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Periodic solutions for a singular damped differential equation

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Abstract
Based on a variational approach, we prove that a second-order singular damped differential equation has at least one periodic solution when some reasonable assumptions are satisfied.
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1 Introduction

The purpose of this paper is to study the existence of T-periodic solutions for second-order singular damped differential equation

$$u''(t) + q(t)u'(t) + f\big(u(t)\big) = g(t), \tag{1.1}$$

where $q, g \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ with $\int_0^T q(t)\,dt = 0$, and the nonlinearity $f \in C((0,\infty), \mathbb{R})$ admits a repulsive singularity at $u = 0$, which means that

$$\lim_{u\to 0^+} f(u) = -\infty.$$

Second-order singular differential equations have attracted many researchers? attention because of the wide applications in applied sciences. For example, they can describe the dynamics of particles under the action of Newtonian-type forces caused by compressed gases [1]. If $q(t) \equiv 0$, then Eq. (1.1) reduces to the following singular differential equation:

$$u''(t) + f\big(u(t)\big) = g(t). \tag{1.2}$$

The existence of periodic solutions for Eq. (1.2) has attracted the attention of many researchers, and some classical tools have been used in the literature, including the method of upper and lower solutions [2, 3], degree theory [4], some fixed point theorems in cones for completely continuous operators [5, 6], Schauder?s fixed point theorem [7], a nonlinear Leray-Schauder alternative principle [7, 8] and variational methods [9–11].

Recently, Eq. (1.1) has also been investigated by several authors; see, for instance, [12, 13] (application of Leray-Schauder alternative principle) and [14] (using Schauder?s fixed point theorem). In general cases, it is very difficult or impossible to apply variational methods to

Eq. (1.1) when $\int_0^T q(t)\,dt > 0$. In this paper, we consider the case $\int_0^T q(t)\,dt = 0$ and under some reasonable assumptions, we establish the corresponding variational framework of T-periodic solutions for Eq. (1.1) on an appropriate Sobolev space and give a new criterion to guarantee the existence of at least one nontrivial T-periodic solution of Eq. (1.1) using a variant of the mountain pass theorem. We refer the reader to [15–17] for the details about variational methods.

In order to state our main result, we need the following assumptions:

(H1) $q, g \in C(\mathbb{R}/T\mathbb{Z})$ with $\int_0^T q(t)\,dt = 0$;

(H2) $f \in C((0,\infty),\mathbb{R})$ has a repulsive singularity at $u = 0$, *i.e.*,

$$\lim_{u\to 0^+} f(u) = -\infty;$$

(H3) $\lim_{u\to 0^+} F(u) = +\infty$, where $F(u) = \int_1^u f(s)\,ds$;

(H4) $M = \sup\{f(s) : 0 < s < +\infty\}$ is bounded;

(H5) $\lim_{u\to +\infty}(F(u) - \bar{g}u) = +\infty$, where $\bar{g}$ is defined by

$$\bar{g} \stackrel{\text{def}}{=} \frac{1}{\int_0^T \exp(\int_0^t q(s)\,ds)\,dt} \int_0^T g(t) \exp\left(\int_0^T q(s)\,ds\right) dt.$$

Theorem 1.1 *Assume that* (H1)-(H5) *are satisfied. Then Eq.* (1.1) *has at least one nontrivial* T*-periodic solution.*

The existence of T-periodic solutions for the following singular damped differential equation

$$u''(t) + q(t)u'(t) + p(t)u(t) + f\big(u(t)\big) = g(t) \tag{1.3}$$

was discussed in [12–14] by using the Leray-Schauder alternative principle or Schauder?s fixed point theorem. However, all of them required that the Green function associated to the linear equation problem

$$\begin{cases} u''(t) + q(t)u'(t) + p(t)u(t) = 0, \\ u(0) = u(T), \qquad u'(0) = u'(T) \end{cases}$$

is positive for all $(t,s) \in [0,T] \times [0,T]$. For example, in [13] and [14] it is supposed that

$$\int_0^T q(t)\,dt > 0. \tag{1.4}$$

In [12], two criteria to make the Green function positive were given. In particular, one criterion was proved when $\int_0^T q(t)\,dt = 0$ and

$$\int_0^T p(t) e^{\int_0^t q(s)\,ds}\,dt > 0. \tag{1.5}$$

Note that in Theorem 1.1, conditions (1.4) and (1.5) do not hold because $p(t) \equiv 0$ and $\int_0^T q(t)\,dt = 0$ in our case.

