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A posteriori error estimates for continuous interior penalty Galerkin approximation of transient convection diffusion optimal control problems

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Abstract

In this paper a posteriori error estimate for continuous interior penalty Galerkin approximation of transient convection dominated diffusion optimal control problems with control constraints is presented. The state equation is discretized by the continuous interior penalty Galerkin method with continuous piecewise linear polynomial space and the control variable is approximated by implicit discretization concept. By use of the elliptic reconstruction technique proposed for parabolic equations, a posteriori error estimates for state variable, adjoint state variable and control variable are proved, which can be used to guide the mesh refinement in the adaptive algorithm.

Keywords: transient convection diffusion optimal control problem; continuous interior penalty Galerkin method; elliptic reconstruction; a posteriori error estimate

1 Introduction

Transient convection diffusion optimal control problems are widely used to model some engineering problems, for example, air pollution problem [1, 2] and waste water treatment [3]. In recent years the numerical approximations of this kind of problems form a hot topic, and many works are contributed to developing effective numerical methods and algorithms. For stabilization methods, we refer to [4–7] and for discontinuous Galerkin methods, we refer to [8, 9]. For more literature, one can refer to the references cited therein.

It is well known that the solutions to convection diffusion problems may have boundary layers with small widths where their gradients change rapidly. Therefore, only using the stable methods to solve convection diffusion optimal control problems is generally not enough. One approach to improve the quality of a numerical solution is to exploit special mesh which is locally refined near the boundary layers, for example, Shishkin-type mesh or adaptive mesh. Note that a priori knowledge of the locations of the boundary layers is necessary to construct Shishkin-type mesh. Using adaptive mesh to resolve the boundary layers seems to be more natural. As we know the key problem of the adaptive finite element method is the a posteriori error estimate. Compared with a posteriori error estimates for stationary convection diffusion optimal control problems (see, [7, 10–12]), the works devoted to a posteriori error estimates for transient convection diffusion optimal control problems are much fewer. In [13] the authors discuss adaptive characteristic finite element approximation of transient convection diffusion optimal control problems with a general diffusion coefficient, where a posteriori error estimates in $L^2(0,T;L^2(\Omega))$ norm are derived by dual argument skill for the state and adjoint state variables.

The primary interest of this paper is to derive a posteriori error estimates for the following transient convection diffusion optimal control problem with dominance convection:

$$\min_{u\in U_{ad}} J(y,u) = \frac{1}{2}\int_{\Omega_T}\big(y(\mathbf{x},t)-y_d(\mathbf{x},t)\big)^2\,dx\,dt + \frac{\gamma}{2}\int_{\Omega_T} u^2(\mathbf{x},t)\,dx\,dt \tag{1.1}$$

subject to

$$\begin{cases} y_t + \boldsymbol{\beta}\cdot\nabla y + \alpha y - \varepsilon\triangle y = f + u, & (\mathbf{x},t)\in\Omega_T = \Omega\times(0,T),\\ y(\mathbf{x},t) = 0, & (\mathbf{x},t)\in\Gamma_T = \partial\Omega\times(0,T),\\ y(\mathbf{x},0) = y_0(\mathbf{x}), & (\mathbf{x},t)\in\Omega. \end{cases} \tag{1.2}$$

The details will be specified in the next section.

In order to improve the quality of the numerical solutions, the continuous interior penalty Galerkin method (CIP Galerkin method) is used to solve the state equation (1.2). This method was firstly proposed in [14]. In [7, 15] the CIP Galerkin method was used to approximate stationary convection diffusion optimal control problems, where a posteriori error estimates in $L^2(\Omega)$ and energy norm were derived. In [16] the CIP Galerkin method combined with Crank-Nicolson scheme was used to solve transient convection diffusion optimal control problems without constraints and a priori error estimates were deduced.

In the present paper, we apply the CIP Galerkin method combined with the backward Euler method to solve control constrained transient convection diffusion optimal control problems (1.1)-(1.2), where the control is discretized by the implicit discretization method developed in [17], and the state is approximated by piecewise linear finite element space. Due to the existence of boundary layer or interior layer for the state and adjoint state as well as limited regularity of control variable, we derive a posteriori error estimates for the state and adjoint state, which can be utilized to guide the mesh refinements in the adaptive algorithm. In contrast to [13], here we use the elliptic reconstruction technique developed in [18] for parabolic problems instead of dual argument skill to deduce the a posterior error estimates for the state and adjoint state. By use of this technique we can take full advantage of the well-established a posteriori error estimates for stationary convection diffusion optimal control problems in [7, 15] to derive the a posterior error estimate for transient convection diffusion optimal control problems.

The paper is organized as follows. In Section 2 we describe the continuous interior penalty Galerkin scheme for the constrained optimal control problem. In Section 3 a posteriori error estimates are derived. Finally, we briefly summarize the method used, results obtained and possible future extensions and challenges.

Throughout this paper $C > 0$ denotes a generic constant independent of mesh parameters and may be different at different occurrence. We use the expression $a \preceq b$ to stand for $a \leq Cb$.

2 The CIP Galerkin approximation scheme

2.1 Problems formulation

Consider the following transient convection diffusion optimal control problems:

$$\min_{u\in U_{ad}} J(y,u) \tag{2.1}$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} + \boldsymbol{\beta}\cdot\nabla y + \alpha y - \varepsilon\triangle y = f + u, & (\mathbf{x},t)\in\Omega_T, \\ y(\mathbf{x},t) = 0, & (\mathbf{x},t)\in\Gamma_T, \\ y(\mathbf{x},0) = y_0(\mathbf{x}), & \mathbf{x}\in\Omega. \end{cases} \tag{2.2}$$

Here Ω is a bounded domain in R^2 with boundary $\partial\Omega$. $f\in L^2(\Omega_T)$ and $y_0(\mathbf{x})\in H_0^1(\Omega)$ is the initial value. $U_{ad} = \{u\in L^2(\Omega_T) : a \le u(\mathbf{x},t) \le b \text{ a.e. in } \Omega_T\}$ is a bounded convex set with two constants satisfying $a < b$. $\alpha > 0$ is the reaction coefficient, $0 < \varepsilon \ll 1$ is a small diffusion coefficient, and $\boldsymbol{\beta}\in (W^{1,\infty}(\Omega))^2$ is a velocity field. We assume that the following coercivity condition holds:

$$\alpha - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} \ge \alpha_0 > 0.$$

To consider the CIP Galerkin approximation of the above optimal control problem, we first derive a weak formulation for the state equation. Let $A(\cdot,\cdot)$ be the bilinear form given by

$$A(y,w) = (\varepsilon\nabla y,\nabla w) + (\boldsymbol{\beta}\cdot\nabla y, w) + (\alpha y, w), \quad \forall y,w\in H_0^1(\Omega). \tag{2.3}$$

It is easy to check

$$A(y,y) \ge \|y\|_*^2, \tag{2.4}$$

where

$$\|y\|_* = \left(\varepsilon\|\nabla y\|_{0,\Omega}^2 + \alpha_0\|y\|_{0,\Omega}^2\right)^{1/2}.$$

Then the weak formulation of state equation (2.2) reads as

$$\left(\frac{\partial y}{\partial t}, w\right) + A(y,w) = (f+u,w), \quad \forall w\in H_0^1(\Omega).$$

