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Ground state solution and multiple solutions to asymptotically linear Schrödinger equations

Xiang-Dong Fang[1,2*] and Zhi-Qing Han[1*]

*Correspondence: fangxd0401@gmail.com; hanzhiq@dlut.edu.cn
[1] School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, P.R. China
[2] State Key Laboratory of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics, Dalian University of Technology, Dalian, 116024, P.R. China

Abstract
In this paper, we consider the Schrödinger equation $-\Delta u + V(x)u = f(x,u)$, $x \in \mathbb{R}^N$, where V and f are periodic in $x_1, \ldots, x_N$, asymptotically linear and satisfies a monotonicity condition. We use the generalized Nehari manifold methods to obtain a ground state solution and infinitely many geometrically distinct solutions when f is odd in u.

Keywords: Schrödinger equation; ground state solution; multiplicity of solutions; asymptotically linear

1 Introduction

We consider the problem

$$-\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where f and V are periodic in $x_1, \ldots, x_N$, asymptotically linear and satisfies a monotonicity condition. In the case that the nonlinear term is asymptotically linear at infinity, there are some results in the literature [1–12] and the references therein, where multiplicity results are considered in [1–3, 9, 10, 12]. As far as we know, there are only a few papers concerned with the existence of infinitely many solutions for the asymptotical linear case when f and V are also periodic in $x_1, \ldots, x_N$; *e.g.* see [2]. Except for [5], there seem to be few results on the existence of a ground state solution in the asymptotically linear case. Motivated by [13], this paper is to present a different approach involving the critical point theory with the discreteness property of the Palais-Smale in search for a ground state solution and multiple solutions for the asymptotically linear Schrödinger equations. It should be pointed out that in [2], they cannot make sure the existence of a ground state solution. Our results can be regarded as complements or different attempts of the results in [2, 5].

Setting $F(x,u) := \int_0^u f(x,s)\,ds$, we suppose that V and f satisfy the following assumptions:

(V) V is continuous, 1-periodic in x_i, $1 \le i \le N$, and there exists a constant $a_0 > 0$ such that $V(x) \ge a_0$ for all $x \in \mathbb{R}^N$.
(f_1) f is continuous, 1-periodic in x_i, $1 \le i \le N$.
(f_2) $f(x,u) = o(u)$ as $u \to 0$, uniformly in x.

(f_3) There is $q(x) > V(x)$, $\forall x \in \mathbb{R}^N$, such that $f(x,u)/u \to q(x)$, as $|u| \to \infty$, where q is continuous, 1-periodic in x_i, $1 \leq i \leq N$.
(f_4) $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.

Let $*$ denote the action of $\mathbb{Z}^N$ on $H^1(\mathbb{R}^N)$ given by

$$(k * u)(x) := u(x-k), \quad k \in \mathbb{Z}^N. \tag{1.2}$$

It follows from (V) and (f_1) that if u_0 is a solution of (1.1), then so is $k * u_0$ for all $k \in \mathbb{Z}^N$. Set

$$\mathcal{O}(u_0) := \{k * u_0 : k \in \mathbb{Z}^N\}.$$

$\mathcal{O}(u_0)$ is called *the orbit* of u_0 with respect to the action of $\mathbb{Z}^N$, and it is called *a critical orbit* for a functional F if u_0 is a critical point of F and F is $\mathbb{Z}^N$*-invariant*, *i.e.*, $F(k*u) = F(u)$ for all $k \in \mathbb{Z}^N$ and all u (then of course all points of $\mathcal{O}(u_0)$ are critical). Two solutions u_1, u_2 of (1.1) are said to be *geometrically distinct* if $\mathcal{O}(u_1) \neq \mathcal{O}(u_2)$.

