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Global nonexistence of solutions for systems of quasilinear hyperbolic equations with damping and source terms

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Abstract

The initial boundary value problem for a class of quasilinear hyperbolic equations system in a bounded domain is studied. We prove that the solutions with positive initial energy blow up in finite time under some conditions. The estimates of the lifespan of solutions are given.

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1 Introduction

In this paper, we are concerned with the blow-up of solutions for the following quasilinear hyperbolic equations system:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) + a|u_t|^{p-2}u_t - \Delta u_t = f_1(u,v), \quad (x,t) \in \Omega \times R^+, \tag{1.1}$$

$$v_{tt} - \operatorname{div}\left(|\nabla v|^{m-2}\nabla v\right) + a|v_t|^{p-2}v_t - \Delta v_t = f_2(u,v), \quad (x,t) \in \Omega \times R^+ \tag{1.2}$$

with the initial boundary value conditions

$$u(x,0) = u_0(x) \in W_0^{1,m}(\Omega), \qquad u_t(x,0) = u_1(x) \in L^2(\Omega), \quad x \in \Omega, \tag{1.3}$$

$$v(x,0) = v_0(x) \in W_0^{1,m}(\Omega), \qquad v_t(x,0) = v_1(x) \in L^2(\Omega), \quad x \in \Omega, \tag{1.4}$$

$$u(x,t) = 0, \qquad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times R^+, \tag{1.5}$$

where Ω is a bounded open domain in R^n with a smooth boundary $\partial\Omega$. $a > 0$ and $m, p \geq 2$ are real numbers, and $f_i(\cdot, \cdot) : R^2 \to R$ $(i = 1, 2)$ are given functions to be determined later.

When $m = 2$, problem (1.1)-(1.5) defines the motion of charged meson in an electromagnetic field and was proposed by Segal [1]. Equations (1.1) and (1.2) with initial boundary conditions (1.3)-(1.5), but without dissipative terms, were early considered by several authors. Medeiros and Miranda [2, 3] showed the existence and uniqueness of global weak solutions. Da Silva Ferreira [4] proved that the first-order energy decays exponentially in the presence of frictional local damping. Cavalcanti *et al.* [5] considered the asymptotic

behavior for an analogous hyperbolic-parabolic system, with boundary damping, using arguments from Komornik and Zuazua [6].

For the initial boundary value problem of a single quasilinear hyperbolic equation

$$u_{tt} - \text{div}\big(|\nabla u|^{m-2}\nabla u\big) + a|u_t|^{p-2}u_t - \Delta u_t = b|u|^{r-2}u, \quad (x,t) \in \Omega \times R^+, \tag{1.6}$$

$$u(x,0) = u_0(x) \in W_0^{1,m}(\Omega), \qquad u_t(x,0) = u_1(x) \in L^2(\Omega), \quad x \in \Omega, \tag{1.7}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R^+, \tag{1.8}$$

Yang and Chen [7–9] studied problem (1.6)-(1.8) and obtained global existence results under the growth assumptions on the nonlinear terms and the initial value. These global existence results have been improved by Liu and Zhao [10] by using a new method. In [11], the author considered a similar problem to (1.6)-(1.8) and proved a blow-up result under the condition $r > \max\{p, m\}$ and that the initial energy is sufficiently negative. Messaoudi and Said-Houari [12] improved the results in [11] and showed that the blow-up takes place for negative initial data only regardless of the size of Ω. By means of the perturbed energy and the potential well methods, Messaoudi [13] gave precise decay rates for the solution of problem (1.6)-(1.8). In particular, he showed that for $m = 2$, the decay is exponential.

