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Some new versions of fractional boundary value problems with slit-strips conditions

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Abstract
We discuss the existence and uniqueness of solutions for a fractional differential equation of order $q \in (n-1, n]$ with slit-strips type boundary conditions. The slit-strips type boundary condition states that the sum of the influences due to finite strips of arbitrary lengths is related to the value of the unknown function at an arbitrary position (nonlocal point) in the slit (a part of the boundary off the two strips). The desired results are obtained by applying standard tools of the fixed point theory and are well illustrated with the aid of examples. We also extend our discussion to the cases of arbitrary number of nonlocal points in the slit, the nonlocal multi-substrips conditions and Riemann-Liouville type slit-strips boundary conditions.
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1 Introduction

In this paper, we introduce some new versions of boundary value problems of fractional order with slit-strips boundary conditions and discuss the existence and uniqueness of solutions for these problems. As a first problem, we consider a Caputo type fractional differential equation

$$ {}^cD^q x(t) = f\big(t, x(t)\big), \quad n-1 < q \le n, t \in [0,1], \tag{1.1}$$

supplemented with slit-strips boundary conditions

$$\begin{cases} x(0) = 0, \qquad x'(0) = 0, \qquad x''(0) = 0, \qquad \dots, \qquad x^{(n-2)}(0) = 0, \\ x(\zeta) = a\int_0^\eta x(s)\,ds + b\int_\xi^1 x(s)\,ds, \quad 0 < \eta < \zeta < \xi < 1, \end{cases} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function, and a and b are real positive constants.

In the problem (1.1)-(1.2), the integral boundary condition describes that the contribution due to finite strips of arbitrary lengths occupying the positions $(0,\eta)$ and $(\xi,1)$ on the interval $[0,1]$ is related to the value of the unknown function at a nonlocal point ζ ($\eta < \zeta < \xi$) located at an arbitrary position in the aperture (slit) - the region of the boundary off the strips. Examples of such boundary conditions include scattering by slits [1–3], silicon strips detectors for scanned multi-slit X-ray imaging [4], acoustic impedance of baffled strips radiators [5], diffraction from an elastic knife-edge adjacent to a strip [6], sound fields of infinitely long strips [7], dielectric-loaded multiple slits in a conducting plane [8], lattice engineering [9], heat conduction by finite regions (Chapter 5, [10]), *etc.*

Next, we introduce finite many points ζ_i, $1 = 1, 2, \dots, m$ on the aperture (off the strips) such that $0 < \eta < \zeta_1 < \zeta_2 < \cdots < \zeta_m < \xi < 1$ and consider the boundary conditions

$$\begin{cases} x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \quad \dots, \quad x^{(n-2)}(0) = 0, \\ \sum_{i=1}^{m} \alpha_i x(\zeta_i) = a \int_0^\eta x(s)\,ds + b \int_\xi^1 x(s)\,ds, \quad \alpha_i \in \mathbb{R}. \end{cases} \tag{1.3}$$

In the third problem, we discuss the situation when the strips occupying the positions $(0, \eta)$ and $(\xi, 1)$ of the domain $[0, 1]$ consist of finitely many non-intersecting substrips and discrete sums of their contributions relates to the value of the unknown function at a nonlocal point ζ $(\eta < \zeta < \xi)$ on the aperture. Precisely, we have the following boundary conditions:

$$\begin{cases} x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \quad \dots, \quad x^{(n-2)}(0) = 0, \\ x(\zeta) = \sum_{i=1}^{m} \delta_i \int_{\eta_i}^{\sigma_i} x(s)\,ds + \sum_{i=1}^{p} \beta_j \int_{\xi_j}^{\rho_j} x(s)\,ds, \end{cases} \tag{1.4}$$

where $0 = \eta_1 < \sigma_1 < \eta_2 < \sigma_2 < \cdots < \eta_m < \sigma_m < \zeta < \xi_1 < \rho_1 < \xi_2 < \rho_2 < \cdots < \xi_p < \rho_p = 1$, with $\sigma_m = \eta$, $\xi_1 = \xi$, $\delta_i, \beta_j \in \mathbb{R}$.