Theorem 1.1 *Suppose that* (V), (f_1)-(f_4) *are satisfied. Then* (1.1) *has a ground state solution. In addition, if f is odd in u, then* (1.1) *admits infinitely many pairs $\pm u$ of geometrically distinct solutions.*

Notation $C, C_1, C_2, \ldots$ will denote different positive constants whose exact value is inessential. The usual norm in the Lebesgue space $L^p(\Omega)$ is denoted by $\|u\|_{p,\Omega}$, and by $\|u\|_p$ if $\Omega = \mathbb{R}^N$. E denotes the Sobolev space $H^1(\mathbb{R}^N)$ and S is the unit sphere in E. It follows from (V) that

$$\|u\| := \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 \right) \right)^{1/2}$$

is an equivalent norm in E. It is more convenient for our purposes than the standard one and will be used henceforth. For a functional I, as in [14], we put

$$I^d := \{u : I(u) \leq d\}, \qquad I_c := \{u : I(u) \geq c\}, \qquad I_c^d := \{u : c \leq I(u) \leq d\}.$$

2 Preliminary results

Consider the functional

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} F(x,u). \tag{2.1}$$

Then I is well defined on E and $I \in C^1(E, \mathbb{R})$ under the hypotheses (V), (f_1)-(f_3). Note also that (V), (f_1) imply I is invariant with respect to the action of $\mathbb{Z}^N$ given by (1.2). It is easy to see that

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + \int_{\mathbb{R}^N} V(x)uv - \int_{\mathbb{R}^N} f(x,u)v \tag{2.2}$$

for all $u, v \in E$.

Let

$$\mathcal{M} := \{u \in E \setminus \{0\} : \langle I'(u), u \rangle = 0\}. \tag{2.3}$$

Recall that $\mathcal{M}$ is called the Nehari manifold. We do not know whether $\mathcal{M}$ is of class C^1 under our assumptions and therefore we cannot use minimax theory directly on $\mathcal{M}$. To overcome this difficulty, we employ the arguments developed in [13, 15, 16].

We assume that (V) and (f_1)-(f_4) are satisfied from now on. First, (f_2) and (f_3) imply that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1} \quad \text{for all } u \in \mathbb{R}, \tag{2.4}$$

where $2 < p < 2^*$, $2^* := 2N/(N-2)$ if $N \ge 3$, $2^* := \infty$ if $N = 1$ or 2.

For $t > 0$, let

$$h(t) := I(tu) = \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \int_{\mathbb{R}^N} F(x,tu).$$

Let

$$\mathcal{E} := \left\{u \in E : \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 < \int_{\mathbb{R}^N} q(x)u^2\right\}.$$

It follows from $q(x) - V(x) > 0$, $\forall x \in \mathbb{R}^N$, that $\mathcal{E} \neq \emptyset$.

Lemma 2.1 *$F(x,u) > 0$ and $\frac{1}{2}f(x,u)u > F(x,u)$ if $u \neq 0$.*

This follows immediately from (f_2) and (f_4).

Lemma 2.2
(1) *For each $u \in \mathcal{E}$ there is a unique $t_u > 0$ such that $h'(t) > 0$ for $0 < t < t_u$ and $h'(t) < 0$ for $t > t_u$. Moreover, $tu \in \mathcal{M}$ if and only if $t = t_u$.*
(2) *If $u \notin \mathcal{E}$, then $tu \notin \mathcal{M}$ for any $t > 0$.*

Proof (1) For each $u \in \mathcal{E}$, due to the Lebesgue dominated convergence theorem and (f_2), (f_3), we get

$$\begin{aligned}\lim_{t\to\infty} \frac{I(tu)}{t^2} &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \lim_{t\to\infty}\int_{u\neq 0} \frac{F(x,tu)}{t^2u^2}u^2 \\ &= \frac{1}{2}\left[\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \int_{\mathbb{R}^N} q(x)u^2\right] < 0\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0} \frac{I(tu)}{t^2} &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \lim_{t\to 0}\int_{u\neq 0} \frac{F(x,tu)}{t^2u^2}u^2 \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 > 0.\end{aligned}$$

Hence h has a positive maximum. The condition $h'(t) = 0$ is equivalent to

$$\|u\|^2 = \int_{u\neq 0} \frac{f(x,tu)}{tu} u^2.$$

By (f_4), the first conclusion holds. The second conclusion follows from $h'(t) = t^{-1}\langle I'(tu), tu\rangle$.