In absence of the strong damping $-\Delta u_t$, equation (1.6) becomes

$$u_{tt} - \text{div}\big(|\nabla u|^{m-2}\nabla u\big) + a|u_t|^{p-2}u_t = b|u|^{r-2}u, \quad (x,t) \in \Omega \times R^+. \tag{1.9}$$

For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial value (see [14–16]). For $a = 0$, the source term causes finite time blow-up of solutions with negative initial energy if $r > m$ (see [17]). When the quasilinear operator $-\text{div}(|\nabla u|^{m-2}\nabla u)$ is replaced by $\Delta^2 u$, Wu and Tsai [18] showed that the solution is global in time under some conditions without the relation between p and r. They also proved that the local solution blows up in finite time if $r > p$ and the initial energy is nonnegative, and gave the decay estimates of the energy function and the lifespan of solutions.

In this paper we show that the local solutions of problem (1.1)-(1.5) with small positive initial energy blow up in finite time. Meanwhile, the lifespan of solutions is given. The main tool of the proof is a technique introduced by paper [19] and some estimates used firstly by Vitillaro [20] in order to study a class of single wave equations.

For simplicity of notations, hereafter we denote by $\|\cdot\|_s$ the space $L^s(\Omega)$ norm, $\|\cdot\|$ denotes $L^2(\Omega)$ norm, and we write an equivalent norm $\|\nabla\cdot\|_m$ instead of $W_0^{1,m}(\Omega)$ norm $\|\cdot\|_{W_0^{1,m}(\Omega)}$. Moreover, C denotes various positive constants depending on the known constants and may be different at each appearance.

2 Preliminaries

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we assume that

$$f_1(u,v) = b_1|u+v|^{2r}(u+v) + b_2|u|^{r-1}u|v|^{r+1},$$

$$f_2(u,v) = b_1|u+v|^{2r}(u+v) + b_2|v|^{r-1}v|u|^{r+1}, \tag{2.1}$$

where $b_1, b_2 > 0$ and $r > 0$ are constants.

It is easy to see that

$$uf_1(u,v) + vf_2(u,v) = 2(r+1)F(u,v), \quad \forall (u,v) \in R^2, \tag{2.2}$$

where

$$F(u,v) = \frac{b_1}{2(r+1)}|u+v|^{2(r+1)} + \frac{b_2}{r+1}|uv|^{r+1}. \tag{2.3}$$

Moreover, a quick computation will show that there exist two positive constants C_0 and C_1 such that the following inequality holds (see [21]):

$$\frac{C_0}{2(r+1)}\big(|u|^{2(r+1)} + |v|^{2(r+1)}\big) \le F(u,v) \le \frac{C_1}{2(r+1)}\big(|u|^{2(r+1)} + |v|^{2(r+1)}\big). \tag{2.4}$$

Now, we define the following energy function associated with a solution $[u,v]$ of problem (1.1)-(1.5):

$$E(t) = \frac{1}{2}\big(\|u_t(t)\|^2 + \|v_t(t)\|^2\big) + \frac{1}{m}\big(\|\nabla u(t)\|_m^m + \|\nabla v(t)\|_m^m\big) - \int_\Omega F(u,v)\,dx \tag{2.5}$$

for $[u,v] \in W_0^{1,m}(\Omega) \times W_0^{1,m}(\Omega)$, and

$$E(0) = \frac{1}{2}\big(\|u_1\|^2 + \|v_1\|^2\big) + \frac{1}{m}\big(\|\nabla u_0\|_m^m + \|\nabla v_0\|_m^m\big) - \int_\Omega F(u_0,v_0)\,dx \tag{2.6}$$

is the initial total energy.

Note that we have from (2.5) that

$$E(t) \ge \frac{1}{m}\big(\|\nabla u(t)\|_m^m + \|\nabla v(t)\|_m^m\big) - \int_\Omega F(u,v)\,dx \tag{2.7}$$

for $[u,v] \in W_0^{1,m}(\Omega) \times W_0^{1,m}(\Omega)$.

Lemma 2.1 *Let s be a number with* $2 \le s < +\infty$ *if* $n \le m$ *and* $2 \le s \le \frac{nm}{n-m}$ *if* $n > m$. *Then there is a constant C depending on* Ω *and s such that*

$$\|u\|_s \le C\|\nabla u\|_m, \quad \forall u \in W_0^{1,m}(\Omega).$$

Lemma 2.2 (Young's inequality) *Let* $a,b \ge 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *for* $1 < p,q < +\infty$, *then one has the inequality*

$$ab \le \delta a^p + C(\delta)b^q,$$

where $\delta > 0$ *is an arbitrary constant, and* $C(\delta)$ *is a positive constant depending on* δ.