The variational formulation of optimal control problem (2.1)-(2.2) then can be written as

$$\min_{u\in U_{ad}} J(y,u) \tag{2.5}$$

subject to

$$\left(\frac{\partial y}{\partial t}, w\right) + A(y,w) = (f+u,w), \quad \forall w\in H_0^1(\Omega). \tag{2.6}$$

The existence and uniqueness of solutions to (2.5)-(2.6) can be guaranteed by the theory in [19]. Moreover, by using the Lagrange functional, the first-order necessary (also sufficient here) optimality condition of (2.5)-(2.6) can be characterized by

$$\begin{cases} (\frac{\partial y}{\partial t}, w) + A(y,w) = (f+u,w), & \forall w \in H_0^1(\Omega), \\ -(\frac{\partial z}{\partial t}, w) + A(w,z) = (y - y_d, w), & \forall w \in H_0^1(\Omega), \\ (\gamma u + z, v - u) \geq 0, & \forall v \in U_{ad}. \end{cases} \tag{2.7}$$

From the second equation in (2.7), we have that the adjoint state z satisfies transient convection diffusion equations with the strong form

$$\begin{cases} -\frac{\partial z}{\partial t} - \varepsilon \triangle z - \nabla \cdot (\boldsymbol{\beta} z) + \alpha z = y - y_d, & (\mathbf{x}, t) \in \Omega_T, \\ z(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Gamma_T, \\ z(\mathbf{x}, T) = 0, & \mathbf{x} \in \Omega. \end{cases} \tag{2.8}$$

In contrast to the state equation, the velocity field of the adjoint equation is $-\boldsymbol{\beta}$.

By the pointwise projection on U_{ad},

$$P_{U_{ad}} : L^2(\Omega_T) \longrightarrow U_{ad}, \quad P_{U_{ad}}(v) = \max\bigl(a, \min\bigl(v(\mathbf{x}, t), b\bigr)\bigr), \tag{2.9}$$

the optimal condition in (2.7) simplifies to

$$u = P_{U_{ad}}\left(-\frac{1}{\gamma} v\right). \tag{2.10}$$

2.2 Semi-discrete discretization

Let T^h be a regular triangulation of Ω, so that $\bar{\Omega} = \bigcup_{K \in T^h} \bar{K}$. Let h_K denote the diameter of the element K. Associated with T^h is a finite dimensional subspace W^h of $C(\bar{\Omega}) \cap H_0^1(\Omega)$, consisting of piecewise linear polynomials.

To control the convective derivative of the discrete solution sufficiently, a symmetric stabilization form S (see, *e.g.*, [14]) on $W^h \times W^h$ was introduced as follows:

$$S(v_h, w_h) = \sigma \sum_{E \in E^h} \int_E h_E^2 [\nabla v_h \cdot \mathbf{n}][\nabla w_h \cdot \mathbf{n}] \, ds,$$

where $\sigma > 0$ is the stabilization parameter. E^h denotes the collection of interior edges of the elements in T^h. h_E is the size of the edge E. $[q]$ denotes the jump of q across E for $E \in E^h$ defined by

$$\bigl[q(x)\bigr] = \lim_{s \to 0^+} \bigl(q(\mathbf{x} + s\mathbf{n}) - q(\mathbf{x} - s\mathbf{n})\bigr),$$

with $\mathbf{n}$ being the outward unit normal.

Using the above stabilization form, a semi-discrete CIP Galerkin approximation of optimal control problem (2.1)-(2.2) is defined by

$$\min_{u_h \in U_{ad}} J(y_h, u_h) \tag{2.11}$$

subject to

$$\begin{cases} (\frac{\partial y_h}{\partial t}, w_h) + A(y_h, w_h) + S(y_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in W^h, \\ y_h(0) = y_0^h \in W^h. \end{cases} \tag{2.12}$$

Here the control variable was approximated by variational discrete concept (see [17]). u_h in general is not a finite element function associated with the space mesh T^h.

By standard argument it can be shown that the following first-order optimality condition holds:

$$\begin{cases} (\frac{\partial y_h}{\partial t}, w_h) + A(y_h, w_h) + S(y_h, w_h) = (f + u_h, w_h), & \forall w_h \in W^h, \\ -(\frac{\partial z_h}{\partial t}, q_h) + A(q_h, z_h) + S(q_h, z_h) = (y_h - y_d, q_h), & \forall q_h \in W^h, \\ (\gamma u_h + z_h, v_h - u_h) \geq 0, & \forall v_h \in U_{ad}, \\ y_h(0) = y_0^h, \qquad z_h(T) = 0. \end{cases} \tag{2.13}$$

By the pointwise projection operator $P_{U_{ad}}$, we have

$$u_h = P_{U_{ad}}\left(-\frac{1}{\gamma} z_h\right). \tag{2.14}$$

2.3 Fully discrete scheme

To define a fully discrete scheme, we introduce a time partition. Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be a time grid with $\tau_n = t_n - t_{n-1}$, $n = 1, 2, \ldots, N$. Set $I_n = (t_{n-1}, t_n]$.

Using variational discretization concept, the fully discrete CIP Galerkin scheme for (2.1)-(2.2) reads as follows:

$$\min_{u_h^n \in U_{ad}} J_h\left(y_h^n, u_h^n\right) = \frac{1}{2} \sum_{n=1}^{N} \tau_n \left(\left\| y_h^n - y_d^n \right\|_{0,\Omega}^2 + \gamma \left\| u_h^n \right\|_{0,\Omega}^2 \right) \tag{2.15}$$

subject to

$$\begin{cases} (\frac{y_h^n - y_h^{n-1}}{\tau_n}, w_h) + A(y_h^n, w_h) + S(y_h^n, w_h) = (f^n + u_h^n, w_h), \quad \forall w_h \in W^h, \\ y_h^0 = y_0^h \in W^h. \end{cases} \tag{2.16}$$

Similar to a semi-discrete scheme, we can derive the discrete first-order optimality condition:

$$\begin{cases} (\frac{y_h^n - y_h^{n-1}}{\tau_n}, w_h) + A(y_h^n, w_h) + S(y_h^n, w_h) = (f^n + u_h^n, w_h), & \forall w_h \in W^h, \\ (\frac{z_h^{n-1} - z_h^n}{\tau_n}, q_h) + A(q_h, z_h^{n-1}) + S(q_h, z_h^{n-1}) = (y_h^n - y_d^n, q_h), & \forall q_h \in W^h, \\ (\gamma u_h^n + z_h^{n-1}, v_h - u_h^n) \geq 0, & \forall v_h \in U_{ad}, \\ y_h^0 = y_0^h, \qquad z_h^N = 0, & n = 1, 2, \ldots, N. \end{cases} \tag{2.17}$$

Again by the pointwise projection operator $P_{U_{ad}}$, we obtain

$$u_h^n = P_{U_{ad}}\left(-\frac{1}{\gamma} z_h^{n-1}\right).$$

We can see that u_h^n is a piecewise constant function in time.