Finally, we introduce Riemann-Liouville type slit-strips boundary conditions

$$\begin{cases} x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \quad \dots, \quad x^{(n-2)}(0) = 0, \\ x(\zeta) = a \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} x(s)\,ds + b \int_\xi^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} x(s)\,ds, \quad 0 < \eta < \zeta < \xi < 1, \beta > 0. \end{cases} \tag{1.5}$$

Boundary value problems for nonlinear differential equations arise in a variety of areas such as applied mathematics, physics, and variational problems of control theory. In recent years, the study of boundary value problems of fractional order has attracted the attention of many scientists and researchers and the subject has been developed in several disciplines. Significant development of the topic over the past few years clearly indicates its popularity. As a matter of fact, the literature on the topic is now well enriched with a variety of results covering theoretical as well as application aspects of the subject. In consequence, fractional calculus has evolved as an interesting topic of research and its tools have played a key role in improving the mathematical modeling of many physical and engineering phenomena. The nonlocal nature of fractional-order operators is one of the salient features accounting for the practical utility of the subject. With the aid of fractional calculus, it has now become possible to trace the history of many important materials and processes. For examples and application details in physics, chemistry, biology, biophysics, blood flow phenomena, control theory, wave propagation, signal and image processing, viscoelasticity, percolation, identification, fitting of experimental data, economics, *etc.*, we refer the reader to the books [11–13]. For some recent work on the topic, see [14–48] and the references therein.

We emphasize that the problems considered in this paper are new and important from application point of view. The existence theory developed for the fractional differential equation (1.1) subject to the boundary conditions (1.2), (1.3), (1.4) and (1.5) will lead to a useful and significant contribution to the existing material on nonlocal fractional-order boundary value problems.

2 Existence results for the problem (1.1)-(1.2)

First of all, we recall some basic definitions of fractional calculus.

Definition 2.1 The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\, ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2 Let $h : [0,\infty) \to \mathbb{R}$ be such that $h \in AC^n[0,T]$. Then the Caputo derivative of fractional order ν for h is defined as

$${}^cD^\nu h(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} h^{(n)}(s)\, ds, \quad n-1 < \nu < n, n = [\nu] + 1,$$

where $[\nu]$ denotes the integer part of the real number ν.

Next we present an auxiliary lemma which plays a key role in proving the main results for the problem (1.1)-(1.2).

Lemma 2.1 *For $y \in C[0,1]$, the solution of the linear fractional differential equation*

$${}^cD^q x(t) = y(t), \quad n-1 < q \le n, t \in [0,1], \tag{2.1}$$

supplemented with the boundary conditions (1.2) *is equivalent to the integral equation*

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds + \frac{t^{n-1}}{A} \left\{ a \int_0^\eta \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)\, du \right) ds \right. \\ & \left. + b \int_\xi^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)\, du \right) ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} y(s)\, ds \right\}, \end{aligned} \tag{2.2}$$

where

$$A = \zeta^{n-1} - \frac{1}{n}\left(b + a\eta^n - b\xi^n\right) \neq 0. \tag{2.3}$$

Proof It is well known that the general solution of the fractional differential equation (2.1) can be written as

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds, \tag{2.4}$$

where $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$ are arbitrary constants.

Applying the boundary conditions (1.2), we find that $c_0 = c_1 = \cdots = c_{n-2} = 0$, and

$$c_{n-1} = \frac{1}{A}\left\{a\int_0^\eta\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\right)ds + b\int_\xi^1\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\right)ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}y(s)\,ds\right\}.$$

Substituting the values of $c_0, c_1, \ldots, c_{n-1}$ in (2.4), we get (2.2). This completes the proof. □

Let $\mathcal{P} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm: $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

In relation to the given problem, we define an operator $\mathcal{H} : \mathcal{P} \to \mathcal{P}$ as

$$\begin{aligned}(\mathcal{H}x)(t) = {}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f\bigl(s,x(s)\bigr)\,ds + \frac{t^{n-1}}{A}\left\{a\int_0^\eta\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}f\bigl(u,x(u)\bigr)\,du\right)ds\right.\\ &\left. + b\int_\xi^1\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}f\bigl(u,x(u)\bigr)\,du\right)ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}f\bigl(s,x(s)\bigr)\,ds\right\},\end{aligned}$$

where A is given by (2.3). Observe that the problem (1.1)-(1.2) has solutions if and only if the operator $\mathcal{H}$ has fixed points.

