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Nonlocal boundary value hyperbolic problems involving integral conditions

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Abstract

Stability estimates for the solution of the nonlocal boundary value problem with two integral conditions for hyperbolic equations in a Hilbert space H are established. In applications, stability estimates for the solution of the nonlocal boundary value problems for hyperbolic equations are obtained.

MSC: 35L10

Keywords: hyperbolic equation; stability; nonlocal boundary value problems

1 Introduction

It is well known that nonlocal boundary value problems with integral conditions are widely used for thermo-elasticity, chemical engineering, heat conduction, and plasma physics [1–4]. Some problems arising in dynamics of ground waters are defined as hyperbolic equations with nonlocal conditions [5] and [6]. The authors of [7] investigate nonclassical problems for multidimensional hyperbolic equation with integral boundary conditions and the uniqueness of classical solution. In [8] a linear second-order hyperbolic equation with forcing and integral constraints on the solution is converted to a nonlocal hyperbolic problem. Using the Riesz representation theorem and the Schauder fixed point theorem, existence and uniqueness of a generalized solution are proved. The solutions of hyperbolic equations with nonlocal integral conditions were investigated in [9–16]. The method of operators as a tool for investigation of the solution to hyperbolic equations in Hilbert and Banach spaces has been used extensively in [17–29].

In [30] the nonlocal boundary value problem

$$\begin{cases} \frac{d^2 u(t)}{dt^2} + Au(t) = f(t), & 0 \le t \le 1, \\ u(0) = \sum_{r=1}^n \alpha_r u(\lambda_r) + \varphi, & u_t(0) = \sum_{r=1}^n \beta_r u_t(\lambda_r) + \psi, \\ 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 1 \end{cases}$$

was investigated. Stability estimates for the solution of the problem were established. First order of accuracy difference schemes for the approximate solutions of the problem were presented. Stability estimates for the solution of these difference schemes were established. Theoretical statements were supported by numerical examples.

In the present paper, we consider the nonlocal boundary value problem with integral conditions

$$
\begin{cases}
\frac{d^2 u(t)}{dt^2} + Au(t) = f(t), & 0 \le t \le 1, \\
u(0) = \int_0^1 \alpha(\rho) u(\rho) \, d\rho + \varphi, \\
u_t(0) = \int_0^1 \beta(\rho) u_t(\rho) \, d\rho + \psi
\end{cases}
\tag{1}
$$

in a Hilbert space H with a self-adjoint positive definite operator A. We are interested in studying the stability of solutions of problem (1) under the assumption

$$
\left| 1 + \int_0^1 \alpha(s)\beta(s) \, ds \right| > \int_0^1 \big(|\alpha(s)| + |\beta(s)| \big) \, ds.
\tag{2}
$$

As in [30], a function $u(t)$ is called a *solution* of problem (1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the interval $(0,1)$ and continuously differentiable on the segment $[0,1]$.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,1]$, and the function $Au(t)$ is continuous on the segment $[0,1]$.

(iii) $u(t)$ satisfies the equation and nonlocal boundary conditions (1).

2 The main theorem

Let H be a Hilbert space, A be a positive definite self-adjoint operator with $A \ge \delta I$, where $\delta > \delta_0 > 0$. Throughout this paper, $\{c(t), t \ge 0\}$ is a strongly continuous cosine operator-function defined by

$$
c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}.
$$

Then, from the definition of sine operator-function $s(t)$,

$$
s(t)u = \int_0^t c(\rho) u \, d\rho
$$

it follows that

$$
s(t) = A^{-1/2} \frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.
$$

For the theory of cosine operator-function we refer to [23] and [24].