For $n = 1, 2, \ldots, N$, let

$$\begin{aligned} Y_h|_{(t_{n-1},t_n]} &= l_n(t)y_h^n + l_{n-1}(t)y_h^{n-1}, \\ Z_h|_{(t_{n-1},t_n]} &= l_n(t)z_h^n + l_{n-1}(t)z_h^{n-1}, \\ U_h|_{(t_{n-1},t_n]} &= u_h^n, \end{aligned}$$

where

$$l_n(t) = \frac{t - t_{n-1}}{\tau_n}, \qquad l_{n-1}(t) = \frac{t_n - t}{\tau_n}.$$

For $\forall \psi \in C(0, T; L^2(\Omega))$, $t \in (t_{n-1}, t_n]$, we set $\widehat{\psi} = \psi(x, t_n)$, $\bar{\psi} = \psi(x, t_{n-1})$. Note that

$$\frac{\partial Y_h}{\partial t} = \frac{y_h^n - y_h^{n-1}}{\tau_n}, \qquad \frac{\partial Z_h}{\partial t} = \frac{z_h^n - z_h^{n-1}}{\tau_n}$$

for $t \in (t_{n-1}, t_n]$. Then the above optimality conditions can be rewritten as

$$\begin{cases} (\frac{\partial Y_h}{\partial t}, w_h) + A(\widehat{Y}_h, w_h) + S(\widehat{Y}_h, w_h) = (f^n + U_h, w_h), & \forall w_h \in W^h, \\ -(\frac{\partial Z_h}{\partial t}, q_h) + A(q_h, \bar{Z}_h) + S(q_h, \bar{Z}_h) = (\widehat{Y}_h - y_d^n, q_h), & \forall q_h \in W^h, \\ (\gamma U_h + \bar{Z}_h, v_h - U_h) \geq 0, & \forall v_h \in U_{ad}. \end{cases} \tag{2.18}$$

3 A posteriori error estimates

The objective of this section is to derive a posteriori error estimates for the state, adjoint state and control.

3.1 The estimate for control

To obtain the estimate for control, we introduce an auxiliary problem. For given U_h, let $(y(U_h), z(U_h))$ be a solution of the following system:

$$\begin{cases} (\frac{\partial y(U_h)}{\partial t}, w) + A(y(U_h), w) = (f + U_h, w), & \forall w \in H_0^1(\Omega), \\ y(U_h)(x, 0) = y_0(x), & x \in \Omega, \\ -(\frac{\partial z(U_h)}{\partial t}, q) + A(q, z(U_h)) = (y(U_h) - y_d, q), & \forall q \in H_0^1(\Omega), \\ z(U_h)(x, T) = 0, & x \in \Omega. \end{cases} \tag{3.1}$$

Lemma 3.1 *Let* (y, z, u) *and* (Y_h, Z_h, U_h) *be the solutions of* (2.7) *and* (2.18), *respectively. Then the following estimate*

$$\|u - U_h\|_{L^2(0,T;L^2(\Omega))} \leq C\big(\|z(U_h) - Z_h\|_{L^2(0,T;L^2(\Omega))} + \|Z_h - \bar{Z}_h\|_{L^2(0,T;L^2(\Omega))}\big)$$

holds.

Proof It follows from (2.7) and (2.18) that

$$\begin{aligned} &\gamma \|u - U_h\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\quad = \int_0^T \int_\Omega \gamma u(u - U_h) - \int_0^T \int_\Omega \gamma U_h(u - U_h) \end{aligned}$$

$$\begin{aligned}
&\le \int_0^T \int_\Omega z(U_h - u) - \int_0^T \int_\Omega \gamma U_h(u - U_h) \\
&= \int_0^T \int_\Omega (z - \bar{Z}_h)(U_h - u) - \int_0^T \int_\Omega (\gamma U_h + \bar{Z}_h)(u - U_h) \\
&= \int_0^T \int_\Omega \bigl(z - z(U_h)\bigr)(U_h - u) + \int_0^T \int_\Omega \bigl(z(U_h) - \bar{Z}_h\bigr)(U_h - u) \\
&\quad - \int_0^T \int_\Omega (\gamma U_h + \bar{Z}_h)(u - U_h) \\
&\le \int_0^T \int_\Omega \bigl(z - z(U_h)\bigr)(U_h - u) + \int_0^T \int_\Omega \bigl(z(U_h) - \bar{Z}_h\bigr)(U_h - u).
\end{aligned}$$

Here the last inequality was fulfilled due to the implicit discretization of the control variable.

By (2.7) and (3.1) we have

$$\begin{aligned}
&\int_0^T \int_\Omega \bigl(z - z(U_h)\bigr)(u - U_h) \\
&\quad = \int_0^T \int_\Omega \frac{\partial (y - y(U_h))}{\partial t}\bigl(z - z(U_h)\bigr) + \int_0^T A\bigl(y - y(U_h), z - z(U_h)\bigr) \\
&\quad = -\int_0^T \int_\Omega \frac{\partial (z - z(U_h))}{\partial t}\bigl(y - y(U_h)\bigr) + \int_0^T A\bigl(y - y(U_h), z - z(U_h)\bigr) \\
&\quad = \int_0^T \int_\Omega \bigl(y - y(U_h)\bigr)^2 \ge 0,
\end{aligned}$$

where $y(U_h)(0) = y_0(x)$ and $z(U_h)(T) = 0$ was used. Thus we arrive at

$$\begin{aligned}
&\gamma \|u - U_h\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad \le \int_0^T \int_\Omega \bigl(z(U_h) - \bar{Z}_h\bigr)(U_h - u) \\
&\quad \le C(\delta) \bigl\| z(U_h) - Z_h \bigr\|^2_{L^2(0,T;L^2(\Omega))} + C(\delta) \|Z_h - \bar{Z}_h\|^2_{L^2(0,T;L^2(\Omega))} \\
&\qquad + C\delta \|u - U_h\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Choosing $\delta = \frac{\gamma}{2C}$ yields the theorem result. □

3.2 The estimate for the state and adjoint state

In this section we shall adopt the elliptic reconstruction technique proposed in [18, 20] to derive a posteriori error estimates for the state and adjoint state.

To this end we first introduce the following elliptic reconstruction definitions for state and adjoint state.

Definition 3.2 For $n = 1, 2, \ldots, N$, we define the elliptic reconstruction $v^n \in H_0^1(\Omega)$ and $\omega^{n-1} \in H_0^1(\Omega)$ satisfying the following elliptic problems:

$$A\bigl(v^n, w\bigr) = \left(f^n + u_h^n - \frac{y_h^n - y_h^{n-1}}{\tau_n}, w\right), \quad \forall w \in H_0^1(\Omega) \tag{3.2}$$

and

$$A\left(q,\omega^{n-1}\right)=\left(y_h^n-y_d^n+\frac{z_h^n-z_h^{n-1}}{\tau_n},q\right),\quad \forall q\in H_0^1(\Omega). \tag{3.3}$$

Noticing that the CIP Galerkin approximation of v^n can be defined as

$$A\left(v_h^n,w_h\right)+S\left(v_h^n,w_h\right)=\left(f^n+u_h^n-\frac{y_h^n-y_h^{n-1}}{\tau_n},w_h\right),\quad \forall w_h\in W_h^n.$$

Then we have

$$A\left(y_h^n-v_h^n,w_h\right)+S\left(y_h^n-v_h^n,w_h\right)=0,$$

which implies $y_h^n=v_h^n$. We can observe a similar property for the CIP Galerkin approximation of ω^{n-1}.

