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Solutions of biharmonic equations with mixed nonlinearity

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Abstract

In this paper, we study the following biharmonic equations with mixed nonlinearity: $\Delta^2 u - \Delta u + V(x)u = f(x,u) + \lambda\xi(x)|u|^{p-2}u$, $x \in \mathbb{R}^N$, $u \in H^2(\mathbb{R}^N)$, where $V \in C(\mathbb{R}^N)$, $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N)$, $1 \le p < 2$, and $\lambda > 0$ is a parameter. The existence of multiple solutions is obtained via variational methods. Some recent results are improved and extended.

MSC: 35J35; 35J60

Keywords: biharmonic equations; mixed nonlinearity; variational methods

1 Introduction and main result

This paper is concerned with the following biharmonic equations:

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = f(x,u) + \lambda\xi(x)|u|^{p-2}u, & x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1}$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $V \in C(\mathbb{R}^N)$, $f \in C(\mathbb{R}^N \times \mathbb{R})$, $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N)$, $\lambda > 0$, and $1 \le p < 2$. There are many results for biharmonic equations, but most of them are on bounded domains; see [1–5]. In addition, biharmonic equations on unbounded domains also have captured a lot of interest; see [6–11] and the references therein. Many of these papers are devoted to the study of the existence and multiplicity of solutions for problem (1). In [6, 7, 9, 11], the authors considered the superlinear case; one considered the sublinear case in [8–10]. However, there are not many works focused on the asymptotically linear case. Motivated by the above facts, in the present paper, we shall study problem (1) with mixed nonlinearity, that is, a combination of superlinear and sublinear terms, or asymptotically linear and sublinear terms. So, the aim of the present paper is to unify and generalize the results of the above papers to a more general case. To the best of our knowledge, there have been no works concerning this case up to now, hence this is an interesting and new research problem. For related results, we refer the readers to [12–14] and the references therein.

More precisely, we make the following assumptions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > 0$, and there exists a constant $l_0 > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\left(\left\{x \in \mathbb{R}^N : |x-y| \le l_0, V(x) \le M\right\}\right) = 0, \quad \forall M > 0,$$

where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;

(F_1) $f(x,u) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, such that $f(x,u) \equiv 0$ for all $u < 0$ and $x \in \mathbb{R}^N$. Moreover, there exists $b \in L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ with $|b|_\infty < \frac{1}{2\gamma_2^2\gamma_0^2}$ such that

$$\lim_{|u|\to 0^+} \frac{f(x,u)}{u} = b(x) \quad \text{uniformly in } x \in \mathbb{R}^N$$

and

$$\frac{f(x,u)}{u^k} \geq b(x) \quad \text{for all } u > 0 \text{ and } x \in \mathbb{R}^N,$$

where γ_2, γ_0 are defined in (3);

(F_2) there exists $q \in L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ with $|q|_\infty > \frac{1}{\gamma_2^2\gamma_0^2}$ such that

$$\lim_{|u|\to \infty} \frac{f(x,u)}{u^k} = q(x) \quad \text{uniformly in } x \in \mathbb{R}^N;$$

(F_3) there exist two constants θ, d_0 satisfying $\theta > 2$ and $0 \leq d_0 < \frac{\theta-2}{2\theta\gamma_2^2\gamma_0^2}$ such that

$$F(x,u) - \frac{1}{\theta} f(x,u)u \leq d_0 u^2 \quad \text{for all } u > 0 \text{ and } x \in \mathbb{R}^N,$$

where $F(x,u) = \int_0^u f(x,s)\,ds$.

Before stating our result, we denote $\xi^\pm = \max\{\pm\xi, 0\}$. The main result of this paper is the following theorem.

Theorem 1.1 *Suppose that* (V), (F_1)-(F_3) *are satisfied.* $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N) \setminus \{0\}$ *with* $\xi^+ \not\equiv 0$. *In addition, for any real number* $k \geq 1$:

(I_1) *If* $k = 1$ *and* $\mu^* < 1$ *with*

$$\mu^* = \inf\left\{ \int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2\right) dx \,\middle|\, u \in H^2(\mathbb{R}^N), \int_{\mathbb{R}^N} q(x)u^2\,dx = 1 \right\}, \quad (2)$$

then there exists $\Lambda_0 > 0$ *such that, for every* $0 < \lambda < \Lambda_0$, *problem* (1) *has at least two solutions*;

(I_2) *If* $k > 1$, *then there exists* $\Lambda_0 > 0$ *such that, for every* $0 < \lambda < \Lambda_0$, *problem* (1) *has at least two solutions*.