For the sake of computational convenience, we set

$$\rho = \frac{1}{|A|\Gamma(q+2)}\bigl\{(q+1)\bigl[|A| + \zeta^q\bigr] + |a|\eta^{q+1} + |b|\bigl(1-\xi^{q+1}\bigr)\bigr\}. \tag{2.5}$$

Theorem 2.1 *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition*:

(H_1) $|f(t,x) - f(t,y)| \le \ell|x-y|$, $\forall t \in [0,1]$, $x,y \in \mathbb{R}$, $\ell > 0$.

Then the problem (1.1)-(1.2) *has a unique solution if $\ell\rho < 1$, where ρ is given by* (2.5).

Proof Setting $\sup_{t\in[0,1]}|f(t,0)| = \wp$, we show that $\mathcal{H}B_r \subset B_r$, where $B_r = \{x \in \mathcal{P} : \|x\| \le r\}$ with $r > \rho\wp(1-\rho\ell)^{-1}$. For $x \in B_r$, $t \in [0,1]$, using $|f(s,x(s))| = |f(s,x(s)) - f(s,0) + f(s,0)| \le \ell r + \wp$, we get

$$\begin{aligned}\bigl\|(\mathcal{H}x)\bigr\| \le {}& \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\bigl|f\bigl(s,x(s)\bigr) - f(s,0) + f(s,0)\bigr|\,ds\right.\\ &+ \frac{t^{n-1}}{A}\left[|a|\int_0^\eta\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}\bigl|f\bigl(s,x(s)\bigr) - f(s,0) + f(s,0)\bigr|\,du\right)ds\right.\\ &+ |b|\int_\xi^1\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}\bigl|f\bigl(s,x(s)\bigr) - f(s,0) + f(s,0)\bigr|\,du\right)ds\\ &\left.\left.+ \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)}\bigl|f\bigl(s,x(s)\bigr) - f(s,0) + f(s,0)\bigr|\,ds\right]\right\}\\ \le {}& (\ell r + \wp)\sup_{t\in[0,1]}\left\{\frac{t^q}{\Gamma(q+1)} + \frac{t^{n-1}}{|A|}\left[\frac{|a|\eta^{q+1} + |b|(1-\xi^{q+1})}{\Gamma(q+2)} + \frac{\zeta^q}{\Gamma(q+1)}\right]\right\}\\ \le {}& (\ell r + \wp)\rho \le r,\end{aligned}$$

which implies that $\mathcal{H}B_r \subset B_r$, where we have used (2.5).

Now, for $x, y \in \mathcal{P}$, and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
\|\mathcal{H}x - \mathcal{H}y\| \leq \sup_{t\in[0,1]} \bigg\{ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big| f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\, ds \\
& + \frac{t^{n-1}}{A}\bigg[|a| \int_0^\eta \bigg(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \big| f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\, du \bigg)\, ds \\
& + |b| \int_\xi^1 \bigg(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \big| f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\, du \bigg)\, ds \\
& + \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} \big| f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\, ds \bigg]\bigg\} \\
\leq {} & \ell\rho\|x-y\|.
\end{aligned}
$$

Since $\ell\rho < 1$ by the given assumption, therefore the operator $\mathcal{H}$ is a contraction. Thus, by Banach's contraction mapping principle, there exists a unique solution for the problem (1.1)-(1.2). This completes the proof. □

The next result is based on Krasnoselskii's fixed point theorem [49].

Theorem 2.2 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (H_1) *and*

(H_2) $|f(t,x)| \leq \mu(t)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$, *and* $\mu \in ([0,1], \mathbb{R}^+)$.

Then the problem (1.1)-(1.2) *has at least one solution on* $[0,1]$ *if*

$$
\frac{\ell}{|A|\Gamma(q+2)} \big\{ (q+1)\zeta^q + |a|\eta^{q+1} + |b|\big(1-\xi^{q+1}\big)\big\} < 1. \tag{2.6}
$$

Proof Fixing $r \geq \|\mu\|\rho$, we consider $B_r = \{x \in \mathcal{P} : \|x\| \leq r\}$ and define the operators ϕ_1 and ϕ_2 on B_r as

$$
\begin{aligned}
(\phi_1 x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\, ds, \\
(\phi_2 x)(t) &= \frac{t^{n-1}}{A} \bigg\{ a \int_0^\eta \bigg(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\, du \bigg)\, ds \\
&\quad + b \int_\xi^1 \bigg(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\, du \bigg)\, ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\, ds \bigg\}.
\end{aligned}
$$

For $x, y \in B_r$, it is easy to show that $\|(\phi_1 x) + (\phi_2 y)\| \leq \|\mu\|\rho \leq r$, which implies that $\phi_1 x + \phi_2 y \in B_r$.