Using the above convention, we define $v(t)$ and $\omega(t)$ as

$$v(t)=l_n(t)v^n+l_{n-1}(t)v^{n-1}$$

and

$$\omega(t)=l_n(t)\omega^n+l_{n-1}(t)\omega^{n-1}$$

for $t\in I_n$ and $n\in[1,N]$. We decompose the error as follows:

$$y(U_h)-Y_h(t)=y(U_h)-v(t)-\left(Y_h(t)-v(t)\right):=\rho_y-\xi_y$$

and

$$z(U_h)-Z_h(t)=z(U_h)-\omega(t)-\left(Z_h(t)-\omega(t)\right):=\rho_z-\xi_z.$$

Nextly we shall derive the estimates of ρ and ξ. For simplicity, we introduce the following notations:

$$\begin{aligned}
&R_{K,y}^n=f^n+u_h^n-\frac{\partial Y_h}{\partial t}-\boldsymbol{\beta}\cdot\nabla y_h^n-\alpha y_h^n,\\
&R_{E,y}^n=\left[\nabla y_h^n\cdot\mathbf{n}\right],\\
&R_{K,z}^n=y_h^n-y_d^n+\frac{\partial Z_h}{\partial t}+\nabla\cdot\left(\boldsymbol{\beta}z_h^{n-1}\right)-\alpha z_h^{n-1},\\
&R_{E,z}^n=\left[\nabla z_h^{n-1}\cdot\mathbf{n}\right],\\
&\bar{\partial}_t\varphi^n=\frac{\varphi^n-\varphi^{n-1}}{\tau_n},\qquad \bar{\partial}_t^2\varphi^n=\frac{\bar{\partial}_t\varphi^n-\bar{\partial}_t\varphi^{n-1}}{\tau_n}.
\end{aligned}$$

Let A_h^s and A_h^a be the discrete operators associated with the state and adjoint state, which are defined by the following for $\forall v\in W^h$:

$$\begin{aligned}
&\left\langle A_h^s v,w_h\right\rangle=A(v,w_h)+S(v,w_h),\quad \forall w_h\in W^h,\\
&\left\langle A_h^a v,w_h\right\rangle=A(w_h,v)+S(w_h,v),\quad \forall w_h\in W^h.
\end{aligned}$$

The time error estimators are characterized by

$$\theta_{y,n} = \begin{cases} (\bar{\partial}_t f^n + \bar{\partial}_t u_h^n - \bar{\partial}_t^2 y_h^n)\tau_n, & 2 \le n \le N, \\ f^1 + u_h^1 - \bar{\partial}_t y_h^1 - A_h^s y_h^0, & n = 1 \end{cases}$$

and

$$\theta_{z,n} = \begin{cases} (\bar{\partial}_t y_h^n - \bar{\partial}_t y_d^n + \bar{\partial}_t^2 z_h^{n-1})\tau_n, & 1 \le n \le N-1, \\ y_h^N - y_d^N + \bar{\partial}_t z_h^N - A_h^a z_h^N, & n = N. \end{cases}$$

Moreover, let $\alpha_K = \min\{\alpha_0^{-\frac{1}{2}}, \varepsilon^{-\frac{1}{2}} h_K\}$, $\alpha_E = \min\{\alpha_0^{-\frac{1}{2}}, \varepsilon^{-\frac{1}{2}} h_E\}$.

By the standard techniques used in a posteriori error estimate for stationary convection diffusion optimal control problems [7, 15], we obtain the following results.

Lemma 3.3 *Let* v^n *and* y_h^n *be the solutions to* (3.2) *and* (2.16). *Then the following a posteriori error estimate holds*:

$$\|\xi_y^n\|_* \preceq \eta_{y,n},$$

where

$$\eta_{y,n}^2 = \sum_{K \in T^h} \alpha_K^2 \|R_{K,y}^n\|_{0,K}^2 + \sum_{E \in E^h} \varepsilon^{-\frac{1}{2}} \alpha_E \|\varepsilon R_{E,y}^n\|_{0,E}^2 + \sum_{E \in E^h} \alpha_E^2 h_E \|R_{E,y}^n\|_{0,E}^2.$$

Lemma 3.4 *Let* ω^{n-1} *and* z_h^{n-1} *be the solutions to* (3.3) *and* (2.17). *Then the following a posteriori error estimate holds*:

$$\|\xi_z^{n-1}\|_* \preceq \eta_{z,n},$$

where

$$\eta_{z,n}^2 = \sum_{K \in T^h} \alpha_K^2 \|R_{K,z}^n\|_{0,K}^2 + \sum_{E \in E^h} \varepsilon^{-\frac{1}{2}} \alpha_E \|\varepsilon R_{E,z}^n\|_{0,E}^2 + \sum_{E \in E^h} \alpha_E^2 h_E \|R_{E,z}^n\|_{0,E}^2.$$

In the following we shall deduce the estimates of ρ_y and ρ_z. By (3.1) and Definition 3.2 we can derive the following error equations for ρ_y and ρ_z.

Lemma 3.5 *Given* $t \in I_n$, *we deduce*

$$\left(\frac{\partial \rho_y}{\partial t}, \psi\right) + A(\rho_y, \psi) = \left(\frac{\partial \xi_y}{\partial t}, \psi\right) + A\big(v^n - v(t), \psi\big) + \big(f - f^n, \psi\big), \quad \forall \psi \in H_0^1(\Omega) \tag{3.4}$$

and

$$-\left(\frac{\partial \rho_z}{\partial t}, \psi\right) + A(\rho_z, \psi) = -\left(\frac{\partial \xi_z}{\partial t}, \psi\right) + A\big(\psi, \omega^{n-1} - \omega(t)\big) + \big(y_d - y_d^n, \psi\big) + \big(y(U_h) - y_h^n, \psi\big), \quad \forall \psi \in H_0^1(\Omega). \tag{3.5}$$

Proof Note that

$$\left(\frac{\partial y(U_h)}{\partial t}, \psi\right) + A\bigl(y(U_h), \psi\bigr) = (f + U_h, \psi), \quad \forall \psi \in H_0^1(\Omega)$$

and

$$A\bigl(v^n, \psi\bigr) = \left(f^n + U_h - \frac{\partial Y_h}{\partial t}, \psi\right), \quad \forall \psi \in H_0^1(\Omega).$$

Then we have

$$\begin{aligned}
&\left(\frac{\partial \rho_y}{\partial t}, \psi\right) + A(\rho_y, \psi) \\
&\quad = \left(\frac{\partial y(U_h)}{\partial t}, \psi\right) + A\bigl(y(U_h), \psi\bigr) - \left(\frac{\partial v(t)}{\partial t}, \psi\right) - A\bigl(v(t), \psi\bigr) \\
&\quad = (f + U_h, \psi) + \left(\frac{\partial \xi_y}{\partial t}, \psi\right) - \left(\frac{\partial Y_h}{\partial t}, \psi\right) - A\bigl(v(t), \psi\bigr) \\
&\quad = \left(f^n + U_h - \frac{\partial Y_h}{\partial t}, \psi\right) + \left(\frac{\partial \xi_y}{\partial t}, \psi\right) + \bigl(f - f^n, \psi\bigr) - A\bigl(v(t), \psi\bigr) \\
&\quad = \left(\frac{\partial \xi_y}{\partial t}, \psi\right) + \bigl(f - f^n, \psi\bigr) + A\bigl(v^n - v(t), \psi\bigr).
\end{aligned}$$

Similarly we can deduce the error equation for ρ_z. □

Before deriving the estimates for ρ_y and ρ_z, we first introduce the following lemma with respect to a Clément-type interpolation operator. The proof can be found in [21, 22].