We get from Minkowski's inequality and Lemma 2.1 that

$$\|u+v\|_{2(r+1)}^2 \le 2\big(\|u\|_{2(r+1)}^2 + \|v\|_{2(r+1)}^2\big) \le 2B^2\big(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\big). \tag{2.8}$$

Also, we have from Hölder's inequality and Lemma 2.2 that

$$\|uv\|_{r+1} \leq \|u(t)\|_{2(r+1)} \cdot \|v(t)\|_{2(r+1)} \leq \frac{1}{2}\big(\|u(t)\|_{2(r+1)}^2 + \|v(t)\|_{2(r+1)}^2\big)$$

$$\leq \frac{B^2}{2}\big(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\big), \tag{2.9}$$

where B is the optimal Sobolev constant from $W_0^{1,m}(\Omega)$ to $L^{2(r+1)}(\Omega)$.

We get from (2.3), (2.8) and (2.9) that

$$\int_\Omega F(u,v)\,dx \leq \frac{C_2 B^{2(r+1)}}{r+1}\big(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\big)^{r+1}, \tag{2.10}$$

where $C_2 = 2^r b_1 + \frac{b_2}{2^{r+1}}$.

Considering the basic inequality $|x+y|^\rho \leq 2^{\rho-1}(|x|^\rho + |y|^\rho)$, $\forall x,y \in R$, $\rho \geq 1$, it follows from (2.7) and (2.10) that

$$E(t) \geq \frac{1}{m2^{\frac{m-2}{2}}}\big(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\big)^{\frac{m}{2}} - \frac{C_2 B^{2(r+1)}}{r+1}\big(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\big)^{r+1}$$

$$= Q\Big(\sqrt{\|\nabla u\|_m^2 + \|\nabla v\|_m^2}\Big), \tag{2.11}$$

where

$$Q(\lambda) = \frac{1}{m2^{\frac{m-2}{2}}}\lambda^m - \frac{C_2 B^{2(r+1)}}{r+1}\lambda^{2(r+1)}.$$

Therefore, we get that

$$Q'(\lambda) = \frac{1}{2^{\frac{m-2}{2}}}\lambda^{m-1} - 2C_2 B^{2(r+1)}\lambda^{2r+1},$$

$$Q''(\lambda) = \frac{m-1}{2^{\frac{m-2}{2}}}\lambda^{m-2} - 2(2r+1)C_2 B^{2(r+1)}\lambda^{2r}.$$

Let $Q'(\lambda) = 0$, which implies that $\lambda_1 = (2^{\frac{m}{2}} C_2 B^{2(r+1)})^{\frac{1}{m-2(r+1)}}$. As $\lambda = \lambda_1$ and $m < 2(r+1)$, an elementary calculation shows that

$$Q''(t) = \frac{m-2(r+1)}{2^{\frac{m-2}{2}}}\big(2^{\frac{m}{2}} C_2 B^{2(r+1)}\big)^{\frac{m-2}{m-2(r+1)}} < 0.$$

Thus, $Q(\lambda)$ has the maximum at λ_1 and the maximum value is

$$d = Q(\lambda_1) = \frac{1}{2^{\frac{m-2}{2}}}\Big(\frac{1}{m} - \frac{1}{2(r+1)}\Big)\big(2^{\frac{m}{2}} C_2 B^{2(r+1)}\big)^{\frac{m}{m-2(r+1)}}. \tag{2.12}$$

In order to prove our main result, we need the following two lemmas.