From (H1), it is obvious that

$$e^{-\|q\|_{L^1}} \leq e^{Q(t)} \leq e^{\|q\|_{L^1}}, \tag{1.6}$$

where $Q(t) = \int_0^t q(s)\,ds$. In addition, it is easy to find the functions $f(u)$ and $g(t)$ which satisfy assumptions (H2)-(H5). For example, if we take

$$f(u) = -\frac{e}{u^\gamma}, \tag{1.7}$$

where $e > 0$ and $\gamma \geq 1$ are constants and choose $g \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ such that

$$\int_0^T g(t)\,dt < 0, \tag{1.8}$$

then (H2)-(H5) are satisfied.

Remark 1 If we take $q(t) \equiv 0$ in (1.1) and $e = 1$ in (1.7), then (1.1) reduces to the following repulsive-type equation:

$$u''(t) - \frac{1}{u^\gamma(t)} = g(t). \tag{1.9}$$

It was proved in [18] that Eq. (1.9) (with $\gamma \geq 1$) has a positive T-periodic solution if and only if (1.8) holds. One open problem is whether we can obtain the sufficient and necessary conditions to guarantee the existence of positive T-periodic solutions for the following special form of Eq. (1.1) with $\gamma \geq 1$:

$$u''(t) + q(t)u'(t) - \frac{1}{u^\gamma(t)} = g(t).$$

The remaining part of this paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, the proof of Theorem 1.1 is given.

2 Preliminary results

In this section, we present some auxiliary results, which will be used in the proof of our main result. First, we define the truncation function $f_\lambda : \mathbb{R} \to \mathbb{R}$, $0 < \lambda < 1$, by

$$f_\lambda(u) = \begin{cases} f(u), & u \geq \lambda, \\ f(\lambda), & u < \lambda. \end{cases}$$

Note that condition (H2) implies that f_λ is continuous with respect to $u \in \mathbb{R}$.

In what follows, for $\lambda \in (0,1)$, we consider the following modified equation:

$$u''(t) + q(t)u'(t) + f_\lambda\big(u(t)\big) = g(t). \tag{2.1}$$

Let

$$Q(t) = \int_0^t q(s)\,ds \quad \text{and} \quad F_\lambda(u) = \int_1^u f_\lambda(s)\,ds.$$

Then the problem of the existence of T-periodic solutions for Eq. (2.1) has a variational structure with corresponding functional Φ_λ given by

$$\Phi_\lambda(u) = \int_0^T e^{Q(t)}\left[\frac{1}{2}u'(t)^2 - F_\lambda\big(u(t)\big) + g(t)u(t)\right]dt, \tag{2.2}$$

and defined on the Hilbert space

$$H_T^1 = \{u : [0,T] \to \mathbb{R} \text{ is absolutely continuous}; u(0) = u(T), u' \in L^2\big([0,T];\mathbb{R}\big)\},$$

equipped with the norm

$$\|u\| = \left(\int_0^T u(t)^2\,dt + \int_0^T u'(t)^2\,dt\right)^{\frac{1}{2}}$$

for $u \in H_T^1$.

Lemma 2.1 [15, Proposition 1.3] (Wirtinger?s inequality) *If $u \in H_T^1$ and $\int_0^T u(t)\,dt = 0$, then*

$$\int_0^T u(t)^2\,dt \le \frac{T^2}{4\pi^2}\int_0^T \dot{u}(t)^2\,dt.$$

Under the conditions of Theorem 1.1, similar to [19, Theorems 2.1 and 2.2], it is easy to verify that Φ_λ is continuously differentiable, weakly lower semicontinuous on H_T^1 and

$$\Phi_\lambda'(u)v = \int_0^T e^{Q(t)}\big[u'(t)v'(t) - f_\lambda\big(u(t)\big)v(t) + g(t)v(t)\big]\,dt. \tag{2.3}$$

Moreover, critical points of Φ_λ on H_T^1 are T-periodic solutions of Eq. (2.1).

In order to obtain the existence of T-periodic solutions of Eq. (2.1), the following version of the mountain pass theorem will be used in our argument.