Lemma 2.1 *The following estimates hold*:

$$
\big\| c(t) \big\|_{H \to H} \le 1, \qquad \big\| A^{1/2} s(t) \big\|_{H \to H} \le 1.
\tag{3}
$$

Lemma 2.2 *Suppose that assumption* (2) *holds. Then the operator* T,

$$
T = \left(1 + \int_0^1 \int_0^1 \alpha(s)\beta(s) \, ds \, ds \right) I - \int_0^1 \big(\beta(s) + \alpha(s) \big) c(s) \, ds,
$$

has the inverse

$$T^{-1} = \left[\left(1 + \int_0^1 \int_0^1 \alpha(s)\beta(s)\,ds\,ds\right)I - \int_0^1 (\beta(s) + \alpha(s))c(s)\,ds\right]^{-1}$$

and the following estimate is satisfied:

$$\|T^{-1}\|_{H\to H} \leq \frac{1}{|1 + \int_0^1 \alpha(s)\beta(s)\,ds| - \int_0^1 (|\alpha(s)| + |\beta(s)|)\,ds}. \tag{4}$$

Proof Applying the triangle inequality and estimates (3), we obtain

$$\left\|\left(1 + \int_0^1 \int_0^1 \alpha(s)\beta(s)\,ds\,ds\right)I - \int_0^1 (\beta(s) + \alpha(s))c(s)\,ds\right\|_{H\to H}$$

$$\geq \left|1 + \int_0^1 \alpha(s)\beta(s)\,ds\right| - \left\|\int_0^1 (\beta(s) + \alpha(s))c(s)\,ds\right\|_{H\to H}$$

$$\geq \left|1 + \int_0^1 \alpha(s)\beta(s)\,ds\right| - \int_0^1 (|\alpha(s)| + |\beta(s)|)\|c(s)\|_{H\to H}\,ds$$

$$\geq \left|1 + \int_0^1 \alpha(s)\beta(s)\,ds\right| - \int_0^1 (|\alpha(s)| + |\beta(s)|)\,ds > 0.$$

Estimate (4) follows from this estimate. Lemma 2.2 is proved. $\qquad\square$

Now, we will obtain the formula for the solution of problem (1) for $\varphi \in D(A)$ and $\psi \in D(A^{1/2})$. It is clear that [23] the initial value problem

$$\frac{d^2u}{dt^2} + Au(t) = f(t), \quad 0 < t < 1, \qquad u(0) = u_0, \qquad u'(0) = u'_0 \tag{5}$$

has a unique solution,

$$u(t) = c(t)u_0 + s(t)u'_0 + \int_0^t s(t - \lambda)f(\lambda)\,d\lambda, \tag{6}$$

where the function $f(t)$ is not only continuous but also continuously differentiable on $[0,1]$, $u_0 \in D(A)$ and $u'_0 \in D(A^{1/2})$.

Using (6) and the nonlocal boundary condition

$$u_0 = \int_0^1 \alpha(y)u(y)\,dy + \varphi,$$

we get

$$u_0 = \int_0^1 \alpha(y)\left(c(y)u_0 + s(y)u'_0 + \int_0^y s(y - \lambda)f(\lambda)\,d\lambda\right)dy + \varphi.$$

Then

$$\left(I - \int_0^1 \alpha(y)c(y)\,dy\right)u_0 - \int_0^1 \alpha(y)s(y)\,dyu'_0 = \int_0^1 \alpha(y)\int_0^y s(y - \lambda)f(\lambda)\,d\lambda\,dy + \varphi. \tag{7}$$

Differentiating both sides of (6), we obtain

$$u'(t) = -As(t)u_0 + c(t)u_0' + \int_0^t c(t - \lambda)f(\lambda)\,d\lambda.$$

Using this formula and the integral condition

$$u_0' = \int_0^1 \beta(y)u'(y)\,dy + \psi,$$

we get

$$u_0' = \int_0^1 \beta(y)\left(-As(y)u_0 + c(y)u_0' + \int_0^y c(y - \lambda)f(\lambda)\,d\lambda\right)dy + \psi.$$

Thus,

$$\int_0^1 \beta(y)As(y)\,dy\,u_0 + \left(I - \int_0^1 \beta(y)c(y)\,dy\right)u_0'$$
$$= \int_0^1 \beta(y)\int_0^y c(y - \lambda)f(\lambda)\,d\lambda\,dy + \psi. \tag{8}$$