Lemma 2.3 *Let $[u,v]$ be a solution to problem* (1.1)-(1.5), *then $E(t)$ is a nonincreasing function for $t > 0$ and*

$$\frac{d}{dt}E(t) = -\big(a\|u_t\|_p^p + a\|v_t\|_p^p + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\big) \leq 0. \tag{2.13}$$

Multiplying equation (1.1) by u_t and (1.2) by v_t, and integrating over $\Omega \times [0, t]$, then adding them together and integrating by parts, we get

$$E(t) - E(0) = -\int_0^t \left(a \|u_t(s)\|_p^p + a \|v_t(s)\|_p^p + \|\nabla u_t(s)\|_2^2 + \|\nabla v_t(s)\|_2^2 \right) ds \tag{2.14}$$

for $t \geq 0$.

Being the primitive of an integrable function, $E(t)$ is absolutely continuous and equality (2.13) is satisfied.

Applying the idea of Vitillaro [20], we have the following lemma.

Lemma 2.4 *Assume that $0 < E(0) < d$.*

(i) *If $(\|\nabla u_0\|_m^2 + \|\nabla v_0\|_m^2)^{\frac{m}{2}} < \lambda_1^m$, then $(\|\nabla u\|_m^2 + \|\nabla v\|_m^2)^{\frac{m}{2}} < \lambda_1^m$ for $t \geq 0$.*

(ii) *If $(\|\nabla u_0\|_m^2 + \|\nabla v_0\|_m^2)^{\frac{m}{2}} > \lambda_1^m$, then there exists $\lambda_2 > \lambda_1$ such that $(\|\nabla u\|_m^2 + \|\nabla v\|_m^2)^{\frac{m}{2}} \geq \lambda_2^m$ for $t \geq 0$.*

For the detailed proof of Lemma 2.4, one can refer to [18].

We conclude this section by stating the local existence and uniqueness of solutions for problem (1.1)-(1.5), which can be obtained by a similar way as done in [7–9, 22]. The result reads as follows.

Theorem 2.1 (Local solution) *Suppose that $[u_0, v_0] \in W_0^{1,m}(\Omega) \times W_0^{1,m}(\Omega)$, $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$ and*

$$2 < m < 2(r+1) \leq \frac{nm}{n-m}, \quad n \geq m; \qquad 2 < m < 2(r+1) < +\infty, \quad n < m, \tag{2.15}$$

then there exists $T > 0$ such that problem (1.1)-(1.5) has a unique local solution $[u(t), v(t)]$ satisfying

$$[u, v] \in L^\infty\left([0, T); W_0^{1,m}(\Omega) \times W_0^{1,m}(\Omega)\right),$$

$$u_t, v_t \in L^\infty\left([0, T); L^2(\Omega)\right) \cap L^p\left(\Omega \times [0, T)\right) \cap L^2\left([0, T); H_0^1(\Omega)\right).$$

Moreover, at least one of the following statements holds true:

(1) $\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|_m^m + \|\nabla v\|_m^m \to +\infty$ *as $t \to T^-$;*

(2) $T = +\infty$.

3 Main result and proof

In this section, we prove that the solutions with positive initial energy blow up in finite time under some conditions and that the estimates of the lifespan of solutions are given. Our main result reads as follows.

Theorem 3.1 *Assume that (2.15) holds, $r > \frac{1}{2}\max\{p, m\} - 1$ and $[u_0, v_0] \in W_0^{1,m}(\Omega) \times W_0^{1,m}(\Omega)$, $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$. If $0 < E(0) < d$ and $(\|\nabla u_0\|_m^2 + \|\nabla v_0\|_m^2)^{\frac{m}{2}} > \lambda_1^m$, then the local solution of problem (1.1)-(1.5) blows up in finite time.*

Proof Let

$$H(t) = h - E(t), \quad t \geq 0, \tag{3.1}$$

where $h = \frac{E(0)+d}{2}$. We see from (2.13) in Lemma 2.3 that $H'(t) \geq 0$. Thus we obtain

$$H(t) \geq H(0) = h - E(0) > 0, \quad t \geq 0. \tag{3.2}$$

Let

$$G(t) = \int_{\Omega} (uu_t + vv_t)\, dx + \frac{1}{2}\big(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\big). \tag{3.3}$$