(2) If $tu \in \mathcal{M}$ for some $t > 0$, then $\langle I'(tu), u\rangle = 0$ and therefore using (f_3) and (f_4)

$$\|u\|^2 = \int_{u\neq 0} \frac{f(x,tu)}{tu} u^2 < \int_{\mathbb{R}^N} q(x)u^2.$$

Hence $u \in \mathcal{E}$. □

Lemma 2.3

(1) *There exists* $\rho > 0$ *such that* $c := \inf_{\mathcal{M}} I \geq \inf_{S_\rho} I > 0$.

(2) $\|u\|^2 \geq 2c$ *for all* $u \in \mathcal{M}$.

Proof (1) Using (2.4) and the Sobolev inequality we have $\inf_{S_\rho} I > 0$ if ρ is small enough. The inequality $\inf_{\mathcal{M}} I \geq \inf_{S_\rho} I$ is a consequence of Lemma 2.2 since for every $u \in \mathcal{M}$ there is $s > 0$ such that $su \in S_\rho$ (and $I(t_u u) \geq I(su)$).

(2) For $u \in \mathcal{M}$, by Lemma 2.1 we have

$$c \leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u) \leq \frac{1}{2}\|u\|^2.$$ □

We do not know whether I is coercive on $\mathcal{M}$. However, we can prove the following.

Lemma 2.4 *All Palais-Smale sequences* $(u_n) \subset \mathcal{M}$ *are bounded.*

Proof Arguing by contradiction, suppose there exists a sequence $(u_n) \subset \mathcal{M}$ such that $\|u_n\| \to \infty$ and $I(u_n) \leq d$ for some $d \in [c,\infty)$. Let $v_n := u_n/\|u_n\|$. Then $v_n \rightharpoonup v$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ after passing to a subsequence. Choose $y_n \in \mathbb{R}^N$ so that

$$\int_{B_1(y_n)} v_n^2 = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} v_n^2. \tag{2.5}$$

Since I and $\mathcal{M}$ are invariant with respect to the action of $\mathbb{Z}^N$ given by (1.2), we may assume that (y_n) is bounded in $\mathbb{R}^N$. If

$$\int_{B_1(y_n)} v_n^2 \to 0 \quad \text{as } n \to \infty, \tag{2.6}$$

then it follows that $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*$ by Lions' lemma (*cf.* [17], Lemma 1.21), and therefore (2.4) implies that $\int_{\mathbb{R}^N} F(x, sv_n) \to 0$ for every $s \in \mathbb{R}$. Lemma 2.2 implies that

$$d \geq I(u_n) \geq I(sv_n) = \frac{s^2}{2} - \int_{\mathbb{R}^N} F(x,sv_n) \to \frac{s^2}{2}.$$

Taking a sufficiently large s, we get a contradiction. Hence (2.6) cannot hold and, since $v_n \to v$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, $v \neq 0$. Hence $|u_n(x)| \to \infty$ if $v(x) \neq 0$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then $\langle I'(u_n), \varphi\rangle \to 0$ and hence

$$\int_{\mathbb{R}^N} \nabla v_n \nabla\varphi + V(x) v_n \varphi - \int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} v_n \varphi \to 0.$$

By the Lebesgue dominated convergence theorem we therefore have

$$\int_{\mathbb{R}^N} \nabla v \nabla\varphi + V(x) v\varphi = \int_{\mathbb{R}^N} q(x) v\varphi.$$

So $v \neq 0$ and $-\Delta v + V(x)v = q(x)v$. This is impossible because $-\Delta + V - q$ has only an absolutely continuous spectrum. The proof is complete. □

Lemma 2.5 *If $\mathcal{V}$ is a compact subset of $\mathcal{E}$, then there exists $R > 0$ such that $I \leq 0$ on $(\mathbb{R}^+\mathcal{V}) \setminus B_R(0)$.*