In view of the condition (2.6), the operator ϕ_2 is a contraction. Continuity of f implies that the operator ϕ_1 is continuous. Also, ϕ_1 is uniformly bounded on B_r as $\|\phi_1 x\| \leq \|\mu\|/\Gamma(q+1)$. Moreover, with $\sup_{(t,x)\in[0,1]\times B_r} |f(t,x)| = f_m < \infty$ and $t_1 < t_2$, $t_1, t_2 \in (0,T]$, we have

$$
\big\|(\phi_1 x)(t_2) - (\phi_1 x)(t_1)\big\| \leq \frac{f_m}{\Gamma(q+1)} \big({t_1}^q - {t_2}^q + 2(t_2-t_1)^q \big),
$$

which tends to zero independent of x as $t_1 \to t_2$. This implies that ϕ_1 is relatively compact on B_r. Hence, by the Arzelá-Ascoli theorem, Φ is compact on B_r. Thus all the assumptions of Krasnoselskii's fixed point theorem are satisfied. So the problem (1.1)-(1.2) has at least one solution on $[0,1]$. This completes the proof. □

Our next result is based on the following fixed point theorem [49].

Theorem 2.3 *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X | u = \epsilon Tu, 0 < \epsilon < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 2.4 *Assume that exists a positive constant L_1 such that $|f(t,x)| \leq L_1$ for all $t \in [0,1]$, $x \in \mathcal{P}$. Then there exists at least one solution for the problem* (1.1)-(1.2).

Proof As a first step, we show that the operator $\mathcal{H}$ is completely continuous. Clearly continuity of $\mathcal{H}$ follows from the continuity of f. Let $D \subset \mathcal{H}$ be bounded. Then, $\forall x \in D$, it is easy to establish that $|(\mathcal{H}x)(t)| \leq L_1\rho = L_2$. Furthermore, we find that

$$\begin{aligned}
\left|(\mathcal{H}x)'(t)\right| &= \left| \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} f\big(s,x(s)\big)\,ds \right. \\
&\quad + \frac{(n-1)t^{n-2}}{A}\left\{ a\int_0^\eta \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du \right) ds \right. \\
&\quad \left.\left. + b\int_\xi^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du \right) ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds \right\} \right| \\
&\leq L_1\left[\frac{1}{\Gamma(q)} + \frac{1}{|A|\Gamma(q+2)}\left\{ (q+1)\zeta^q + |a|\eta^{q+1} + |b|\big(1-\xi^{q+1}\big) \right\} \right] \\
&= L_3.
\end{aligned}$$

Hence, for $t_1, t_2 \in [0,1]$, it follows that

$$\left|(\mathcal{H}x)(t_1) - (\mathcal{H}x)(t_2)\right| \leq \int_{t_1}^{t_2} \left|(\mathcal{H}x)'(s)\right|\,ds \leq L_3(t_2 - t_1).$$

Therefore, $\mathcal{H}$ is equicontinuous on $[0,1]$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{H}$ is completely continuous.

Next, we consider the set $V = \{x \in \mathcal{P} : x = \epsilon\mathcal{H}x, 0 < \epsilon < 1\}$. To show that V is bounded, let $x \in V$, $t \in [0,1]$. Then

$$\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds + \frac{t^{n-1}}{A}\left\{ a\int_0^\eta \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du \right) ds \right. \\
&\quad \left. + b\int_\xi^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du \right) ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds \right\},
\end{aligned}$$

and $|x(t)| = \epsilon|(\mathcal{H}x)(t)| \leq L_1\rho = L_2$. Hence, $\|x\| \leq L_2$, $\forall x \in V$, $t \in [0,1]$. So V is bounded. Thus, Theorem 2.3 applies and in consequence the problem (1.1)-(1.2) has at least one solution. This completes the proof. □

Our final result is based on the Leray-Schauder nonlinear alternative.