By differentiating both sides of (3.3) on t, we get from (1.1) and (1.2) that

$$G'(t) = \big(\|u_t\|^2 + \|v_t\|^2\big) - \big(\|\nabla u\|_m^m + \|\nabla v\|_m^m\big)$$
$$+ 2(r+1)\int_{\Omega} F(u,v)\, dx - a \int_{\Omega} \big(|u_t|^{p-2}u_t u + |v_t|^{p-2}v_t v\big)\, dx. \tag{3.4}$$

We have from (2.5), (3.1) and (3.4) that

$$G'(t) = (r+2)\big(\|u_t\|^2 + \|v_t\|^2\big) + \frac{2(r+1)-m}{m}\big(\|\nabla u\|_m^m + \|\nabla v\|_m^m\big)$$
$$- a \int_{\Omega} \big(|u_t|^{p-2}u_t u + |v_t|^{p-2}v_t v\big)\, dx + 2(r+1)H(t) - 2(r+1)h. \tag{3.5}$$

We obtain from Lemma 2.4 that

$$\frac{2(r+1)-m}{m}\big(\|\nabla u\|_m^m + \|\nabla v\|_m^m\big) - 2(r+1)h$$
$$\geq \frac{2(r+1)-m}{m} \cdot \frac{(\lambda_2^m - \lambda_1^m)}{\lambda_2^m} \cdot \big(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\big)^{\frac{m}{2}}$$
$$+ \frac{2(r+1)-m}{m}\lambda_1^m \cdot \frac{(\|\nabla u\|_m^2 + \|\nabla v\|_m^2)^{\frac{m}{2}}}{\lambda_2^m} - 2(r+1)h$$
$$\geq \frac{2(r+1)-m}{m} \cdot \frac{(\lambda_2^m - \lambda_1^m)}{\lambda_2^m} \cdot \big(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\big)^{\frac{m}{2}}$$
$$+ \frac{2(r+1)-m}{m}\lambda_1^m - 2(r+1)h. \tag{3.6}$$

We have from Lemma 2.4 that $\frac{2(r+1)-m}{m} \cdot \frac{(\lambda_2^m - \lambda_1^m)}{\lambda_2^m} > 0$, and by (2.12) and (3.2), we see that

$$\frac{2(r+1)-m}{m}\lambda_1^2 - 2(r+1)h = \frac{2(r+1)-m}{m}\big(2^{\frac{m}{2}}C_2 B^{2(r+1)}\big)^{\frac{2}{m-2(r+1)}} - 2(r+1)h$$
$$= 2(r+1)(d-h) = 2(r+1)\big(h - E(0)\big) > 0. \tag{3.7}$$

It follows from (3.5), (3.6) and (3.7) that

$$G'(t) \geq (r+2)\big(\|u_t\|^2 + \|v_t\|^2\big) + \frac{2(r+1)-m}{m} \cdot \frac{(\lambda_2^m - \lambda_1^m)}{\lambda_2^m} \cdot \big(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\big)^{\frac{m}{2}}$$
$$- a \int_{\Omega} \big(|u_t|^{p-2}u_t u + |v_t|^{p-2}v_t v\big)\, dx + 2(r+1)H(t). \tag{3.8}$$

We have from Hölder's inequality that

$$a\left|\int_\Omega \left(|u_t|^{p-2}u_t u + |v_t|^{p-2}v_t v\right) dx\right|$$
$$\leq C_2\left(\|u\|_{2(r+1)}^{1-\frac{2(r+1)}{p}}\|u\|_{2(r+1)}^{\frac{2(r+1)}{p}}\|u_t\|_p^{p-1} + \|v\|_{2(r+1)}^{1-\frac{2(r+1)}{p}}\|v\|_{2(r+1)}^{\frac{2(r+1)}{p}}\|v_t\|_p^{p-1}\right). \tag{3.9}$$

