{1}{2}c(x)\right)dx \le C\int_\Omega \bar{M}(c(x))\,dx < \infty. \tag{50}$$

Since $\bar{M}^{-1}(M(|Du_n|)) \le 2\frac{M(|Du_n|)}{|Du_n|}$, we have

$$\frac{1}{2}\bar{M}^{-1}(M(|Du_n|)) \le \frac{M(|Du_n|)}{|Du_n|}.$$

Hence

$$\int_\Omega Q\circ H^{-1}\left(\frac{1}{2}\bar{M}^{-1}\big(M(|Du_n|)\big)\right)dx \le \int_\Omega Q\circ H^{-1}\left(\frac{M(|Du_n|)}{|Du_n|}\right)dx = \int_\Omega Q\big(|Du_n|\big)\,dx,$$

and

$$\int_\Omega Q\circ H^{-1}\left(\frac{1}{2}\bar{M}^{-1}\big(M(|u_n|)\big)\right)dx \le \int_\Omega Q\big(|u_n|\big)\,dx.$$

For $t > 0$, by taking $T_h(u_n - T_t(u_n))$ as a test function in (32), from (14) and (46), we can deduce that

$$\int_{\{t<|u_n|\le t+h\}} a(x,u_n,Du_n)Du_n\,dx \le Ch,$$

where C is a constant independent of n, h, t, which gives

$$\frac{1}{h}\int_{\{t<|u_n|\le t+h\}} M\big(|Du_n|\big)\,dx \le C,$$

and by letting $h \to 0$,

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} M\big(|Du_n|\big)\,dx \le C.$$

Let now $B \in \mathcal{P}_M$. Following the lines of [35], it is easy to deduce that

$$\int_\Omega B\big(|Du_n|\big)\,dx \le C, \quad \forall n.$$

This implies that $\{u_n\}$ is bounded in $W_0^1L_Q(\Omega)$ and converges to u strongly in $L_Q(\Omega)$. Consequently, using the convexity of $Q\circ H^{-1}$ and by (50), we have

$$\begin{aligned}
&\int_\Omega Q\circ H^{-1}\left(\frac{|a(x,u_n,Du_n)|}{6\beta}\right)dx \\
&\quad\le \frac{1}{3}\int_\Omega Q\circ H^{-1}\left(\frac{1}{2}c(x)\right)dx + \frac{1}{3}\int_\Omega Q\circ H^{-1}\left(\frac{1}{2}\bar{M}^{-1}\big(M(|u_n|)\big)\right)dx \\
&\qquad + \frac{1}{3}\int_\Omega Q\circ H^{-1}\left(\frac{1}{2}\bar{M}^{-1}\big(M(|Du_n|)\big)\right)dx \\
&\quad\le \frac{1}{3}\left[C\int_\Omega \bar{M}\big(c(x)\big)\,dx + \int_\Omega Q\big(|u_n|\big)\,dx + \int_\Omega Q\big(|Du_n|\big)\,dx\right] \le C,
\end{aligned}$$

where C is independent of n. Thus we get

$$a(x,u_n,Du_n) \rightharpoonup a(x,u,Du) \quad \text{weakly for } \sigma\big(\Pi L_{Q\circ H^{-1}}\Pi E_{\overline{Q\circ H^{-1}}}\big). \tag{51}$$

Thanks to (45) and (51) we can pass to the limit in (32) and we obtain that u is a solution of (15). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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