Now, we have a system of equations (7) and (8) for the solution of u_0 and u_0'. Solving it, we get

$$u_0 = T^{-1}\left\{\left[I - \int_0^1 \beta(y)c(y)\,dy\right]\left[\int_0^1 \alpha(y)\int_0^y s(y - \lambda)f(\lambda)\,d\lambda\,dy + \varphi\right]\right.$$
$$\left.+ \int_0^1 \alpha(y)s(y)\,dy\left[\int_0^1 \beta(y)\int_0^y c(y - \lambda)f(\lambda)\,d\lambda\,dy + \psi\right]\right\} \tag{9}$$

and

$$u_0' = T^{-1}\left\{\left[I - \int_0^1 \alpha(y)c(y)\,dy\right]\left[\int_0^1 \beta(y)\int_0^y c(y - \lambda)f(\lambda)\,d\lambda\,dy + \psi\right]\right.$$
$$\left.- \int_0^1 \beta(y)As(y)\,dy\left[\int_0^1 \alpha(y)\int_0^y s(y - \lambda)f(\lambda)\,d\lambda\,dy + \varphi\right]\right\}. \tag{10}$$

Hence, for the solution of the nonlocal boundary value problem (1) we have (6), (9), and (10).

Theorem 2.1 *Suppose that $\varphi \in D(A)$, $\psi \in D(A^{1/2})$, and $f(t)$ is a continuously differentiable on $[0, 1]$; assumption (2) holds. Then there is a unique solution of problem (1) and the following stability inequalities*:

$$\max_{0 \le t \le 1}\|u(t)\|_H \le M\left[\|\varphi\|_H + \|A^{-1/2}\psi\|_H + \int_0^1 \|A^{-1/2}f(\lambda)\|_H\,d\lambda\right], \tag{11}$$

$$\max_{0 \le t \le 1}\left\|\frac{du(t)}{dt}\right\|_H + \max_{0 \le t \le 1}\|A^{1/2}u(t)\|_H$$
$$\le M\left[\|A^{1/2}\varphi\|_H + \|\psi\|_H + \int_0^1 \|f(\lambda)\|_H\,d\lambda\right], \tag{12}$$

$$\max_{0 \le t \le 1} \left\| \frac{d^2 u(t)}{dt^2} \right\|_H + \max_{0 \le t \le 1} \|Au(t)\|_H$$

$$\le M \left\{ \|A\varphi\|_H + \|A^{1/2}\psi\|_H + \|f(0)\|_H + \int_0^1 \|f'(\lambda)\|_H \, d\lambda \right\} \tag{13}$$

are valid, where M does not depend on $f(t)$, $t \in [0,1]$, φ, and ψ.

Proof We take the estimates

$$\max_{0 \le t \le 1} \|u(t)\|_H \le \left[\|u(0)\|_H + \|A^{-1/2}u'(0)\|_H + \int_0^1 \|A^{-1/2}f(t)\|_H \, dt \right], \tag{14}$$

$$\max_{0 \le t \le 1} \|u'(t)\|_H + \max_{0 \le t \le 1} \|A^{1/2}u(t)\|_H$$

$$\le M \left[\|A^{1/2}u(0)\|_H + \|u'(0)\|_H + \int_0^1 \|f(t)\|_H \, dt \right], \tag{15}$$

$$\max_{0 \le t \le 1} \left\| \frac{d^2 u(t)}{dt^2} \right\|_H + \max_{0 \le t \le 1} \|Au(t)\|_H + \max_{0 \le t \le 1} \|A^{1/2}u'(t)\|_H$$

$$\le M \left[\|Au(0)\|_H + \|A^{1/2}u'(0)\|_H + \|f(0)\|_H + \int_0^1 \|f'(t)\|_H \, dt \right] \tag{16}$$

from [26] for the solution of problem (5). The proof of Theorem 2.1 is based on estimates (14), (15), (16), and the estimates for the norms of u_0, $A^{-1/2}u_0'$, $A^{1/2}u_0$, u_0', Au_0, $A^{1/2}u_0'$.