We get from (2.7), (3.1) and Lemma 2.4 that

$$H(t) \leq h - \frac{1}{m2^{\frac{m-2}{2}}}\left(\|\nabla u\|_m^2 + \|\nabla v\|_m^2\right)^{\frac{m}{2}} + \int_\Omega F(u,v)\,dx$$
$$\leq d - \frac{1}{m2^{\frac{m-2}{2}}}\lambda_1^m + \int_\Omega F(u,v)\,dx. \tag{3.10}$$

Since

$$d - \frac{1}{m2^{\frac{m-2}{2}}}\lambda_1^m = -\frac{1}{2^{\frac{m}{2}}(r+1)}\left(2^{\frac{m}{2}}C_2 B^{2(r+1)}\right)^{\frac{m}{m-2(r+1)}} < 0, \tag{3.11}$$

so we have from (3.2), (3.10), (3.11), (2.4) and Lemma 2.1 that

$$0 < H(0) \leq H(t) \leq \int_\Omega F(u,v)\,dx$$
$$\leq \frac{C_1}{2(r+1)}\left(\|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)}\right)$$
$$\leq \frac{C_1 B^{2(r+1)}}{2(r+1)}\left(\|\nabla u\|_m^{2(r+1)} + \|\nabla v\|_m^{2(r+1)}\right), \quad t \geq 0. \tag{3.12}$$

We obtain from (3.9) and (3.12) that

$$a\left|\int_\Omega \left(|u_t|^{p-2}u_t u + |v_t|^{p-2}v_t v\right) dx\right|$$
$$\leq C_3 H(t)^{\frac{1}{2(r+1)}-\frac{1}{p}}\left(\|u\|_{2(r+1)}^{\frac{2(r+1)}{p}}\|u_t\|_p^{p-1} + \|v\|_{2(r+1)}^{\frac{2(r+1)}{p}}\|v_t\|_p^{p-1}\right). \tag{3.13}$$

We get from (3.13), Lemma 2.2, Lemma 2.3 and (3.1) that

$$a\left|\int_\Omega \left(|u_t|^{p-2}u_t u + |v_t|^{p-2}v_t v\right) dx\right|$$
$$\leq C_3\left[\varepsilon^p\left(\|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)}\right) + \varepsilon^{-\frac{p}{p-1}}H'(t)\right]H(t)^{-\alpha}, \tag{3.14}$$

where $\alpha = \frac{1}{p} - \frac{1}{2(r+1)}$, $\varepsilon > 0$. Let $0 < \rho < \alpha$, then we have from (3.2) and (3.14) that

$$a\left|\int_\Omega \left(|u_t|^{p-2}u_t u + |v_t|^{p-2}v_t v\right) dx\right|$$
$$\leq C_3\left[\varepsilon^r H(0)^{-\alpha}\left(\|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)}\right) + \varepsilon^{-\frac{p}{p-1}}H(0)^{\rho-\alpha}H(t)^{-\rho}H'(t)\right]. \tag{3.15}$$

Now, we define $L(t)$ as follows:

$$L(t) = H(t)^{1-\rho} + \delta G(t), \quad t \geq 0, \tag{3.16}$$

where δ is a positive constant to be determined later. By differentiating (3.16), we see from (3.8) and (3.15) that

$$
\begin{aligned}
L'(t) &= (1-\rho)H(t)^{-\rho}H'(t) + \delta G'(t) \\
&\geq \left[1 - \rho - \delta C_3 \varepsilon^{-\frac{p}{p-1}} H(0)^{\rho-\alpha}\right] H(t)^{-\rho}H'(t) \\
&\quad + \delta\left[(r+2)\left(\|u_t\|^2 + \|v_t\|^2\right) + 2(r+1)H(t)\right] \\
&\quad + \frac{2(r+1)-m}{m} \cdot \frac{(\lambda_2^m - \lambda_1^m)}{\lambda_2^m}\delta\left(\|\nabla u\|_m^m + \|\nabla v\|_m^m\right) \\
&\quad - C_3\delta\varepsilon^r H(0)^{-\alpha}\left(\|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)}\right).
\end{aligned}
\tag{3.17}
$$