Proof We may assume without loss of generality that $\mathcal{V} \subset S$. Arguing by contradiction, suppose there exist $u_n \in \mathcal{V}$ and $w_n = t_n u_n$, where $u_n \to u$, $t_n \to \infty$ and $I(w_n) \geq 0$. We have

$$\begin{aligned} 0 \leq \frac{I(t_n u_n)}{t_n^2} &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)u_n^2 - \int_{u_n \neq 0} \frac{F(x,t_n u_n)}{t_n^2 u_n^2} u_n^2 \\ &\to \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \frac{1}{2}\int_{\mathbb{R}^N} q(x)u^2 < 0. \end{aligned}$$

□

Let $U := \mathcal{E} \cap S$ and define the mapping $m : U \to \mathcal{M}$ by setting

$$m(w) := t_w w,$$

where t_w is as in Lemma 2.2.

Lemma 2.6 *U is an open subset of S.*

Proof Obvious because $\mathcal{E}$ is open in E. □

Lemma 2.7 *Assume $u_n \in U$, $u_n \to u_0 \in \partial U$, and $t_n u_n \in \mathcal{M}$, then $I(t_n u_n) \to \infty$.*

Proof Since $u_0 \in \partial U$, $\int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 = \int_{\mathbb{R}^N} q(x)u_0^2$. Using this, we have

$$\begin{aligned} I(tu_0) &= \frac{1}{2}t^2 \int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 - t^2 \int_{\mathbb{R}^N} \frac{F(x,tu_0)}{t^2 u_0^2} u_0^2 \\ &= \frac{1}{2}t^2 \int_{\mathbb{R}^N} \left(q(x) - \frac{2F(x,tu_0)}{t^2 u_0^2}\right) u_0^2 \\ &= \frac{1}{2}t^2 \int_{\mathbb{R}^N} \left(q(x) - \frac{f(x,tu_0)}{tu_0}\right) u_0^2 \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{2} f(x,tu_0)tu_0 - F(x,tu_0). \end{aligned}$$

Note that by (f_4), we have for large enough s, there is $\delta > 0$ such that

$$\frac{1}{2} f(x,s)s - F(x,s) \geq \delta$$

(see [4], Remark 1.5). So $I(tu_0) \to \infty$, as $t \to \infty$ (we have used Fatou's lemma). Given $C > 0$, choose $t > 0$ such that $I(tu_0) \geq C$. Since $u_n \to u_0$,

$$\lim_{n\to\infty} I(t_n u_n) \geq \lim_{n\to\infty} I(tu_n) = I(tu_0) \geq C$$

and hence $I(t_n u_n) \to \infty$. □

The following lemmas are taken from [13, 15].

Below we shall use the notations

$$K := \{w \in S : \Psi'(w) = 0\},$$
$$K_d := \{w \in K : \Psi(w) = d\}.$$

Since f is odd in u, we can choose a subset $\mathcal{F}$ of K such that $\mathcal{F} = -\mathcal{F}$ and each orbit $\mathcal{O}(w) \subset K$ has a unique representative in $\mathcal{F}$. We must show that the set $\mathcal{F}$ is infinite. Arguing indirectly, assume

$$\mathcal{F} \text{ is a finite set.} \tag{2.7}$$

Lemma 2.8 *The mapping m is a homeomorphism between U and $\mathcal{M}$, and the inverse of m is given by $m^{-1}(u) = \frac{u}{\|u\|}$.*

We consider the functional $\Psi : U \to \mathbb{R}$ given by

$$\Psi(w) := I(m(w)).$$

Lemma 2.9

(1) $\Psi \in C^1(U, \mathbb{R})$ *and*

$$\langle \Psi'(w), z\rangle = \|m(w)\| \langle I'(m(w)), z\rangle \quad \textit{for all } z \in T_w(U).$$

(2) *If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for I. If $(u_n) \subset \mathcal{M}$ is a bounded Palais-Smale sequence for I, then $(m^{-1}(u_n))$ is a Palais-Smale sequence for Ψ.*

(3) *w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of I. Moreover, the corresponding values of Ψ and I coincide and* $\inf_U \Psi = \inf_{\mathcal{M}} I$.