Lemma 2.2 [15, Theorem 4.10] *Let X be a Banach space, and let $\varphi \in C^1(X,\mathbb{R})$. Assume that there exist $x_0, x_1 \in X$ and a bounded open neighborhood Ω of x_0 such that $x_1 \in X\backslash\overline{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega}\varphi(u).$$

Let

$$\Gamma = \{h \in C\big([0,1],X\big) : h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h\in\Gamma}\max_{s\in[0,1]}\varphi\big(h(s)\big).$$

If φ satisfies the (*PS*)*-condition* (*that is, a sequence $\{u_n\}$ in X satisfying $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$ has a convergent subsequence*), *then c is a critical value of φ and $c > \max\{\varphi(x_0), \varphi(x_1)\}$.*

3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1.

Proof The proof will be divided into four steps.

Step 1. Φ_λ satisfies the (PS)-condition.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in H_T^1 such that $\{\Phi'_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\Phi'_\lambda(u_n)\to 0$ as $n\to+\infty$. Then there exist a constant $c_1>0$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^+$ with $\epsilon_n\to 0$ as $n\to+\infty$ such that, for all n,

$$\left|\int_0^T e^{Q(t)}\left[\frac{1}{2}u_n'(t)^2 - F_\lambda\bigl(u_n(t)\bigr) + g(t)u_n(t)\right]dt\right| \le c_1, \tag{3.1}$$

and for every $v\in H_T^1$,

$$\left|\int_0^T e^{Q(t)}\bigl[u_n'(t)v'(t) - f_\lambda\bigl(u_n(t)\bigr)v(t) + g(t)v(t)\bigr]dt\right| \le \epsilon_n\|v\|_{H_T^1}. \tag{3.2}$$

Using a standard argument, it is sufficient to show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H_T^1, and this will be enough to derive the (PS)-condition.

Taking $v(t)\equiv -1$ in (3.2), we obtain that

$$\left|\int_0^T e^{Q(t)}\bigl[f_\lambda\bigl(u_n(t)\bigr) - g(t)\bigr]dt\right| \le \epsilon_n\sqrt{T}.$$

So that

$$\begin{aligned}\left|\int_0^T e^{Q(t)}f_\lambda\bigl(u_n(t)\bigr)\,dt\right| &\le \epsilon_n\sqrt{T} + \left|\int_0^T e^{Q(t)}g(t)\,dt\right| \\ &\le \epsilon_n\sqrt{T} + e^{\|q\|_{L^1}}\int_0^T \bigl|g(t)\bigr|\,dt \\ &= \epsilon_n\sqrt{T} + e^{\|q\|_{L^1}}\|g\|_{L^1} := c_2.\end{aligned} \tag{3.3}$$

Let

$$I_{1,n} = \bigl\{t\in[0,T] : f_\lambda\bigl(u_n(t)\bigr)\ge 0\bigr\},$$

and

$$I_{2,n} = \bigl\{t\in[0,T] : f_\lambda\bigl(u_n(t)\bigr) < 0\bigr\}.$$

It follows from (3.3) that

$$\left|\int_{I_{2,n}} e^{Q(t)}f_\lambda\bigl(u_n(t)\bigr)\,dt\right| \le c_2 + \int_{I_{1,n}} e^{Q(t)}f_\lambda\bigl(u_n(t)\bigr)\,dt \le c_2 + TMe^{\|q\|_{L^1}},$$

where M is defined in (H4). Hence, there exists $c_3>0$ such that

$$\int_0^T e^{Q(t)}\bigl|f_\lambda\bigl(u_n(t)\bigr)\bigr|\,dt \le c_3 \quad \text{for all } n. \tag{3.4}$$

On the other hand, if we take, in (3.2), $v(t) \equiv w_n(t) := u_n(t) - \bar{u}_n$, where $\bar{u}_n$ is the average of u_n over the interval $[0, T]$, we get (taking into account (3.4))

$$
\begin{aligned}
c_4 \|w_n\|_{H_T^1} &\geq \left| \int_0^T e^{Q(t)} \left[\frac{1}{2} w_n'(t)^2 - f_\lambda\big(u_n(t)\big) w_n(t) + g(t) w_n(t) \right] dt \right| \\
&\geq \frac{e^{-\|q\|_{L^1}}}{2} \|w_n'\|_{L^2}^2 - \big(c_3 + e^{\|q\|_{L^1}} \|g\|_{L^1}\big) \|w_n\|_{L^\infty} \\
&\geq \frac{e^{-\|q\|_{L^1}}}{2} \|w_n'\|_{L^2}^2 - c_5 \|w_n\|_{H_T^1}.
\end{aligned}
$$