Remark 1.2 It is easy to check that $f(x,u)$ is asymptotically linear at infinity in u when $k = 1$ and $f(x,u)$ is superlinear at infinity in u when $k > 1$. Together with $\lambda > 0$ and $1 \leq q < 2$, we see easily that our nonlinearity is a more general mixed nonlinearity, that is, a combination of sublinear, superlinear, and asymptotically linear terms. Therefore, our result unifies and sharply improves some recent results.

2 Variational setting and proof of the main result

Now we establish the variational setting for our problem (1). Let

$$E = \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2\right) dx < +\infty \right\},$$

equipped with the inner product

$$(u,v)=\int_{\mathbb{R}^N}\big(\Delta u\Delta v+\nabla u\cdot\nabla v+V(x)uv\big)\,dx,\quad u,v\in E,$$

and the norm

$$\|u\|=\left(\int_{\mathbb{R}^N}\big(|\Delta u|^2+|\nabla u|^2+V(x)|u|^2\big)\,dx\right)^{\frac{1}{2}},\quad u\in E.$$

Lemma 2.1 ([15]) *Under assumptions* (V), *the embedding* $E\hookrightarrow L^s(\mathbb{R}^N)$ *is compact for any* $s\in[2,2_*)$, *where* $2_*=\frac{2N}{N-4}$ *if* $N\ge 5$, $2_*=\infty$ *if* $N<5$.

Clearly, E is continuously embedded into $H^2(\mathbb{R}^N)$ and from Lemma 2.1, there exist $\gamma_s>0$ and $\gamma_0>0$ such that

$$\|u\|_s\le\gamma_s\|u\|_{H^2(\mathbb{R}^N)}\le\gamma_s\gamma_0\|u\|,\quad\forall u\in E,2\le s<2_*.\tag{3}$$

Now, on E we define the following functional:

$$\Phi(u)=\frac{1}{2}\int_{\mathbb{R}^N}\big(|\Delta u|^2+|\nabla u|^2+V(x)u^2\big)\,dx-\int_{\mathbb{R}^N}F(x,u)\,dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}\xi(x)|u|^p\,dx.\tag{4}$$

By a standard argument, it is easy to verify that $\Phi\in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle\Phi'(u),v\rangle&=\int_{\mathbb{R}^N}\big[\Delta u\Delta v+\nabla u\cdot\nabla v+V(x)uv\big]\,dx\\&\quad-\int_{\mathbb{R}^N}f(x,u)v\,dx-\lambda\int_{\mathbb{R}^N}\xi(x)|u|^{p-2}uv\,dx\end{aligned}\tag{5}$$

for all $u,v\in E$.

Lemma 2.2 $\mu^*>0$, *and this is achieved by some* $\phi_1\in H^2(\mathbb{R}^N)$ *with* $\int_{\mathbb{R}^N}q\phi_1^2\,dx=1$, *where* μ^* *is given in* (2).

Proof By Lemma 2.1 and standard arguments, it is easy to prove this lemma, so we omit the proof here. □

Next, we give a useful theorem. It is the variant version of the mountain pass theorem, which allows us to find a $(C)_c$ sequence.

