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Solvability for system of nonlinear singular differential equations with integral boundary conditions

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Abstract

By using the Banach contraction principle and the Leggett-Williams fixed point theorem, this paper investigates the uniqueness and existence of at least three positive solutions for a system of mixed higher-order nonlinear singular differential equations with integral boundary conditions:

$$\begin{cases} u^{(n_1)}(t) + a_1(t)f_1(t,u(t),v(t)) = 0, & 0 < t < 1, \\ v^{(n_2)}(t) + a_2(t)f_2(t,u(t),v(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n_1-2)}(0) = 0, & u(1) = g_1(\beta_1[u],\beta_1[v]), \\ v(0) = v'(0) = \cdots = v^{(n_2-2)}(0) = 0, & v(1) = g_2(\beta_2[u],\beta_2[v]), \end{cases}$$

where the nonlinear terms f_i, g_i satisfy some growth conditions, $\beta_i[\cdot]$ are linear functionals given by $\beta_i[w] = \int_0^1 w(s)\,\mathrm{d}\phi_i(s)$, involving Stieltjes integrals with positive measures, and $i = 1,2$. We give an example to illustrate our result.

MSC: 34B16; 34B18

Keywords: positive solutions; integral boundary conditions; higher-order differential equations; fixed point theorem

1 Introduction

The purpose of this paper is to establish the uniqueness and existence of at least three positive solutions for a system of mixed higher-order nonlinear singular differential equations with integral boundary conditions,

$$\begin{cases} u^{(n_1)}(t) + a_1(t)f_1(t,u(t),v(t)) = 0, & 0 < t < 1, \\ v^{(n_2)}(t) + a_2(t)f_2(t,u(t),v(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n_1-2)}(0) = 0, & u(1) = g_1(\beta_1[u],\beta_1[v]), \\ v(0) = v'(0) = \cdots = v^{(n_2-2)}(0) = 0, & v(1) = g_2(\beta_2[u],\beta_2[v]), \end{cases} \tag{1.1}$$

where $n_i \geq 3$, $a_i(t) \in C((0,1),[0,+\infty))$, $a_i(t)$ are allowed to be singular at $t = 0$ and/or $t = 1$, $f_i \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, $g_i \in C([0,+\infty)\times[0,+\infty),[0,+\infty))$, $a_i(t)f_i(t,0,0)$ do not vanish identically on any subinterval of $(0,1)$, the functionals $\beta_i[\cdot]$ are linear functionals given by $\beta_i[w] = \int_0^1 w(s)\,\mathrm{d}\phi_i(s)$, involving Stieltjes integrals with positive measures, and $i = 1,2$.

The theory of boundary value problems with integral conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, and plasma physics can be reduced to boundary value problems with integral conditions, which included, as special cases, two-point, three-point and multi-point boundary value problems considered by many authors (see [1–5]).

In recent years, to the best of our knowledge, although there are many papers concerning the existence of positive solutions for nth order boundary value problems with different kinds of boundary conditions for system (see [6–10] and the references therein), results for the system (1.1) are rarely seen. Moreover, the methods mainly depend on the Krasonsel'skii fixed point theorem, fixed point index theory, the upper and lower solution technique, some new fixed point theorem for cones, *etc.* For example, in [7], by applying the Krasonsel'skii fixed point theorem, Henderson and Ntouyas studied the existence of at least one positive solution for the following system:

$$\begin{cases} u^{(n)}(t) + \lambda h_1(t)f_1(v(t)) = 0, \quad 0 < t < 1, \\ v^{(n)}(t) + \lambda h_2(t)f_2(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \alpha u(\eta), \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \qquad v(1) = \alpha u(\eta). \end{cases} \tag{1.2}$$