Lemma 2.2 (Nonlinear alternative for single valued maps [50]) *Let E be a Banach space, E_1 a closed, convex subset of E, V an open subset of E_1, and $0 \in V$. Suppose that $\mathcal{U} : \overline{V} \to E_1$ is a continuous, compact (that is, $\mathcal{U}(\overline{V})$ is a relatively compact subset of E_1) map. Then either*

(i) *$\mathcal{U}$ has a fixed point in $\overline{V}$, or*
(ii) *there is a $x \in \partial V$ (the boundary of V in E_1) and $\kappa \in (0,1)$ with $x = \kappa\mathcal{U}(x)$.*

Theorem 2.5 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_3) *there exist a function $p \in \mathcal{C}([0,1],\mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,x)| \leq p(t)\psi(\|x\|)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$;*
(H_4) *there exists a constant $M > 0$ such that*

$$M\left[\frac{\psi(M)\|p\|}{|A|\Gamma(q+2)}\left\{(q+1)\left[|A| + \zeta^q\right] + |a|\eta^{q+1} + |b|\left(1 - \xi^{q+1}\right)\right\}\right]^{-1} > 1.$$

Then the problem (1.1)-(1.2) *has at least one solution on* $[0,1]$.

Proof Let us consider the operator $\mathcal{H} : \mathcal{P} \to \mathcal{P}$ defined by (3.1) and show that $\mathcal{H}$ maps bounded sets into bounded sets in $\mathcal{P}$. For a positive number r, let $B_r = \{x \in \mathcal{P} : \|x\| \leq r\}$ be a bounded set in $\mathcal{P}$. Then, for $x \in B_r$ together with (H_3), we obtain

$$\begin{aligned}
\left|(\mathcal{H}x)(t)\right| \leq{}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\psi\left(\|x\|\right) ds + \frac{t^{n-1}}{A}\left\{a\int_0^\eta \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} p(s)\psi\left(\|x\|\right) du\right) ds\right. \\
&\left. + b\int_\xi^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} p(s)\psi\left(\|x\|\right) du\right) ds + \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} p(s)\psi\left(\|x\|\right) ds\right\} \\
\leq{}& \frac{\psi(M)\|p\|}{|A|\Gamma(q+2)}\left\{(q+1)\left[|A| + \zeta^q\right] + |a|\eta^{q+1} + |b|\left(1 - \xi^{q+1}\right)\right\}.
\end{aligned}$$

Next, it will be shown that $\mathcal{H}$ maps bounded sets into equicontinuous sets of $\mathcal{P}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_r$. Then

$$\begin{aligned}
&\left|(\mathcal{H}x)(t_2) - (\mathcal{H}x)(t_1)\right| \\
&\quad = \left|\int_0^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f\left(s,x(s)\right) ds - \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} f\left(s,x(s)\right) ds\right. \\
&\qquad + \frac{(t_2^{n-1} - t_2^{n-1})}{A}\left\{a\int_0^\eta \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\left(u,x(u)\right) du\right) ds\right. \\
&\qquad \left.\left. + b\int_\xi^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\left(u,x(u)\right) du\right) ds - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f\left(s,x(s)\right) ds\right\}\right| \\
&\quad \leq \psi(r)\|p\|\left[\frac{|t_2^q - t_1^q|}{\Gamma(q+1)} + \frac{|t_2^{n-1} - t_2^{n-1}|}{|A|\Gamma(q+2)}\left\{(q+1)\zeta^q + |a|\eta^{q+1} + |b|\left(1 - \xi^{q+1}\right)\right\}\right].
\end{aligned}$$

Clearly, the right-hand side tends to zero independently of $x \in B_r$ as $t_2 \to t_1$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{H}$ is completely continuous.

Let x be a solution for the given problem. Then, for $\lambda \in (0,1)$, using the method of computation employed to show the boundedness of the operator $\mathcal{H}$, we obtain

$$\|x(t)\| = \|\lambda(\mathcal{H}x)(t)\| \leq \frac{\psi(\|x\|)\|p\|}{|A|\Gamma(q+2)}\{(q+1)[|A|+\zeta^q]+|a|\eta^{q+1}+|b|(1-\xi^{q+1})\},$$

which implies that

$$\|x\|\left[\frac{\psi(\|x\|)\|p\|}{|A|\Gamma(q+2)}\{(q+1)[|A|+\zeta^q]+|a|\eta^{q+1}+|b|(1-\xi^{q+1})\}\right]^{-1} \leq 1.$$

In view of (H_4), there exists M such that $\|x\| \neq M$. Let us choose $\mathcal{N} = \{x \in \mathcal{P} : \|x\| < M+1\}$.