First of all, let us find an estimate for $\|u(0)\|_H$. By using (9) and estimates (3), (4), we obtain

$$\|u(0)\|_H \le M_1 \left[\int_0^1 \|A^{-1/2}f(\lambda)\|_H \, d\lambda + \|\varphi\|_H + \|A^{-1/2}\psi\|_H \right]. \tag{17}$$

Applying $A^{-1/2}$ to (10), we get

$$A^{-1/2}u_0' = T^{-1} \left\{ \left[I - \int_0^1 \alpha(y)c(y) \, dy \right] \left[\int_0^1 \beta(y) \int_0^y c(y - \lambda)A^{-1/2}f(\lambda) \, d\lambda \, dy + A^{-1/2}\psi \right] \right.$$

$$- \int_0^1 \beta(y)A^{1/2}s(y) \, dy$$

$$\left. \times \left[\int_0^1 \alpha(y) \int_0^y A^{1/2}s(y - \lambda)A^{-1/2}f(\lambda) \, d\lambda \, dy + \varphi \right] \right\}.$$

Using estimates (3), (4), we obtain

$$\|A^{-1/2}u'(0)\|_H \le M_2 \left[\int_0^1 \|A^{-1/2}f(\lambda)\|_H \, d\lambda + \|A^{-1/2}\psi\|_H + \|\varphi\|_H \right]. \tag{18}$$

Thus, estimates (14), (17), and (18) yield estimate (11).

Second, applying operator $A^{1/2}$ to (9), we get

$$A^{1/2}u_0 = T^{-1} \left\{ \left[I - \int_0^1 \beta(y)c(y) \, dy \right] \left[\int_0^1 \alpha(y) \int_0^y A^{1/2}s(y - \lambda)f(\lambda) \, d\lambda \, dy + A^{1/2}\varphi \right] \right.$$

$$\left. + \int_0^1 \alpha(y)A^{1/2}s(y) \, dy \left[\int_0^1 \beta(y) \int_0^y c(y - \lambda)f(\lambda) \, d\lambda \, dy + \psi \right] \right\}.$$

Using estimates (3) and (4), we obtain

$$\left\| A^{1/2} u(0) \right\|_H \le M_1 \left[\int_0^1 \left\| f(\lambda) \right\|_H d\lambda + \left\| A^{1/2} \varphi \right\|_H + \| \psi \|_H \right]. \tag{19}$$

Using (10), and estimates (3), (4), we get

$$\left\| u'(0) \right\|_H \le M_2 \left[\int_0^1 \left\| f(\lambda) \right\|_H d\lambda + \| \psi \|_H + \left\| A^{1/2} \varphi \right\|_H \right]. \tag{20}$$

Then estimate (12) follows from estimates (15), (19), and (20).

Third, applying A to (9) and using Abel's formula, we have

$$
\begin{aligned}
Au_0 = T^{-1} \Bigg\{ &\left[I - \int_0^1 \beta(y) c(y) \, dy \right] \\
&\times \left[\int_0^1 \alpha(y) \left[f(y) - c(y) f(0) - \int_0^y c(y - \lambda) f'(\lambda) \, d\lambda \right] dy + A\varphi \right] \\
&+ \int_0^1 \alpha(y) A^{1/2} s(y) \, dy \\
&\times \left[\int_0^1 \beta(y) \left[A^{1/2} s(y) f(0) + \int_0^y A^{1/2} s(y - \lambda) f'(\lambda) \, d\lambda \right] dy + A^{1/2} \psi \right] \Bigg\}
\end{aligned}
$$

and using estimates (3), (4), we get

$$\left\| Au(0) \right\|_H \le M_3 \left[\| A\varphi \|_H + \left\| A^{1/2} \psi \right\|_H + \| f(0) \|_H + \int_0^1 \left\| f'(\lambda) \right\|_H d\lambda \right]. \tag{21}$$

In the same manner, applying $A^{1/2}$ to (10) and using Abel's formula, and estimates (3), (4), we obtain

$$\left\| A^{1/2} u'(0) \right\|_H \le M_4 \left[\| A\varphi \|_H + \left\| A^{1/2} \psi \right\|_H + \| f(0) \|_H + \int_0^1 \left\| f'(\lambda) \right\|_H d\lambda \right]. \tag{22}$$