Letting $k = \min\{r+1, \frac{2(r+1)-m}{m} \cdot \frac{(\lambda_2^m - \lambda_1^m)}{\lambda_2^m}\}$ and decomposing $2\delta(r+1)H(t)$ in (3.17) by

$$
2\delta(r+1)H(t) = 2k\delta H(t) + 2\delta(r+1-k)H(t),
\tag{3.18}
$$

we find from (2.4) that

$$
\int_\Omega F(u,v)\,dx \geq \frac{C_0}{2(r+1)}\left(\|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)}\right).
\tag{3.19}
$$

Combining (3.1)-(3.2) and (3.17)-(3.19), we obtain that

$$
\begin{aligned}
L'(t) &\geq \left[1 - \rho - \delta C_3 \varepsilon^{-\frac{p}{p-1}} H(0)^{\rho-\alpha}\right] H(t)^{-\rho}H'(t) \\
&\quad + \delta\left[(r+2) - k\right]\left(\|u_t\|^2 + \|v_t\|^2\right) + 2\delta(r+1-k)H(t) \\
&\quad + \left[\frac{2(r+1)-m}{m} \cdot \frac{(\lambda_2^m - \lambda_1^m)}{\lambda_2^m} - k\right]\delta\left(\|\nabla u\|_m^m + \|\nabla v\|_m^m\right) \\
&\quad + \left[\frac{kC_0}{r+1} - C_3\varepsilon^p H(0)^{-\alpha}\right]\delta\left(\|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)}\right).
\end{aligned}
\tag{3.20}
$$

Choosing $\varepsilon > 0$ small enough such that $\varepsilon^p < \frac{kC_0}{(r+1)C_3}H(0)^\alpha$ and $0 < \delta < \frac{1-\rho}{C_3}\varepsilon^{\frac{p}{p-1}}H(0)^{\alpha-\rho}$, we have from (3.20) that

$$
L'(t) \geq C_4\delta\left[\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|_m^m + \|\nabla v\|_m^m + \|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)} + H(t)\right],
\tag{3.21}
$$

where $C_4 = \{r+2-k, 2(r+1-k), \frac{kC_0}{2(r+1)}, \frac{2(r+1)-m}{m} \cdot \frac{(\lambda_2^m - \lambda_1^m)}{\lambda_2^m} - k\}$. Therefore, $L(t)$ is a nondecreasing function for $t \geq 0$. Letting δ in (3.16) be small enough, we get $L(0) > 0$. Consequently, we obtain that $L(t) \geq L(0) > 0$ for $t \geq 0$.

Since $0 < \rho < \alpha < 1$, it is evident that $1 < \frac{1}{1-\rho} < \frac{1}{1-\alpha}$. We deduce from (3.3) and (3.16) that

$$
L(t)^{\frac{1}{1-\rho}} \leq C_5\left[H(t) + \left(\delta\int_\Omega(uu_t + vv_t)\,dx\right)^{\frac{1}{1-\rho}} + \left(\delta\left[\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right]\right)^{\frac{1}{1-\rho}}\right].
\tag{3.22}
$$

On the other hand, for $r > 0$, we have from Hölder's inequality and Lemma 2.2 that

$$
\begin{aligned}
\left(\delta\int_\Omega(uu_t + vv_t)\,dx\right)^{\frac{1}{1-\rho}} &\leq C_6\left(\|u_t\|^{\frac{1}{1-\rho}}\|u\|_{2(r+1)}^{\frac{1}{1-\rho}} + \|v_t\|^{\frac{1}{1-\rho}}\|v\|_{2(r+1)}^{\frac{1}{1-\rho}}\right) \\
&\leq C_7\left(\|u\|_{2(r+1)}^{\frac{\mu}{1-\rho}} + \|v\|_{2(r+1)}^{\frac{\mu}{1-\rho}} + \|u_t\|^{\frac{\nu}{1-\rho}} + \|v_t\|^{\frac{\nu}{1-\rho}}\right),
\end{aligned}
\tag{3.23}
$$