Lemma 3.6 *Let I_h be a quasi-interpolation operator of Clément type. The following estimates hold for all elements K, all faces E and all functions $v \in H_0^1(\Omega)$*:

$$\begin{aligned}
&\|v - I_h v\|_{0,K} \preccurlyeq \alpha_K \|v\|_{*,N(K)}, \\
&\|v - I_h v\|_{0,E} \preccurlyeq \varepsilon^{-\frac{1}{4}} \alpha_E^{\frac{1}{2}} \|v\|_{*,N(E)}, \\
&\|I_h v\|_{*,K} \preccurlyeq \|v\|_{*,N(K)},
\end{aligned}$$

where $N(K)$ and $N(E)$ denote the union of all elements that share at least one point with K and E.

Then we arrive at the following.

Lemma 3.7 *The following estimate holds*:

$$\begin{aligned}
&\max_{t \in [0,T]} \|\rho_y\|^2 + \int_0^T \|\rho_y\|_*^2 \\
&\quad \preceq \bigl\|\rho_y(0)\bigr\|^2 + \left(\frac{1}{2}\sum_{n=1}^{N} \tau_n \|\theta_{y,n}\|\right)^2 + \left(\sum_{n=1}^{N} \int_{I_n} \bigl\|f(t) - f^n\bigr\|\right)^2
\end{aligned}$$

$$+\left[\sum_{n=1}^{N}\int_{I_n}\left(\sum_{K\in T^h}\alpha_K\left\|\bar{\partial}_t R^n_{K,y}\right\|_{0,K}+\sum_{E\in E^h}\alpha_E^{\frac{1}{2}}\varepsilon^{-\frac{1}{4}}\left\|\varepsilon\bar{\partial}_t R^n_{E,y}\right\|_{0,E}\right.\right.$$
$$\left.\left.+\sum_{E\in E^h}\alpha_E h_E^{\frac{1}{2}}\left\|\bar{\partial}_t R^n_{E,y}\right\|_{0,E}\right)\right]^2.$$

Proof Setting $\psi = \rho_y$ in (3.4) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\rho_y(t)\right\|^2+A(\rho_y,\rho_y)\\&\quad=\left(\frac{\partial\xi_y}{\partial t},\rho_y\right)+A\left(v^n-v(t),\rho_y\right)+\left(f-f^n,\rho_y\right).\end{aligned}$$

Integrating in time from 0 to T gives

$$\begin{aligned}&\frac{1}{2}\left(\left\|\rho_y(T)\right\|^2-\left\|\rho_y(0)\right\|^2\right)+\int_0^T A(\rho_y,\rho_y)\\&\quad=\int_0^T\left(\frac{\partial\xi_y}{\partial t},\rho_y\right)+\int_0^T A\left(v^n-v(t),\rho_y\right)+\int_0^T\left(f-f^n,\rho_y\right)\\&\quad\le\int_0^T\left|\left(\frac{\partial\xi_y}{\partial t},\rho_y\right)\right|+\int_0^T\left|A\left(v^n-v(t),\rho_y\right)\right|+\int_0^T\left|\left(f-f^n,\rho_y\right)\right|.\end{aligned}$$

Assume that

$$\left\|\rho_y\left(t_m^*\right)\right\|=\max_{t\in[0,T]}\|\rho_y\|.$$

Again integrating in time from 0 to t_m^* results in

$$\begin{aligned}&\frac{1}{2}\left(\left\|\rho_y\left(t_m^*\right)\right\|^2-\left\|\rho_y(0)\right\|^2\right)+\int_0^{t_m^*} A(\rho_y,\rho_y)\\&\quad=\int_0^{t_m^*}\left(\frac{\partial\xi_y}{\partial t},\rho_y\right)+\int_0^{t_m^*} A\left(v^n-v(t),\rho_y\right)+\int_0^{t_m^*}\left(f-f^n,\rho_y\right)\\&\quad\le\int_0^T\left|A\left(v^n-v(t),\rho_y\right)\right|+\int_0^T\left|\left(f-f^n,\rho_y\right)\right|+\int_0^T\left|\left(\frac{\partial\xi_y}{\partial t},\rho_y\right)\right|.\end{aligned}$$

Combining the above two inequalities yields

$$\begin{aligned}&\frac{1}{2}\left(\left\|\rho_y\left(t_m^*\right)\right\|^2-\left\|\rho_y(0)\right\|^2\right)+\int_0^T A(\rho_y,\rho_y)\\&\quad\le 2\int_0^T\left|A\left(v^n-v(t),\rho_y\right)\right|+2\int_0^T\left|\left(f-f^n,\rho_y\right)\right|+2\int_0^T\left|\left(\frac{\partial\xi_y}{\partial t},\rho_y\right)\right|\\&\quad:=2\sum_{i=1}^{3}T_i.\end{aligned}\tag{3.6}$$

In the following we shall derive the estimates of T_i. By the definition of the elliptic reconstruction, we can bound T_1 as follows:

$$\begin{aligned} T_1 &= \sum_{n=1}^{N} \int_{I_n} l_{n-1}(t) \big|(\theta_{y,n}, \rho_y)\big| \\ &\leq \sum_{n=1}^{N} \int_{I_n} l_{n-1}(t) \|\theta_{y,n}\| \|\rho_y\| \\ &\leq \frac{1}{2} \sum_{n=1}^{N} \tau_n \|\theta_{y,n}\| \big\|\rho_y\big(t_m^*\big)\big\|, \end{aligned}$$

which implies

$$T_1 \leq C(\delta) \left(\frac{1}{2} \sum_{n=1}^{N} \tau_n \|\theta_{y,n}\| \right)^2 + \delta \big\|\rho_y\big(t_m^*\big)\big\|^2$$

with an arbitrarily positive constant δ.