(4) *Ψ is even (because I is).*

By (2.4), the following lemma also holds.

Lemma 2.10 *Let $d \geq c$. If $(v_n^1), (v_n^2) \subset \Psi^d$ are two Palais-Smale sequences for Ψ, then either $\|v_n^1 - v_n^2\| \to 0$ as $n \to \infty$ or $\limsup_{n\to\infty} \|v_n^1 - v_n^2\| \geq \rho(d) > 0$, where $\rho(d)$ depends on d but not on the particular choice of Palais-Smale sequences.*

It is well known that Ψ admits a pseudo-gradient vector field $H : U \setminus K \to TU$ (see *e.g.* [18], p.86). Moreover, since Ψ is even, we may assume H is odd. Let $\eta : \mathcal{G} \to U \setminus K$ be the flow defined by

$$\begin{cases} \frac{d}{dt}\eta(t,w) = -H(\eta(t,w)), \\ \eta(0,w) = w, \end{cases} \tag{2.8}$$

where

$$\mathcal{G} := \{(t,w) : w \in U \setminus K, T^-(w) < t < T^+(w)\}$$

and $(T^-(w), T^+(w))$ is the maximal existence time for the trajectory $t \mapsto \eta(t,w)$. Note that η is odd in w because H is and $t \mapsto \Psi(\eta(t,w))$ is strictly decreasing by the properties of a pseudogradient.

Let $P \subset U$, $\delta > 0$ and define $U_\delta(P) := \{w \in U : \operatorname{dist}(w,P) < \delta\}$.

Lemma 2.11 *Let $d \geq c$. Then for every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that*
(a) $\Psi_{d-\varepsilon}^{d+\varepsilon} \cap K = K_d$ *and*
(b) $\lim_{t \to T^+(w)} \Psi(\eta(t,w)) < d - \varepsilon$ *for* $w \in \Psi^{d+\varepsilon} \setminus U_\delta(K_d)$.

Part (a) is an immediate consequence of (2.7) and (b) has been proved in [15]; see Lemmas 2.15 and 2.16 there. The argument is exactly the same except that S should be replaced by U. We point out that an important role in the proof of Lemma 2.11 is played by the discreteness property of the Palais-Smale sequences expressed in Lemma 2.10.

3 Proof of Theorem 1.1

Proof of Theorem 1.1 Taking a similar argument as in the proof of Theorem 1.1 in [15], it is easy to get a ground state solution. Noting that by Lemma 2.7 and Ekeland's variational principle, it can make sure the existence of a $(PS)_c$ sequence belonging to U.

For the multiplicity the argument is the same as in Theorem 1.2 (*cf.* [15]). However, there are details which need to be clarified.

Let η be the flow given by (2.8). If $T^+(w) < \infty$, then $\lim_{t \to T^+(w)} \eta(t,w)$ exists (*cf.* [15], Lemma 2.15, Case 1) but unlike the situation in [15], this limit may be a point $w_0 \in \partial U$. This possibility is ruled out by Lemma 2.7.

Finally, we need to show that U contains sets of arbitrarily large genus. Since the spectrum of $-\Delta + V - q$ in $L^2(\mathbb{R}^N)$ is absolutely continuous, $\mathcal{E} \cup \{0\}$ contains an infinite-dimensional subspace E_0. Hence $E_0 \cap S \subset U$ and $\gamma(E_0 \cap S) = \infty$. □

Remark 3.1 There is a small gap in the proof of Theorem 1.2 in [13]. Lemma 4.6 as stated there does not exclude the possibility of $\eta(t,w)$ approaching the boundary as $t \to T^+(w)$ (because we only know that $\eta(t,w)$ goes to infinity). But it is easy to prove that $I(\eta(t,w))$ goes to infinity as well in [13]. In Lemma 2.7 of this paper we make some proper modifications which also apply to [13] and were proposed by Andrzej Szulkin.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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