where $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Let $0 < \rho < \min\{\alpha, \frac{1}{2} - \frac{1}{2(r+1)}, 1 - \frac{2}{m}\}$, $\nu = 2(1 - \rho)$, then $\frac{\mu}{1-\rho} = \frac{2}{1-2\rho} < 2(r + 1)$. It follows from (3.12) that

$$\left(\frac{C_1}{2(r+1)H(0)}\right)^{\frac{1}{2(r+1)}} \|u\|_{2(r+1)} \geq 1, \qquad \left(\frac{C_1}{2(r+1)H(0)}\right)^{\frac{1}{2(r+1)}} \|v\|_{2(r+1)} \geq 1, \qquad (3.24)$$

$$\left(\frac{C_1 B^{2(r+1)}}{2(r+1)H(0)}\right)^{\frac{1}{2(r+1)}} \|\nabla u\|_m \geq 1, \qquad \left(\frac{C_1 B^{2(r+1)}}{2(r+1)H(0)}\right)^{\frac{1}{2(r+1)}} \|\nabla v\|_m \geq 1. \qquad (3.25)$$

Thus, we get from (3.24) that

$$\|u\|_{2(r+1)}^{\frac{\mu}{1-\rho}} = \|u\|_{2(r+1)}^{\frac{2}{1-2\rho}} \leq \left(\frac{C_1}{2(r+1)H(0)}\right)^{1-\frac{1}{(1-2\rho)(r+1)}} \|u\|_{2(r+1)}^{2(r+1)},$$
$$\|v\|_{2(r+1)}^{\frac{\mu}{1-\rho}} = \|v\|_{2(r+1)}^{\frac{2}{1-2\rho}} \leq \left(\frac{C_1}{2(r+1)H(0)}\right)^{1-\frac{1}{(1-2\rho)(r+1)}} \|v\|_{2(r+1)}^{2(r+1)}. \qquad (3.26)$$

We obtain from (3.23) and (3.26) that

$$\left(\delta \int_\Omega (uu_t + vv_t)\,dx\right)^{\frac{1}{1-\rho}} \leq C_8 \left(\|u_t\|^2 + \|v_t\|^2 + \|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)}\right). \qquad (3.27)$$

Similarly, we have from Hölder's inequality and (3.25) that

$$\left(\delta\left[\|\nabla u\|^2 + \|\nabla v\|^2\right]\right)^{\frac{1}{1-\rho}} \leq C_9 \left(\|\nabla u\|_m^{\frac{2}{1-\rho}} + \|\nabla v\|_m^{\frac{2}{1-\rho}}\right) \leq C_{10}\left(\|\nabla u\|_m^m + \|\nabla v\|_m^m\right). \qquad (3.28)$$

Combining (3.22), (3.27) and (3.28), we find that

$$L(t)^{\frac{1}{1-\rho}} \leq C_{11}\left[\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|_m^m + \|\nabla v\|_m^m + \|u\|_{2(r+1)}^{2(r+1)} + \|v\|_{2(r+1)}^{2(r+1)} + H(t)\right]. \qquad (3.29)$$

We obtain from (3.21) and (3.29) that

$$L'(t) \geq C_{12} L(t)^{\frac{1}{1-\rho}}, \quad t \geq 0, \qquad (3.30)$$

where $C_{12} = \frac{C_4 \delta}{C_{11}}$. Integrating both sides of (3.30) over $[0, t]$ yields

$$L(t) \geq \left(L(0)^{\frac{\rho}{1-\rho}} - C_{12}\rho t\right)^{-\frac{\rho}{1-\rho}}. \qquad (3.31)$$

Note that $L(0) > 0$, then there exists $T* = T_{\max} = \frac{(1-\rho)L(0)^{\frac{\rho}{1-\rho}}}{C_{12}\rho}$ such that $L(t) \to +\infty$ as $t \to +\infty$. Namely, the solutions of problem (1.1)-(1.5) blow up in finite time. $\qquad \square$

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