Thus, estimate (13) follows from estimates (16) and (21), and (22). □

3 Applications

Now, we consider the applications of Theorem 2.1. First, a nonlocal boundary value problem for a hyperbolic equation

$$
\begin{cases}
u_{tt} - (a(x) u_x)_x + \sigma u = f(t, x), & 0 < t < 1, 0 < x < 1, \\
u(0, x) = \int_0^1 \alpha(\rho) u(\rho, x) \, d\rho + \varphi(x), \\
u_t(0, x) = \int_0^1 \beta(\rho) u_t(\rho, x) \, d\rho + \psi(x), & 0 \le x \le 1, \\
u(t, 0) = u(t, 1), \qquad u_x(t, 0) = u_x(t, 1), & 0 \le t \le 1
\end{cases}
\tag{23}
$$

under assumption (2) is considered. Problem (23) has a unique smooth solution $u(t, x)$ for (2), smooth functions $a(x) \ge a > 0$ ($x \in (0, 1)$), $a(0) = a(1)$, $\varphi(x)$, $\psi(x)$ ($x \in [0, 1]$) and $f(t, x)$ ($t, x \in [0, 1]$), σ a positive constant and under some conditions. This allows us to reduce problem (23) to nonlocal boundary value problem (1) in the Hilbert space $H = L_2[0, 1]$ with a self-adjoint positive definite operator A^x defined by (23).

Theorem 3.1 *For the solution of problem* (23), *we have the following stability inequalities*:

$$\max_{0 \le t \le 1} \|u_x(t, \cdot)\|_{L_2[0,1]} \le M \Big[\max_{0 \le t \le 1} \|f(t, \cdot)\|_{L_2[0,1]} + \|\varphi_x\|_{L_2[0,1]} + \|\psi\|_{L_2[0,1]} \Big], \tag{24}$$

$$\max_{0 \le t \le 1} \|u_{xx}(t, \cdot)\|_{L_2[0,1]} + \max_{0 \le t \le 1} \|u_{tt}(t, \cdot)\|_{L_2[0,1]}$$

$$\le M \Big[\max_{0 \le t \le 1} \|f_t(t, \cdot)\|_{L_2[0,1]} + \|f(0, \cdot)\|_{L_2[0,1]} + \|\varphi_{xx}\|_{L_2[0,1]} + \|\psi_x\|_{L_2[0,1]} \Big], \tag{25}$$

where M does not depend on $\varphi(x)$, $\psi(x)$, *and* $f(t, x)$.

The proof of Theorem 3.1 is based on Theorem 2.1 and the symmetry properties of the space operator generated by problem (23).

Proof Problem (23) can be written in the abstract form

$$\begin{cases} \frac{d^2 u(t)}{dt^2} + Au(t) = f(t) & (0 \le t \le 1), \\ u(0) = \int_0^1 \alpha(\rho) u(\rho) \, d\rho + \varphi, \\ u_t(0) = \int_0^1 \beta(\rho) u_t(\rho) \, d\rho + \psi \end{cases} \tag{26}$$

in the Hilbert space $L_2[0,1]$ of all square integrable functions defined on $[0,1]$ with a self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -\big(a(x) u_x\big)_x + \sigma u(x)$$

with the domain

$$D\big(A^x\big) = \begin{cases} u(x) : u, u_x, (a(x) u_x)_x \in L_2[0,1], \\ \quad u(0) = u(1), u'(0) = u'(1) \end{cases}.$$

Here, $f(t) = f(t, x)$ and $u(t) = u(t, x)$ are known and unknown abstract functions defined on $[0,1]$ with the values in $H = L_2[0,1]$. Therefore, estimates (24) and (25) follow from estimates (11), (12), and (13). Thus, Theorem 3.1 is proved. □