In [9], by using fixed point index theory, Xu and Yang extended the results of [7, 8] and established the existence of at least one and two positive solutions for the following system:

$$\begin{cases} u^{(n)}(t) + a_1(t)f_1(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ v^{(n)}(t) + a_2(t)f_2(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = 0, \\ v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \qquad v(1) = 0, \end{cases} \tag{1.3}$$

where $h_i(t)$ and $a_i(t)$ are nonsingular. In [10], $u(1) = 0$, $v(1) = 0$ of the system (1.3) are replaced by $u(1) = \alpha u(\eta)$, $v(1) = \beta v(\eta)$, and in [6], $u(1) = 0$, $v(1) = 0$ of the system (1.3) are replaced by $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i)$, $v(1) = \sum_{i=1}^{m-2} \beta_i v(\eta_i)$, where $a_i(t)$ is singular. By using fixed point index theory and the Krasonsel'skii fixed point theorem, the existence of one and/or two positive solutions is established.

On the other hand, Webb [11] gave a unified method of tackling many nonlocal boundary value problems, which have been applied to the study of the problem with Stieltjes integrals,

$$\begin{cases} u^{(n)}(t) + g_1(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \alpha[u]. \end{cases} \tag{1.4}$$

We mention that Stieltjes integrals are also used in the framework of nonlinear boundary conditions in several papers (see [12–17] and the references therein). In particular, Yang [12] studied the existence of positive solutions for the following system by using fixed point index theory in a cone:

$$\begin{cases} u''(t) + f_1(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ v''(t) + f_2(t,u(t),v(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = H_1(\int_0^1 u(s)\,\mathrm{d}B_1(s)), \\ v(0) = 0, \qquad v(1) = H_2(\int_0^1 v(s)\,\mathrm{d}B_2(s)). \end{cases} \tag{1.5}$$

Infante and Pietramala [14] studied the following system as a special case to illustrate the obtained theory:

$$\begin{cases} u''(t) + f_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\ v''(t) + f_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\ u(0) = H_{11}(\beta_{12}[u]), & u(1) = H_{12}(\beta_{12}[u]), \\ v(0) = H_{21}(\beta_{21}[v]), & v(1) = H_{22}(\beta_{21}[v]). \end{cases} \tag{1.6}$$

By constructing a special cone and using fixed point index theory, Cui and Sun [15] studied the existence of at least one positive solution for the system with Stieltjes integrals,

$$\begin{cases} u''(t) + f_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\ v''(t) + f_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \beta_{12}[v], \\ v(0) = 0, & v(1) = \beta_{22}[u]. \end{cases} \tag{1.7}$$

By using fixed point index theory and *a priori* estimates achieved by utilizing some properties of concave functions, Xu and Yang [16] showed the existence and multiplicity positive solutions for the system of the generalized Lidstone problems, where the system are mixed higher-order differential equations.

Motivated by the work of the above papers, we aim to investigate the solvability for the system (1.1). The main features are as follows: Firstly, the method we adopt, which has been widely used, is different from [5–12, 14–17]. Secondly, the nonlinear terms f_i we considered here satisfy some growth conditions. In [6–8, 10, 11, 15, 17], the sublinear or superlinear conditions are used for f_i. Moreover, the form of the Stieltjes integrals we consider here is quite general, which involves that of the Stieltjes integrals in [11–13, 15, 17] and is different from [14]. This implies that the case of boundary conditions (1.1) covers the multi-point boundary conditions and also the integral boundary conditions in a single framework.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and several lemmas. In Section 3, by applying the fixed-point theorem, we obtain the uniqueness and existence of at least three positive solutions for the system (1.1). In Section 4, we give an example to illustrate our result.

2 Preliminaries and lemmas

Definition 2.1 Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone, which satisfies the following conditions:

(1) $\forall x \in P, \lambda > 0 \Rightarrow \lambda x \in P$;
(2) $x, -x \in P \Rightarrow x = 0$.

Definition 2.2 Let E be a real Banach space with cone P. A map $\beta : P \to [0, +\infty)$ is said to be a non-negative continuous concave functional on P if β is continuous and

$$\beta\bigl(tx + (1-t)y\bigr) \geq t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$ and $t \in [0,1]$.