Theorem 2.3 ([16]) *Let E be a real Banach space, with dual space* E^*, *and suppose that* $\Phi\in C^1(E,\mathbb{R})$ *satisfies*

$$\max\big\{\Phi(0),\Phi(e)\big\}\le\mu<\eta\le\inf_{\|u\|=\rho}\Phi(u),$$

for some $\mu<\eta$, $\rho>0$ *and* $e\in E$ *with* $\|e\|>\rho$. *Let* $\hat{c}\ge\eta$ *be characterized by*

$$\hat{c}=\inf_{\beta\in\Gamma}\max_{0\le\tau\le1}\Phi\big(\beta(\tau)\big),$$

where $\Gamma = \{\beta \in C([0,1],E) : \beta(0) = 0, \beta(1) = e\}$ *is the set of continuous paths joining* 0 *and* e, *then there exists a sequence* $\{u_n\} \subset E$ *such that*

$$\Phi(u_n) \to \hat{c} \geq \eta \quad \textit{and} \quad \bigl(1 + \|u_n\|\bigr)\bigl\|\Phi'(u_n)\bigr\|_{E^*} \to 0 \quad \textit{as } n \to \infty.$$

Lemma 2.4 *For any real number* $k \geq 1$, *assume that* (F_1) *and* (F_2) *are satisfied, and* $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N) \setminus \{0\}$ *with* $\xi^+ \not\equiv 0$. *Then there exists* $\Lambda_0 > 0$ *such that, for every* $\lambda \in (0, \Lambda_0)$, *there exist two positive constants* ρ, η *such that* $\Phi(u)|_{\|u\|=\rho} \geq \eta > 0$.

Proof For any $\varepsilon > 0$, it follows from the conditions (F_1) and (F_2) that there exist $C_\varepsilon > 0$ and $\max\{2, k\} < r < 2_*$ such that

$$F(x,u) \leq \frac{|b|_\infty + \varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{r}|u|^r, \quad \text{for all } u \in E. \tag{6}$$

Thus, from (3), (6), and the Sobolev inequality, we have, for all $u \in E$,

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u)\,dx &\leq \frac{|b|_\infty + \varepsilon}{2}\int_{\mathbb{R}^N} u^2\,dx + \frac{C_\varepsilon}{r}\int_{\mathbb{R}^N} |u|^r\,dx \\ &\leq \frac{(|b|_\infty + \varepsilon)\gamma_2^2\gamma_0^2}{2}\|u\|^2 + \frac{C_\varepsilon \gamma_r^r \gamma_0^r}{r}\|u\|^r,\end{aligned}$$

which implies that

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} \xi(x)|u|^p\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{(|b|_\infty + \varepsilon)\gamma_2^2\gamma_0^2}{2}\|u\|^2 - \frac{C_\varepsilon\gamma_r^r\gamma_0^r}{r}\|u\|^r - \frac{\lambda\gamma_2^p\gamma_0^p}{p}\|\xi\|_{\frac{2}{2-p}}\|u\|^p \\ &= \|u\|^p\biggl[\frac{1}{2}\bigl(1 - \bigl(|b|_\infty + \varepsilon\bigr)\gamma_2^2\gamma_0^2\bigr)\|u\|^{2-p} - \frac{C_\varepsilon\gamma_r^r\gamma_0^r}{r}\|u\|^{r-p} - \frac{\lambda\gamma_2^p\gamma_0^p}{p}\|\xi\|_{\frac{2}{2-p}}\biggr].\end{aligned} \tag{7}$$

Take $\varepsilon = \frac{1}{2\gamma_2^2\gamma_0^2} - |b|_\infty$ and define

$$g(t) = \frac{1}{4}t^{2-p} - \frac{C_\varepsilon\gamma_r^r\gamma_0^r}{r}t^{t-p}, \quad \text{for } t \geq 0.$$

It is easy to prove that there exists $\rho > 0$ such that

$$\max_{t\geq 0} g(t) = g(\rho) = \frac{r-2}{4(r-p)}\biggl[\frac{(2-p)r}{4C_\varepsilon\gamma_r^r\gamma_0^r(r-p)}\biggr]^{\frac{2-p}{r-2}}.$$

Then it follows from (7) that there exists $\Lambda_0 > 0$ such that, for every $\lambda \in (0, \Lambda_0)$, there exists $\eta > 0$ such that $\Phi(u)|_{\|u\|=\rho} \geq \eta$. □

Lemma 2.5 *For any real number* $k \geq 1$, *assume that* (F_1), (F_2) *are satisfied, and* $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N) \setminus \{0\}$ *with* $\xi^+ \not\equiv 0$. *Let* $\rho, \Lambda_0 > 0$ *be as in Lemma* 2.4. *Then we have the following results*:

(i) *If* $k = 1$ *and* $\mu^* < 1$, *then there exists* $e \in E$ *with* $\|e\| > \rho$ *such that* $\Phi(e) < 0$ *for all* $\lambda \in (0, \Lambda_0)$;

(ii) *if* $k > 1$, *then there exists* $e \in E$ *with* $\|e\| > \rho$ *such that* $\Phi(e) < 0$ *for all* $\lambda \in (0, \Lambda_0)$.