Second, let Ω be the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m$: $\{x = (x_1, \dots, x_m) : 0 < x_j < 1, 1 \le j \le m\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, let us consider a boundary value problem for the multidimensional hyperbolic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^m (a_r(x) u_{x_r})_{x_r} + \sigma u(x) = f(t, x), \\ \quad x = (x_1, \dots, x_m) \in \Omega, 0 < t < 1, \\ u(0, x) = \int_0^1 \alpha(\rho) u(\rho, x) \, d\rho + \varphi(x), \quad x \in \overline{\Omega}, \\ u_t(0, x) = \int_0^1 \beta(\rho) u_t(\rho, x) \, d\rho + \psi(x), \quad x \in \overline{\Omega}, \\ u(t, x) = 0, \quad x \in S, \end{cases} \tag{27}$$

under assumption (2). Here, $a_r(x)$ $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$ $(x \in \overline{\Omega})$ and $f(t, x)$, $t \in (0, 1)$, $x \in \Omega$ are given smooth functions and $a_r(x) \ge a > 0$.

Let us introduce the Hilbert space $L_2(\overline{\Omega})$ of all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int_{x\in\overline{\Omega}} |f(x)|^2 \, dx_1 \cdots dx_m \right\}^{\frac{1}{2}}.$$

Theorem 3.2 *For the solution of problem* (27), *the following stability inequalities hold*:

$$\max_{0\le t\le 1} \sum_{r=1}^{m} \left\| u_{x_r}(t,\cdot) \right\|_{L_2(\overline{\Omega})}$$

$$\le M \left[\max_{0\le t\le 1} \left\| f(t,\cdot) \right\|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \|\varphi_{x_r}\|_{L_2(\overline{\Omega})} + \|\psi\|_{L_2(\overline{\Omega})} \right], \tag{28}$$

$$\max_{0\le t\le 1} \sum_{r=1}^{m} \left\| u_{x_r x_r}(t,\cdot) \right\|_{L_2(\overline{\Omega})} + \max_{0\le t\le 1} \left\| u_{tt}(t,\cdot) \right\|_{L_2(\overline{\Omega})}$$

$$\le M \left[\max_{0\le t\le 1} \left\| f_t(t,\cdot) \right\|_{L_2(\overline{\Omega})} + \left\| f(0,\cdot) \right\|_{L_2(\overline{\Omega})} \right.$$

$$\left. + \sum_{r=1}^{m} \|\varphi_{x_r x_r}\|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \|\psi_{x_r}\|_{L_2(\overline{\Omega})} \right], \tag{29}$$

where M does not depend on $\varphi(x)$, $\psi(x)$, and $f(t,x)$ ($t \in (0,1)$, $x \in \Omega$).

Proof Problem (27) can be written in the abstract form (26) in the Hilbert space $L_2(\overline{\Omega})$ with a self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = - \sum_{r=1}^{m} \left(a_r(x) u_{x_r} \right)_{x_r} + \sigma u(x)$$

with domain

$$D(A^x) = \left\{ u(x) : u, u_{x_r}, \left(a_r(x) u_{x_r} \right)_{x_r} \in L_2(\overline{\Omega}), 1 \le r \le m, u(x) = 0, x \in S \right\}.$$

Here, $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions defined on $\overline{\Omega}$ with the values in $H = L_2(\overline{\Omega})$. So, estimates (28) and (29) follow from estimates (11), (12), (13), and the following theorem. $\qquad\square$

Theorem 3.3 [29] *For the solution of the elliptic differential problem*

$$A^x u(x) = \omega(x), \quad x \in \Omega,$$

$$u(x) = 0, \quad x \in S,$$

the following coercivity inequality holds:

$$\sum_{r=1}^{m} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \le M \|\omega\|_{L_2(\overline{\Omega})},$$

where M is independent of ω.

4 Conclusion

This work is devoted to the study of the stability of the nonlocal boundary value problem with integral conditions for hyperbolic equations. For the solution of nonlocal boundary problem (1) in a Hilbert space H with a self-adjoint positive definite operator A, Theorem 2.1 is established. Two applications of Theorem 2.1 are given. Of course, stable two-step difference schemes for approximate solution of problem (1) can be presented. The methods given above permit us to establish the stability of these difference schemes. Applying [31], we can give a numerical support of the theoretical results.

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