For the second term T_2, we can bound it as follows:

$$\begin{aligned} T_2 &\leq \sum_{n=1}^{N} \int_{I_n} \big\|f(t) - f^n\big\| \|\rho_y\| \\ &\leq \big\|\rho_y\big(t_m^*\big)\big\| \sum_{n=1}^{N} \int_{I_n} \big\|f(t) - f^n\big\| \\ &\leq C(\delta) \left(\sum_{n=1}^{N} \int_{I_n} \big\|f(t) - f^n\big\| \right)^2 + \delta \big\|\rho_y\big(t_m^*\big)\big\|^2. \end{aligned}$$

Now it remains to estimate T_3. Note that

$$T_3 = \sum_{n=1}^{N} \int_{I_n} \tau_n^{-1} \big|\big(v^n - v^{n-1} - y_h^n + y_h^{n-1}, \rho_y\big)\big|.$$

This term can be estimated by the techniques used in a posterior error estimates for the stationary problem. To this end we introduce an auxiliary problem

$$\begin{cases} -\varepsilon \triangle \phi - \nabla \cdot (\boldsymbol{\beta} \phi) + \alpha \phi = \rho_y, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

For the above auxiliary problem, the following stability estimates (see, *e.g.*, [23]) hold:

$$\varepsilon^{\frac{3}{2}} \|\phi\|_2 + \varepsilon^{\frac{1}{2}} \|\phi\|_1 + \|\phi\| \leq C \|\rho_y\|. \tag{3.8}$$

Using the above auxiliary problem, we have

$$\big(v^n - v^{n-1} - y_h^n + y_h^{n-1}, \rho_y\big) = A\big(v^n - v^{n-1} - y_h^n + y_h^{n-1}, \phi\big).$$

By the definitions of v^n and y_n^n, we can deduce

$$\begin{aligned}
&\big(v^n - v^{n-1} - y_h^n + y_h^{n-1}, \rho_y\big) \\
&\quad = \big(f^n + u_h^n - \bar{\partial}_t y_h^n - f^{n-1} - u_h^{n-1} + \bar{\partial}_t y_h^{n-1}, \phi\big) - A\big(y_h^n - y_h^{n-1}, \phi - I_h\phi\big) \\
&\qquad - A\big(y_h^n - y_h^{n-1}, I_h\phi\big) - S\big(y_h^n - y_h^{n-1}, I_h\phi\big) + S\big(y_h^n - y_h^{n-1}, I_h\phi\big) \\
&\quad = \big(f^n + u_h^n - \bar{\partial}_t y_h^n - f^{n-1} - u_h^{n-1} + \bar{\partial}_t y_h^{n-1}, \phi - I_h\phi\big) \\
&\qquad - A\big(y_h^n - y_h^{n-1}, \phi - I_h\phi\big) + S\big(y_h^n - y_h^{n-1}, I_h\phi\big),
\end{aligned}$$

where $I_h\phi$ denotes the Clément interpolation of ϕ. Further, we have

$$\begin{aligned}
&\big(v^n - v^{n-1} - y_h^n + y_h^{n-1}, \rho_y\big) \\
&\quad = \tau_n \sum_{K\in T^h} \int_K \big(\bar{\partial}_t f^n + \bar{\partial}_t u_h^n - \bar{\partial}_t^2 y_h^n - \bar{\partial}_t\big(\boldsymbol{\beta}\cdot\nabla y_h^n + \alpha y_h^n\big), \phi - I_h\phi\big) \\
&\qquad + \tau_n \sum_{E\in E^h} \int_E [\varepsilon\bar{\partial}_t \nabla y_h \cdot \mathbf{n}](I_h\phi - \phi)\,ds + \tau_n S(\bar{\partial}_t y_h, I_h\phi) \\
&\quad \le \tau_n \sum_{K\in T^h} \big\|\bar{\partial}_t R_{K,y}^n\big\|_{0,K} \|\phi - I_h\phi\|_{0,K} + \tau_n \sum_{E\in E^h} \big\|\varepsilon\bar{\partial}_t R_{E,y}^n\big\|_{0,E} \|I_h\phi - \phi\|_{0,E} \\
&\qquad + \tau_n\sigma \sum_{E\in E^h} \big\|h_E^2 \bar{\partial}_t R_{E,y}^n\big\|_{0,E} \big\|\big[\mathbf{n}\cdot\nabla(I_h\phi)\big]\big\|_{0,E}.
\end{aligned}$$

Note that

$$\big\|\big[\nabla(I_h\phi)\big]\big\|_{0,E} \le C h_E^{-\frac{1}{2}} \big\|\nabla(I_h\phi)\big\|_{0,N(E)},$$

and

$$\big\|\big[\nabla(I_h\phi)\big]\big\|_{0,E} \le C h_E^{-\frac{3}{2}} \|I_h\phi\|_{0,N(E)}.$$

Then we derive

$$\tau_n \sum_{E\in E^h} \big\|h_E^2 \bar{\partial}_t R_{E,y}^n\big\|_{0,E} \big\|\big[\mathbf{n}\cdot\nabla(I_h\phi)\big]\big\|_{0,E} \le C\tau_n \sum_{E\in E^h} h_E^{\frac{3}{2}} \big\|\bar{\partial}_t R_{E,y}^n\big\|_{0,E} \big\|\nabla(I_h\phi)\big\|_{0,N(E)}$$

or

$$\tau_n \sum_{E\in E^h} \big\|h_E^2 \bar{\partial}_t R_{E,y}^n\big\|_{0,E} \big\|\big[\mathbf{n}\cdot\nabla(I_h\phi_1)\big]\big\|_{0,E} \le C\tau_n \sum_{E\in E^h} h_E^{\frac{1}{2}} \big\|\bar{\partial}_t R_{E,y}^n\big\|_{0,E} \|I_h\phi\|_{0,N(E)}.$$

This implies

$$\tau_n \sum_{E\in E^h} \big\|h_E^2 \bar{\partial}_t R_{E,y}^n\big\|_{0,E} \big\|\big[\mathbf{n}\cdot\nabla(I_h\phi_1)\big]\big\|_{0,E} \le C\tau_n \sum_{E\in E^h} \alpha_E h_E^{\frac{1}{2}} \big\|\bar{\partial}_t R_{E,y}^n\big\|_{0,E} \|I_h\phi\|_{*,N(E)}.$$

It follows from Lemma 3.6 and (3.8) that

$$\begin{aligned}&\left(w^n - w^{n-1} - y_h^n + y_h^{n-1}, \rho_y\right)\\&\quad\le C\tau_n\Biggl(\sum_{K\in T^h}\alpha_K\left\|\bar\partial_t R_{K,y}^n\right\|_{0,K} + \sum_{E\in E^h}\alpha_E^{\frac12}\varepsilon^{-\frac14}\left\|\varepsilon\bar\partial_t R_{E,y}^n\right\|_{0,E}\\&\qquad+\sum_{E\in E^h}\alpha_E h_E^{\frac12}\left\|\bar\partial_t R_{E,y}^n\right\|_{0,E}\Biggr)\|\rho_y\|_{0,\Omega}.\end{aligned}\tag{3.9}$$

Thus we arrive at

$$\begin{aligned}T_3 &\le C\sum_{n=1}^N\int_{I_n}\Biggl(\sum_{K\in T^h}\alpha_K\left\|\bar\partial_t R_{K,y}^n\right\|_{0,K} + \sum_{E\in E^h}\alpha_E^{\frac12}\varepsilon^{-\frac14}\left\|\varepsilon\bar\partial_t R_{E,y}^n\right\|_{0,E}\\&\quad+\sum_{E\in E^h}\alpha_E h_E^{\frac12}\left\|\bar\partial_t R_{E,y}^n\right\|_{0,E}\Biggr)\left\|\rho_y\left(t_m^*\right)\right\|\\&\le C(\delta)\Biggl[\sum_{n=1}^N\int_{I_n}\Biggl(\sum_{K\in T^h}\alpha_K\left\|\bar\partial_t R_{K,y}^n\right\|_{0,K} + \sum_{E\in E^h}\alpha_E^{\frac12}\varepsilon^{-\frac14}\left\|\varepsilon\bar\partial_t R_{E,y}^n\right\|_{0,E}\\&\quad+\sum_{E\in E^h}\alpha_E h_E^{\frac12}\left\|\bar\partial_t R_{E,y}^n\right\|_{0,E}\Biggr)\Biggr]^2 + C\delta\left\|\rho_y\left(t_m^*\right)\right\|^2.\end{aligned}$$