Proof (i) In the case $k = 1$, since $\mu^* < 1$, we can choose a nonnegative function $\varphi \in E$ with

$$\int_{\mathbb{R}^N} q(x)\varphi^2\,dx = 1 \quad \text{such that} \quad \int_{\mathbb{R}^N} \bigl(|\Delta\varphi|^2 + |\nabla\varphi|^2 + V(x)|\varphi|^2\bigr)\,dx < 1.$$

Therefore, from (F_2) and Fatou's lemma, we have

$$\begin{aligned}\lim_{t\to+\infty}\frac{\Phi(t\varphi)}{t^2} &= \frac{1}{2}\|\varphi\|^2 - \lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,t\varphi)}{t^2\varphi^2}\varphi^2\,dx - \lim_{t\to+\infty}\frac{\lambda}{pt^{2-p}}\int_{\mathbb{R}^N}\xi(x)|\varphi|^p\,dx \\ &\le \frac{1}{2}\|\varphi\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} q(x)\varphi^2\,dx = \frac{1}{2}\bigl(\|\varphi\|^2 - 1\bigr) < 0.\end{aligned}$$

So, if $\Phi(t\varphi) \to -\infty$ as $t \to +\infty$, then there exists $e \in E$ with $\|e\| > \rho$ such that $\Phi(e) < 0$.

(ii) In the case $k > 1$, since $q \in L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ with $q^+ \not\equiv 0$, we can choose a nonnegative function $\omega \in E$ such that $\int_{\mathbb{R}^N} q(x)\omega^{k+1}\,dx > 0$. Thus, from (F_2) and Fatou's lemma, we have

$$\begin{aligned}\lim_{t\to+\infty}\frac{\Phi(t\omega)}{t^{k+1}} &= \lim_{t\to+\infty}\frac{\|\omega\|^2}{2t^{k-1}} - \lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(x,t\omega)}{t^{k+1}\omega^{k+1}}\omega^{k+1}\,dx \\ &\quad - \lim_{t\to+\infty}\frac{\lambda}{pt^{k+1-p}}\int_{\mathbb{R}^N}\xi(x)|\omega|^p\,dx \\ &\le -\frac{1}{k+1}\int_{\mathbb{R}^N} q(x)\omega^{k+1}\,dx < 0.\end{aligned}$$

So, if $\Phi(t\omega) \to -\infty$ as $t \to +\infty$, then there exists $e \in E$ with $\|e\| > \rho$ such that $\Phi(e) < 0$. This completes the proof. □

Next, we define

$$\alpha = \inf_{\beta\in\Gamma}\max_{0\le t\le 1}\Phi\bigl(\beta(t)\bigr),$$

where $\Gamma = \{\beta \in C([0,1],E) : \beta(0) = 0, \beta(1) = e\}$. Then by Theorem 2.3 and Lemmas 2.4 and 2.5, there exists a sequence $\{u_n\} \subset E$ such that

$$\Phi(u_n) \to \alpha > 0 \quad \text{and} \quad \bigl(1 + \|u_n\|\bigr)\bigl\|\Phi'(u_n)\bigr\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{8}$$

Lemma 2.6 *For any real number $k \ge 1$, assume that* (V) *and* (F_1)-(F_3) *are satisfied, and* $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N) \setminus \{0\}$ *with* $\xi^+ \not\equiv 0$. *Let* $\Lambda_0 > 0$ *be as in Lemma* 2.4. *Then* $\{u_n\}$ *defined by* (8) *is bounded in* E *for all* $\lambda \in (0, \Lambda_0)$.