Using the Poincaré-Wirtinger inequality for zero mean functions in the Sobolev space H_T^1, we know that there exists $c_6 > 0$ such that

$$
\|u_n'\|_{L^2} \leq \|w_n\|_{H_T^1} \leq c_6. \tag{3.5}
$$

Now suppose that

$$
\|u_n\|_{H_T^1} \to +\infty \quad \text{as } n \to +\infty.
$$

Since (3.5) holds, we have, passing to a subsequence if necessary, that either

$$
\begin{aligned}
&m_n = \min u_n \to -\infty \quad \text{as } n \to +\infty, \quad \text{or} \\
&M_n = \max u_n \to +\infty \quad \text{as } n \to +\infty.
\end{aligned}
$$

(i) Assume that the second possibility occurs. We have

$$
\begin{aligned}
&\int_0^T e^{Q(t)} \big[F_\lambda\big(u_n(t)\big) - g(t) u_n(t)\big] dt \\
&\quad = \int_0^T e^{Q(t)} \left[\left(\int_1^{u_n(t)} f_\lambda(s)\, ds \right) - g(t) u_n(t) \right] dt \\
&\quad = \int_0^T e^{Q(t)} \left[\left(\int_1^{M_n} f_\lambda(s)\, ds - \int_{u_n(t)}^{M_n} f_\lambda(s)\, ds \right) - g(t) u_n(t) \right] dt \\
&\quad = \int_0^T e^{Q(t)} \big[F_\lambda(M_n) - M_n g(t)\big] dt - \int_0^T \left[\int_{u_n(t)}^{M_n} e^{Q(t)} \big(f_\lambda(s) - g(t)\big) ds \right] dt \\
&\quad \geq \int_0^T e^{Q(t)} \big[F_\lambda(M_n) - M_n g(t)\big] dt - e^{\|q\|_{L^1}} \|M - g\|_{L^1} \|M_n - u_n\|_C.
\end{aligned}
$$

Thus, using the Sobolev and Poincaré inequalities to $M_n - u_n(\cdot)$, we have, from (3.5),

$$
\begin{aligned}
\int_0^T e^{Q(t)} \big[F_\lambda(M_n) - M_n g(t)\big] dt &\leq \int_0^T e^{Q(t)} \big[F_\lambda\big(u_n(t)\big) - g(t) u_n(t)\big] dt \\
&\quad + e^{\|q\|_{L^1}} \|M - g\|_{L^1} \|M_n - u_n\|_C \\
&\leq \int_0^T e^{Q(t)} \big[F_\lambda\big(u_n(t)\big) - g(t) u_n(t)\big] dt \\
&\quad + e^{\|q\|_{L^1}} \|M - g\|_{L^1} \sqrt{T} c_6.
\end{aligned}
$$

In view of (3.1), we see that

$$\int_0^T e^{Q(t)}\big[F_\lambda(M_n) - M_n g(t)\big]\,dt$$

is bounded, which contradicts (H5).

(ii) Assume that the first possibility occurs, *i.e.*, $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by $-m_n$ in the preceding arguments, and we also get a contradiction.

Therefore, Φ_λ satisfies the (PS) condition. This completes the proof of the claim.

Step 2. In what follows, let

$$\Omega = \big\{u \in H_T^1 : \min u > 1\big\},$$

and

$$\partial\Omega = \big\{u \in H_T^1 : u(t) \geq 1 \text{ for all } t \in [0,T], \exists t_u \in (0,T) \text{ such that } u(t_u) = 1\big\}.$$

We show that there exists $m > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\lambda(u) \geq -m$ whenever $\lambda \in (0,1)$.