Inserting the estimates of T_1, T_2 and T_3 into (3.6) and setting δ small enough leads to the theorem results. □

Theorem 3.8 *Let $y(U_h)$ and Y_h be the solutions of* (2.7) *and* (2.13), *respectively. Then the following estimate holds*:

$$\begin{aligned}&\left(\int_0^T\left\|y(U_h) - Y_h\right\|_*^2\right)^{\frac12}\\&\quad\preceq\left\|\rho_y(0)\right\| + \frac12\sum_{n=1}^N\tau_n\|\theta_{y,n}\| + \sum_{n=1}^N\int_{I_n}\left\|f(t) - f^n\right\|\\&\qquad+\sum_{n=1}^N\int_{I_n}\Biggl(\sum_{K\in T^h}\alpha_K\left\|\bar\partial_t R_{K,y}^n\right\|_{0,K} + \sum_{E\in E^h}\alpha_E^{\frac12}\varepsilon^{-\frac14}\left\|\varepsilon\bar\partial_t R_{E,y}^n\right\|_{0,E}\\&\qquad+\sum_{E\in E^h}\alpha_E h_E^{\frac12}\left\|\bar\partial_t R_{E,y}^n\right\|_{0,E}\Biggr) + \left(\frac13\sum_{n=1}^N\tau_n\left(\left\|\xi_y^n\right\|_*^2 + \left\|\xi_y^{n-1}\right\|_*^2\right)\right)^{\frac12}.\end{aligned}\tag{3.10}$$

Proof Note that

$$\begin{aligned}\int_0^T\left\|y(U_h) - Y_h\right\|_*^2 &\preceq \int_0^T\|\rho_y\|_*^2 + \int_0^T\|\xi_y\|_*^2 \preceq \int_0^T\|\rho_y\|_*^2 + \int_0^T\|\xi_y\|_*^2\\&\preceq\int_0^T\|\rho_y\|_*^2 + \frac13\tau_n\sum_{n=1}^N\left(\left\|\xi_y^n\right\|_*^2 + \left\|\xi_y^{n-1}\right\|_*^2\right).\end{aligned}$$

Then by Lemmas 3.3 and 3.7 we can deduce the estimates of $\int_0^T\|y(U_h) - Y_h\|_*^2$. □

Remark 3.9 Note that

$$\|y(U_h) - Y_h\| \le \|\rho_y\| + \|\xi_y\|. \tag{3.11}$$

The second term is the elliptic reconstruction error, which can bounded as follows for $t \in I_n$:

$$\|\xi_y\| = \|l_n \xi_y^n + l_{n-1}\xi_y^{n-1}\| \le \max(\|\xi_y^n\|, \|\xi_y^{n-1}\|). \tag{3.12}$$

Then

$$\|y(U_h) - Y_h\|_{L^\infty(0,T;L^2(\Omega))} \le \max_t \|\rho_y(t)\| + \max_t \|\xi_y\| \le \max_t \|\rho_y(t)\| + \max_n \|\xi_y^n\|.$$

Combining Lemmas 3.3 and 3.7, we can deduce

$$\begin{aligned}
&\|y(U_h) - Y_h\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad\preceq \|\rho_y(0)\| + \frac{1}{2}\sum_{n=1}^N \tau_n \|\theta_{y,n}\| + \sum_{n=1}^N \int_{I_n} \|f(t) - f^n\| \\
&\qquad + \sum_{n=1}^N \tau_n \int_{I_n} \biggl(\sum_{K\in T^h} \alpha_K \|\bar{\partial}_t R_{K,y}^n\|_{0,K} + \sum_{E\in E^h} \alpha_E^{\frac{1}{2}} \varepsilon^{-\frac{1}{4}} \|\varepsilon \bar{\partial}_t R_{E,y}^n\|_{0,E} \\
&\qquad + \sum_{E\in E^h} \alpha_E h_E^{\frac{1}{2}} \|\bar{\partial}_t R_{E,y}^n\|_{0,E} \biggr) + \max_{n\in[0,N]} \eta_{y,n}.
\end{aligned} \tag{3.13}$$

Now we turn our attention to estimate $z(U_h) - Z_h$. The argument skills are similar to those used in the estimate of $y(U_h) - Y_h$. Therefore we just sketch the proof.

Setting $\psi = \rho_z$ in (3.5) leads to

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\|\rho_z(t)\|^2 + A(\rho_z, \rho_z) = -\left(\frac{\partial \xi_z}{\partial t}, \rho_z\right) + A(\rho_z, \omega^{n-1} - \omega(t)) + (y_d - y_d^n, \rho_z) \\
+ (y(U_h) - y_h^n, \rho_z).
\end{aligned}$$

Let

$$\max_{t\in[0,T]} \|\rho_z\| = \|\rho_z(t_n^*)\|.$$

Then integrating the above equation from t_n^* to T and 0 to T, respectively, leads to

$$\begin{aligned}
&\frac{1}{2}(\|\rho_z(t_n^*)\|^2 - \|\rho_z(T)\|^2) + \int_0^T A(\rho_z, \rho_z) \\
&\quad\le 2\int_0^T \left|\left(\frac{\partial \xi_z}{\partial t}, \rho_z\right)\right| + 2\int_0^T |A(\rho_z, \omega^{n-1} - \omega)| + 2\int_0^T |(y_d - y_d^n, \rho_z)| \\
&\qquad + 2\int_0^T |(y(U_h) - y_h^n, \rho_z)|.
\end{aligned}$$

In an analogous way to Lemma 3.7, we can derive the estimate for ρ_z.

Lemma 3.10 *We have*

$$\begin{aligned}
&\max_{t\in[0,T]} \|\rho_z\|^2 + \int_0^T \|\rho_z\|_*^2 \\
&\quad\preceq \|\rho_z(T)\|^2 + \left(\frac{1}{2}\sum_{n=1}^N \tau_n\|\theta_{z,n}\|\right)^2 + \left(\sum_{n=1}^N \int_{I_n} \|y_d - y_d^n\|\right)^2 \\
&\qquad + \left[\sum_{n=1}^N \int_{I_n} \left(\sum_{K\in T^h} \alpha_K \|\bar\partial_t R_{K,z}^n\|_{0,K} + \sum_{E\in E^h} \alpha_E \varepsilon^{-\frac{1}{4}} \|\varepsilon\bar\partial_t R_{E,z}^n\|_{0,E}\right.\right. \\
&\qquad \left.\left. + \sum_{E\in E^h} \alpha_E h_E^{\frac{1}{2}} \|\bar\partial_t R_{E,z}^n\|_{0,E}\right)\right]^2 + \left(\sum_{n=1}^N \int_{I_n} \|Y_h - y_h^n\|\right)^2 + \|y(U_h) - Y_h\|_{L^2(0,T;L^2(\Omega))}^2.
\end{aligned}$$

Collecting Lemmas 3.4 and 3.10 and using similar arguments to Theorem 3.8 yields the following.