Observe that the operator $\mathcal{H} : \overline{\mathcal{N}} \to \mathcal{P}$ is continuous and completely continuous. From the choice of $\mathcal{N}$, there is no $x \in \partial\mathcal{N}$ such that $x = \lambda\mathcal{H}(x)$ for some $\lambda \in (0,1)$. Consequently, by Lemma 2.2, we deduce that the operator $\mathcal{H}$ has a fixed point $x \in \overline{\mathcal{N}}$ which is a solution of the problem (1.1)-(1.2). This completes the proof. □

Example 2.1 Consider a fractional boundary value problem given by

$$\begin{cases} {}^cD^{\frac{9}{2}}x(t) = e^t\tan^{-1}x + \frac{1}{\sqrt{(t+4)}}\sin x + (t+1)^{3/2}, \quad t \in [0,1], \\ x(0) = 0, \qquad x'(0) = 0, \qquad x''(0) = 0, \qquad x'''(0), \\ x(1/2) = a\int_0^{1/3} x(s)\,ds + b\int_{2/3}^1 x(s)\,ds. \end{cases} \tag{2.7}$$

Here, $q = 9/2$, $a = 1$, $b = 1$, $\eta = 1/3$, $\zeta = 1/2$, $\xi = 2/3$, and $f(t,x) = e^t\tan^{-1}x + \frac{1}{\sqrt{(t+4)}}\sin x + (t+1)^{3/2}$. With the given data, $\ell = e + 1/2$,

$$|A| = \left|\zeta^{n-1} - \frac{1}{n}(b + a\eta^n - b\xi^n)\right| = 0.111986,$$

$$\rho = \frac{1}{|A|\Gamma(q+2)}\{(q+1)[|A|+\zeta^q]+|a|\eta^{q+1}+|b|(1-\xi^{q+1})\} = 0.0544011.$$

Clearly $\ell < 1/\rho$. Thus all the conditions of Theorem 2.1 are satisfied and, consequently, there exists a unique solution for the problem (2.7).

Example 2.2 Consider the problem (2.7) with

$$f(t,x) = \frac{(t+2)(x+2)^2}{1+(x+2)^2} + (t+1)\sin x. \tag{2.8}$$

Clearly $|f(t,x)| \leq p(t)\psi(|x|)$, where $p(t) = (t+2)$, $\psi(|x|) = (1+\|x\|)$. By the assumption:

$$M\left[\frac{\psi(M)\|p\|}{|A|\Gamma(q+2)}\{(q+1)[|A|+\zeta^q]+|a|\eta^{q+1}+|b|(1-\xi^{q+1})\}\right]^{-1} > 1,$$

we find that $M > 0.195033$. Thus, by Theorem 2.5, there exists at least one solution for the problem (2.7) with $f(t,x)$ given by (2.8).

3 Nonlocal multi-point case on the aperture

As argued in Lemma 2.1, the integral solution of the problem (1.1) and (1.3) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds + \frac{t^{n-1}}{A_m}\left\{a\int_0^\eta \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right)ds \right.$$
$$\left. + b\int_\xi^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right)ds - \sum_{i=1}^m \alpha_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds\right\},$$

where

$$A_m = \sum_{i=1}^m \alpha_i \zeta_i^{n-1} - \frac{1}{n}\big(b + a\eta^n - b\xi^n\big) \neq 0. \tag{3.1}$$

The value analog to ρ (given by (2.5)), denoted by ρ_m is

$$\rho_m = \frac{1}{|A_m|\Gamma(q+2)}\left\{(q+1)\left[|A_m| + \sum_{i=1}^m \alpha_i \zeta_i^q\right] + |a|\eta^{q+1} + |b|\big(1-\xi^{q+1}\big)\right\}. \tag{3.2}$$

The results analogous to Section 2 for the problem (1.1) and (1.3) can be obtained by replacing the operator $\mathcal{H}$ with $\mathcal{H}_m$ in Section 2, where

$$(\mathcal{H}_m x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\,ds + \frac{t^{n-1}}{A_m}\left\{a\int_0^\eta \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)\,du\right)ds \right.$$
$$\left. + b\int_\xi^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)\,du\right)ds - \sum_{i=1}^m \alpha_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{q-1}}{\Gamma(q)} y(s)\,ds\right\}.$$