For any $u \in \partial\Omega$, we have $\min u = u(t_u) = 1$ for some t_u. By (2.2), we obtain that

$$\begin{aligned}\Phi_\lambda(u) &= \int_{t_u}^{t_u+T} e^{Q(t)}\left[\frac{1}{2}u'(t)^2 - F_\lambda\big(u(t)\big) + g(t)u(t)\right]dt \\ &\geq \int_{t_u}^{t_u+T} \frac{1}{2}e^{Q(t)}u'(t)^2\,dt - \left[\int_{t_u}^{t_u+T} e^{Q(t)}\big(M - g(t)\big)\big(u(t) - 1\big)\,dt \right.\\ &\quad \left. - \int_{t_u}^{t_u+T} e^{Q(t)}g(t)\,dt\right].\end{aligned}$$

The Hölder inequality and the fact that $u'(t) = (u(\cdot) - 1)'(t)$ imply that

$$\begin{aligned}\Phi_\lambda(u) \geq{}& \frac{e^{-\|q\|_{L^1}}}{2}\big\|\big(u(\cdot) - 1\big)'\big\|_{L^2}^2 - e^{\|q\|_{L^1}}\|M - g\|_{L^2}\big\|\big(u(\cdot) - 1\big)\big\|_{L^2} \\ &- e^{\|q\|_{L^1}}\|g\|_{L^1}.\end{aligned}$$

Applying the Poincaré inequality to $u(\cdot) - 1$, we get

$$\Phi_\lambda(u) \geq \frac{e^{-\|q\|_{L^1}}}{2}\|u'\|_{L^2}^2 - T^{\frac{3}{2}}e^{\|q\|_{L^1}}\|M - g\|_{L^2}\|u'\|_{L^2} - e^{\|q\|_{L^1}}\|g\|_{L^1}.$$

The above inequality shows that

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u'\|_{L^2} \to +\infty.$$

Since $\min u = 1$, we have that $\|u(\cdot) - 1\|_{H_T^1} \to +\infty$ is equivalent to $\|u'\|_{L^2} \to +\infty$. Hence

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u\|_{H_T^1} \to +\infty, u \in \partial\Omega,$$

which yields that Φ_λ is coercive. Thus it has a minimizing sequence. The weak lower semicontinuity of Φ_λ implies that

$$\inf_{u\in\partial\Omega}\Phi_\lambda(u) > -\infty.$$

It follows that there exists $m>0$ such that $\inf_{u\in\partial\Omega}\Phi_\lambda(u)\geq -m$ for all $\lambda\in(0,1)$.

Step 3. We show that there exists $\lambda_0\in(0,1)$ with the property that, for every $\lambda\in(0,\lambda_0)$, any solution u of Eq. (2.1) satisfying $\Phi_\lambda(u)\geq -m$ is such that $\min u\geq\lambda_0$, and hence u is a solution of Eq. (1.1).

On the contrary, assume that there are sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that

(i) $\lambda_n\leq\frac{1}{n}$;
(ii) u_n is a solution of Eq. (2.1) with $\lambda=\lambda_n$;
(iii) $\Phi_{\lambda_n}(u_n)\geq -m$;
(iv) $\min u_n<\frac{1}{n}$.

Since

$$\int_0^T e^{Q(t)}\bigl[f_{\lambda_n}\bigl(u_n(t)\bigr)-g(t)\bigr]\,dt=0, \tag{3.6}$$

we have

$$\bigl\|e^{Q(\cdot)}f_{\lambda_n}\bigl(u_n(\cdot)\bigr)\bigr\|_{L^1}\leq c_7\quad\text{for some constant } c_7>0.$$

On the other hand, since $u_n(0)=u_n(T)$, there exists $\tau_n\in(0,T)$ such that

$$u_n'(\tau_n)=0.$$

Therefore, we obtain that

$$e^{Q(t)}u_n'(t)-e^{Q(\tau_n)}u_n'(\tau_n)=\int_{\tau_n}^t e^{Q(s)}\bigl[f_\lambda\bigl(u_n(s)\bigr)-g(s)\bigr]\,ds,$$

which, from (3.6), yields that

$$\|u_n'\|_{L^\infty}\leq c_8\quad\text{for some constant } c_8>0. \tag{3.7}$$

Since $\Phi_{\lambda_n}(u_n)>-m$, it follows that there exist two constants R_1 and R_2 with $0<R_1<R_2$ such that

$$\max\bigl\{u_n(t):t\in[0,T]\bigr\}\in[R_1,R_2].$$

If not, u_n would tend uniformly to 0 or $+\infty$. In both cases, in view of (H3), (H5) and (3.7), we have

$$\Phi_{\lambda_n}(u_n)\to-\infty\quad\text{as } n\to+\infty,$$

which contradicts the fact that $\Phi_{\lambda_n}(u_n)\geq -m$.