Proof For n large enough, from (F_2), (3), the Hölder inequality, and Lemma 2.4, we have

$$\begin{aligned}\alpha + 1 &\ge \Phi(u_n) - \frac{1}{\theta}\bigl\langle\Phi'(u_n), u_n\bigr\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \int_{\mathbb{R}^N}\left[F(x,u_n) - \frac{1}{\theta}f(x,u_n)u_n\right]dx \\ &\quad - \lambda\left(\frac{1}{p} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N}\xi(x)|u_n|^p\,dx \\ &\ge \frac{\theta-2}{2\theta}\|u_n\|^2 - d_0\int_{\mathbb{R}^N} u_n^2\,dx - \frac{\lambda(\theta-p)}{p\theta}\|\xi\|_{\frac{2}{2-p}}\|u_n\|_2^p\end{aligned}$$

$$\geq \frac{\theta-2}{2\theta}\|u_n\|^2 - d_0\gamma_2^2\gamma_0^2\|u_n\|^2 - \frac{\lambda(\theta-p)\gamma_2^p\gamma_0^p}{p\theta}\|\xi\|_{\frac{2}{2-p}}\|u_n\|^p$$
$$> \left(\frac{\theta-2}{2\theta} - d_0\gamma_2^2\gamma_0^2\right)\|u_n\|^2 - \frac{\lambda(\theta-p)\gamma_2^p\gamma_0^p}{p\theta}\|\xi\|_{\frac{2}{2-p}}\|u_n\|^p,$$

which implies that $\{u_n\}$ is bounded in E since $1 \leq p < 2$. □

Lemma 2.7 *For any real number $k \geq 1$, assume that* (V) *and* (F_1)-(F_2) *are satisfied, and $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N) \setminus \{0\}$ with $\xi^+ \not\equiv 0$. Let $\Lambda_0 > 0$ be as in Lemma* 2.4. *Then for every $\lambda \in (0, \Lambda_0)$, there exists $u_0 \in E$ such that*

$$\Phi(u_0) = \inf\{\Phi(u) : u \in \bar{B}_\rho\} < 0$$

and u_0 is a nontrivial solution of problem (1).

Proof Since $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N) \setminus \{0\}$ with $\xi^+ \not\equiv 0$, we can choose a function $\phi \in E$ such that

$$\int_{\mathbb{R}^N} \xi(x)|\phi|^p\,dx > 0. \tag{9}$$

By (9), for $t > 0$, we have

$$\Phi(t\phi) = \frac{t^2}{2}\|\phi\|^2 - \int_{\mathbb{R}^N} F(x, t\phi)\,dx - \frac{\lambda t^p}{p}\int_{\mathbb{R}^N} \xi(x)|\phi|^p\,dx$$
$$\leq \frac{t^2}{2}\|\phi\|^2 - \frac{\lambda t^p}{p}\int_{\mathbb{R}^N} \xi(x)|\phi|^p\,dx, \quad \text{for } t > 0 \text{ small enough.}$$

Hence, $\theta_0 := \inf\{\Phi(u) : u \in \bar{B}_\rho\} < 0$. By Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset \bar{B}_\rho$ such that $\Phi(u_n) \to \theta_0$ and $\Phi'(u_n) \to 0$ as $n \to \infty$. Hence, Lemma 2.1 implies that there exists $u_0 \in E$ such that $\Phi'(u_0) = 0$ and $\Phi(u_0) = \theta_0 < 0$. □

Proof of Theorem 1.1 From Lemmas 2.1 and 2.6, there exists a constant $\tilde{u} \in E$ such that, up to a subsequence,

$$u_n \rightharpoonup \tilde{u} \quad \text{in } E, \qquad u_n \to \tilde{u} \quad \text{in } L^s(\mathbb{R}^N) \text{ for } s \in [2, 2_*).$$

By using a standard procedure, we can prove that $u_n \to \tilde{u}$ in E. Moreover, $\Phi(\tilde{u}) = \alpha > 0$ and $\tilde{u}$ is another nontrivial solution of problem (1). Therefore, combining with Lemma 2.7, we can prove that problem (1) has at least two nontrivial solutions $u_0, \tilde{u} \in E$ satisfying $\Phi(u_0) < 0$ and $\Phi(\tilde{u}) > 0$. □

Competing interests

The author declares that they have no competing interests.

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