Theorem 3.11 *Let* $z(U_h)$ *and* Z_h *be the solutions of* (2.7) *and* (2.13), *respectively. Then the following estimates hold*:

$$\begin{aligned}
&\left(\int_0^T \|z(U_h) - Z_h\|_*^2\right)^{\frac{1}{2}} \\
&\quad\preceq \|\rho_z(T)\| + \frac{1}{2}\sum_{n=1}^N \tau_n \|\theta_{z,n}\| + \sum_{n=1}^N \int_{I_n} \|y_d - y_d^n\| + \sum_{n=1}^N \int_{I_n} \|Y_h - y_h^n\| \\
&\qquad + \left[\sum_{n=1}^N \int_{I_n} \sum_{K\in T^h} \alpha_K \|\bar\partial_t R_{K,z}^n\|_{0,K} + \sum_{E\in E^h} \alpha_E \varepsilon^{-\frac{1}{4}} \|\varepsilon\bar\partial_t R_{E,z}^n\|_{0,E} + \sum_{E\in E^h} \alpha_E h_E^{\frac{1}{2}} \|\bar\partial_t R_{E,z}^n\|_{0,E}\right] \\
&\qquad + \left(\frac{1}{3}\sum_{n=1}^N \tau_n\left(\|\xi_z^n\|_*^2 + \|\xi_z^{n-1}\|_*^2\right)\right)^{\frac{1}{2}} + \|y(U_h) - Y_h\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Remark 3.12 Similar to Remark 3.9, we can also derive the posteriori error estimates of

$$\begin{aligned}
&\|z(U_h) - Z_h\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad\preceq \|\rho_z(T)\| + \frac{1}{2}\sum_{n=1}^N \tau_n \|\theta_{z,n}\| + \sum_{n=1}^N \int_{I_n} \|y_d - y_d^n\| + \sum_{n=1}^N \int_{I_n} \|Y_h - y_h^n\| \\
&\qquad + \left[\sum_{n=1}^N \int_{I_n} \sum_{K\in T^h} \alpha_K \|\bar\partial_t R_{K,z}^n\|_{0,K} + \sum_{E\in E^h} \alpha_E \varepsilon^{-\frac{1}{4}} \|\varepsilon\bar\partial_t R_{E,z}^n\|_{0,E} + \sum_{E\in E^h} \alpha_E h_E^{\frac{1}{2}} \|\bar\partial_t R_{E,z}^n\|_{0,E}\right] \\
&\qquad + \max_{n\in[0,N]} \eta_{z,n} + \|y(U_h) - Y_h\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

3.3 The main results

By (2.7) and (3.1) we can also derive

$$\int_0^T \|y - y(U_h)\|_*^2 \preceq \|u - U_h\|_{L^2(0,T;L^2(\Omega))}^2 \tag{3.14}$$

and

$$\int_0^T \left\| z - z(U_h) \right\|_*^2 \preceq \|u - U_h\|_{L^2(0,T;L^2(\Omega))}^2. \tag{3.15}$$

Therefore, combining Lemma 3.1, Theorems 3.8, 3.11, (3.14) and (3.15), we can deduce the following estimates.

Theorem 3.13 *Let* (y,p,u) *and* (Y_h, Z_h, U_h) *be the solutions of* (2.7) *and* (2.13), *respectively. Then the following estimate*

$$\|u - U_h\|_{L^2(0,T;L^2(\Omega))} + \left(\int_0^T \|y - Y_h\|_*^2\right)^{\frac{1}{2}} + \left(\int_0^T \|z - Z_h\|_*^2\right)^{\frac{1}{2}} \preceq \widetilde{\eta}_y + \widetilde{\eta}_z$$

holds, where

$$\begin{aligned}\widetilde{\eta}_y = {} & \left\| \rho_y(0) \right\| + \frac{1}{2} \sum_{n=1}^N \tau_n \|\theta_{y,n}\| + \sum_{n=1}^N \int_{I_n} \left\| f(t) - f^n \right\| \\ & + \sum_{n=1}^N \int_{I_n} \Bigg(\sum_{K \in T^h} \alpha_K \left\| \bar{\partial}_t R_{K,y}^n \right\|_{0,K} + \sum_{E \in E^h} \alpha_E{}^{\frac{1}{2}} \varepsilon^{-\frac{1}{4}} \left\| \varepsilon \bar{\partial}_t R_{E,y}^n \right\|_{0,E} \\ & + \sum_{E \in E^h} \alpha_E h_E^{\frac{1}{2}} \left\| \bar{\partial}_t R_{E,y}^n \right\|_{0,E} \Bigg) + \sum_{n=1}^N \int_{I_n} \left\| Y_h - y_h^n \right\| + \left(\frac{1}{3} \sum_{n=1}^N \tau_n \left(\left\| \xi_y^n \right\|_*^2 + \left\| \xi_y^{n-1} \right\|_*^2 \right) \right)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\eta}_z = {} & \left\| \rho_z(T) \right\| + \frac{1}{2} \sum_{n=1}^N \tau_n \|\theta_{z,n}\| + \sum_{n=1}^N \int_{I_n} \left\| y_d - y_d^n \right\| + \|Z_h - \bar{Z}_h\|_{L^2(0,T;L^2(\Omega))} \\ & + \Bigg[\sum_{n=1}^N \int_{I_n} \sum_{K \in T^h} \alpha_K \left\| \bar{\partial}_t R_{K,z}^n \right\|_{0,K} + \sum_{E \in E^h} \alpha_E \varepsilon^{-\frac{1}{4}} \left\| \varepsilon \bar{\partial}_t R_{E,z}^n \right\|_{0,E} \\ & + \sum_{E \in E^h} \alpha_E h_E^{\frac{1}{2}} \left\| \bar{\partial}_t R_{E,z}^n \right\|_{0,E} \Bigg] + \left(\frac{1}{3} \sum_{n=1}^N \tau_n \left(\left\| \xi_z^n \right\|_*^2 + \left\| \xi_z^{n-1} \right\|_*^2 \right) \right)^{\frac{1}{2}}.\end{aligned}$$

Remark 3.14 It follows from (2.7) and (3.1) that

$$\left\| y - y(U_h) \right\|_{L^\infty(0,T;L^2(\Omega))} \preceq \|u - U_h\|_{L^2(0,T;L^2(\Omega))}$$

and

$$\left\| z - z(U_h) \right\|_{L^\infty(0,T;L^2(\Omega))} \preceq \|u - U_h\|_{L^2(0,T;L^2(\Omega))}$$

Using the above estimate and Lemma 3.1, we can derive the posteriori error estimates of $\|u - U_h\|_{L^2(0,T;L^2(\Omega))} + \|y - Y_h\|_{L^\infty(0,T;L^2(\Omega))} + \|z - Z_h\|_{L^\infty(0,T;L^2(\Omega))}$.

4 Conclusion

In this paper a posteriori error estimates were established for time-dependent convection diffusion optimal control problems by the elliptic reconstruction technique. By introducing the elliptic reconstruction, we can take full advantage of the well-established a posteriori error estimates for stationary convection diffusion optimal control problems. There are still many issues needed to be addressed, such as optimal control problems with state constraints and pointwisely imposed control problems. The applications of our approach to these settings will be postponed to our future work.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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