Let τ_n^1, τ_n^2 be such that, for n large enough,

$$u_n(\tau_n^1) = \frac{1}{n} < R_1 = u_n(\tau_n^2).$$

Multiplying Eq. (2.1) by u_n' and integrating the resulting equation on $[\tau_n^1, \tau_n^2]$ (or $[\tau_n^2, \tau_n^1]$), we get

$$\begin{aligned} J &:= \int_{\tau_n^1}^{\tau_n^2} u_n''(t)u_n'(t)\,dt + \int_{\tau_n^1}^{\tau_n^2} q(t)u_n'(t)^2\,dt + \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(t))u_n'(t)\,dt \\ &= \int_{\tau_n^1}^{\tau_n^2} g(t)u_n'(t)\,dt. \end{aligned}$$

It is clear that

$$J = J_1 + \frac{1}{2}\left[u_n'^2(\tau_n^1) - u_n'^2(\tau_n^2)\right] + \int_{\tau_n^1}^{\tau_n^2} q(t)u_n'^2(t)\,dt,$$

where

$$J_1 = \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(t))u_n'(t)\,dt.$$

Since q is bounded, g is integrable and $\|u_n'\|_{L^\infty} \le c_8$ (see (3.7)), it follows that J is bounded, and consequently, J_1 is bounded. On the other hand, we have

$$f_{\lambda_n}(u_n(t))u_n'(t) = \frac{d}{dt}\left[F_{\lambda_n}(u_n(t))\right],$$

which yields that

$$J_1 = F_{\lambda_n}(R_1) - F_{\lambda_n}\left(\frac{1}{n}\right).$$

However, due to (H3), it follows that J_1 is unbounded. This is a contradiction.

Step 4. We prove that Φ_λ has a mountain pass geometry for $\lambda \le \lambda_0$.

Fix $\lambda \in (0, \lambda_0]$ such that $f(\lambda) < 0$. It is possible because of (H2). Therefore, we have

$$\begin{aligned} F_\lambda(0) &= \int_1^0 f_\lambda(s)\,ds = -\int_0^1 f_\lambda(s)\,ds \\ &= -\int_0^\lambda f_\lambda(s)\,ds - \int_\lambda^1 f_\lambda(s)\,ds \\ &= -\int_0^\lambda f(\lambda)\,ds - \int_\lambda^1 f_\lambda(s)\,ds \\ &= -\lambda f(\lambda) - \int_\lambda^1 f_\lambda(s)\,ds. \end{aligned}$$

This implies that

$$F_\lambda(0) > -\int_\lambda^1 f_\lambda(s)\,ds = \int_1^\lambda f_\lambda(s)\,ds = F_\lambda(\lambda).$$

Hence

$$\begin{aligned}\Phi_\lambda(0) &= -\int_0^T e^{Q(t)} F_\lambda(0)\,dt + \int_0^T e^{Q(t)} g(t) 0\,dt \\ &\leq -\int_0^T e^{Q(t)} F_\lambda(\lambda)\,dt. \end{aligned} \tag{3.8}$$

By (H3), choose $\lambda \in (0, \lambda_0]$ such that

$$F_\lambda(\lambda) > \frac{m}{T} e^{\|q\|_{L^1}} \quad \text{for all } t \in [0, T].$$

It follows from (3.8) that $\Phi_\lambda(0) < -m$.

Also, using (H5), we can find R large enough such that $R > 1$ and

$$-\int_0^T e^{Q(t)} \bigl[F_\lambda(R) - g(t)R\bigr]\,dt < -m,$$

which implies that

$$\Phi_\lambda(R) < -m.$$

Since Ω is a neighborhood of R, $0 \notin \Omega$ and

$$\max\bigl\{\Phi_\lambda(0), \Phi_\lambda(R)\bigr\} < \inf_{u \in \partial\Omega} \Phi_\lambda(u).$$

Step 1 and Step 4 imply that Φ_λ has a critical point u_λ such that

$$\Phi_\lambda(u_\lambda) = \inf_{\eta \in \Gamma} \max_{0 \leq s \leq 1} \Phi_\lambda\bigl(\eta(s)\bigr) \geq \inf_{u \in \partial\Omega} \Phi_\lambda(u),$$

where $\Gamma = \{\eta \in C([0,1], H_T^1) : \eta(0) = 0, \eta(1) = R\}$.

Since $\inf_{u \in \partial\Omega} \Phi_\lambda(u) \geq -m$, it follows from Step 3 that u_λ is a solution of Eq. (1.1). Now the proof is finished. □

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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