4 Nonlocal multi-substrips case

As before (see Lemma 2.1), the integral solution of the problem (1.1) and (1.4) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds + \frac{t^{n-1}}{A_{mp}}\left\{\sum_{i=1}^m \delta_i \int_{\eta_i}^{\sigma_i} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right)ds \right.$$
$$+ \sum_{j=1}^p \beta_i \int_{\xi_i}^{\rho_i} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right)ds$$
$$\left. - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds\right\}, \tag{4.1}$$

where

$$A_{mp} = \zeta^{n-1} - \frac{1}{n}\left[\sum_{i=1}^m \delta_i\big(\sigma_i^n - \eta_i^n\big) + \sum_{j=1}^p \beta_j\big(\rho_j^n - \xi_i^n\big)\right] \neq 0. \tag{4.2}$$

In this case, the value analog to ρ (given by (2.5)), denoted by ϱ_{mp} is

$$\begin{aligned} \varrho_{mp} &= \frac{1}{\Gamma(q+1)} \\ &\quad + \frac{1}{A_{mp}\Gamma(q+2)}\left[\sum_{i=1}^{m}\delta_i\left(\sigma_i^{q+1}-\eta_i^{q+1}\right)+\sum_{j=1}^{p}\beta_j\left(\rho_j^{q+1}-\xi_i^{q+1}\right)+(q+1)\zeta^q\right]. \end{aligned} \tag{4.3}$$

In view of (4.1), we define a fixed point operator equation equivalent to the problem (1.1)-(1.4) as $\mathcal{H}_{mp}x = x$, where the operator $\mathcal{H}_{mp} : \mathcal{P} \to \mathcal{P}$ is given by

$$\begin{aligned} (\mathcal{H}_{mp}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds \\ &\quad + \frac{t^{n-1}}{A_{mp}}\left\{\sum_{i=1}^{m}\delta_i\int_{\eta_i}^{\sigma_i}\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right)ds\right. \\ &\quad + \sum_{j=1}^{p}\beta_i\int_{\xi_i}^{\rho_i}\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right)ds \\ &\quad \left. - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds\right\}, \end{aligned}$$

where A_{mp} is given by (4.2). With the above setting, the existence results, analogous to the ones given in Section 2, can be obtained for the problem (1.1) and (1.4) in a similar way.

5 Riemann-Liouville slit-strips boundary conditions

The integral solution of the problem (1.1) and (1.5) is

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds \\ &\quad + \frac{t^{n-1}}{A_{RL}}\left\{a\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right)ds\right. \\ &\quad + b\int_\xi^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right)ds \\ &\quad \left. - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds\right\}, \end{aligned} \tag{5.1}$$

where

$$A_{RL} = \left(\zeta^{n-1} - a\frac{\eta^{\beta+n-1}\Gamma(n)}{\Gamma(\beta+n)} - b\Delta\right) \neq 0, \tag{5.2}$$

$$\begin{aligned} \Delta &= \frac{(1-\xi)^\beta\xi^{n-1}}{\Gamma(\beta+1)} + \frac{(n-1)(1-\xi)^{\beta+1}\xi^{n-2}}{\Gamma(\beta+2)} + \frac{(n-1)(n-2)(1-\xi)^{\beta+2}\xi^{n-3}}{\Gamma(\beta+3)} \\ &\quad + \cdots + \frac{\Gamma(n)(1-\xi)^{\beta+n-1}}{\Gamma(\beta+n)}. \end{aligned}$$

In this case, the operator equation is $\mathcal{H}_{RL}x = x$, where the operator $\mathcal{H}_{RL} : \mathcal{P} \to \mathcal{P}$ is given by

$$\begin{aligned}
(\mathcal{H}_{RL}x)(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds \\
& + \frac{t^{n-1}}{A_{RL}} \biggl\{ a \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \biggl(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\bigl(u,x(u)\bigr)\,du \biggr)\,ds \\
& + b \int_\xi^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \biggl(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\bigl(u,x(u)\bigr)\,du \biggr)\,ds \\
& - \int_0^\zeta \frac{(\zeta-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds \biggr\}.
\end{aligned}$$

The existence results for the problem (1.1) and (1.5) can be obtained by employing the procedure used in Section 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA and RPA contributed to each part of this work equally and read and approved the final version of the manuscript.

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