Let a, b be two numbers such that $0 < a < b$ and β be a non-negative continuous concave functional on P. We define the following convex sets:

$$P_a = \bigl\{x \in P : \|x\| < a\bigr\}, \qquad P(\beta, a, b) = \bigl\{x \in P : a \leq \beta(x), \|x\| \leq b\bigr\}.$$

Lemma 2.3 (see [18]) *Let $A : \overline{P}_c \to \overline{P}_c$ be completely continuous operator and β be a nonnegative continuous concave functional on P such that $\beta(x) \le \|x\|$ for $x \in \overline{P}_c$. Suppose there exist $0 < a < b < d \le c$ such that*

(A_1) $\{x \in P(\beta, b, d) : \beta(x) > b\} \neq \phi$ *and* $\beta(Ax) > b$ *for* $x \in P(\beta, b, d)$,
(A_2) $\|Ax\| < a$ *for* $\|x\| \le a$,
(A_3) $\beta(Ax) > b$ *for* $x \in P(\beta, b, c)$ *with* $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2, x_3 in $\overline{P}_c$ such that

$$\|x_1\| < a, \qquad b < \beta(x_2) \quad \text{and} \quad \|x_3\| > a \quad \text{with } \beta(x_3) < b.$$

Definition 2.4 $(u, v) \in C^{n_1}([0,1], [0, +\infty)) \times C^{n_2}([0,1], [0, +\infty))$ is said to be a positive solution of the system (1.1) if and only if (u, v) satisfies the system (1.1) and $u(t) \ge 0$, $v(t) \ge 0$, for any $t \in [0,1]$.

Lemma 2.5 *Let $x(t), y(t) \in C[0,1]$, then the boundary value problem*

$$\begin{cases} u^{(n_1)}(t) + x(t) = 0, \qquad v^{(n_2)}(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n_1-2)}(0) = 0, \qquad u(1) = g_1(\beta_1[u], \beta_1[v]), \\ v(0) = v'(0) = \cdots = v^{(n_2-2)}(0) = 0, \qquad v(1) = g_2(\beta_2[u], \beta_2[v]), \end{cases} \tag{2.1}$$

has the integral representation

$$\begin{cases} u(t) = \int_0^1 K_1(t,s)x(s)\,\mathrm{d}s + t^{n_1-1} g_1(\beta_1[u], \beta_1[v]), \\ v(t) = \int_0^1 K_2(t,s)y(s)\,\mathrm{d}s + t^{n_2-1} g_2(\beta_2[u], \beta_2[v]), \end{cases} \tag{2.2}$$

where

$$K_i(t,s) = \frac{1}{(n_i-1)!} \begin{cases} t^{n_i-1}(1-s)^{n_i-1} - (t-s)^{n_i-1}, & 0 \le s \le t \le 1, \\ t^{n_i-1}(1-s)^{n_i-1}, & 0 \le t \le s \le 1, \end{cases} \quad i = 1, 2. \tag{2.3}$$

Proof By Taylor's formula, we have

$$\begin{aligned} u(t) &= u(0) + tu'(0) + \cdots + \frac{t^{n_1-1}}{(n_1-1)!} u^{(n_1-1)}(0) \\ &\quad + \frac{1}{(n_1-1)!} \int_0^t (t-s)^{n_1-1} u^{(n_1)}(s)\,\mathrm{d}s, \\ v(t) &= v(0) + tv'(0) + \cdots + \frac{t^{n_2-1}}{(n_2-1)!} v^{(n_2-1)}(0) \\ &\quad + \frac{1}{(n_2-1)!} \int_0^t (t-s)^{n_2-1} v^{(n_2)}(s)\,\mathrm{d}s, \end{aligned}$$

so, we reduce the equation of problem (2.1) to an equivalent integral equation,

$$u(t) = -\frac{1}{(n_1-1)!} \int_0^t (t-s)^{n_1-1} x(s)\,\mathrm{d}s + \frac{t^{n_1-1}}{(n_1-1)!} u^{(n_1-1)}(0), \tag{2.4}$$

$$v(t) = -\frac{1}{(n_2-1)!} \int_0^t (t-s)^{n_2-1} y(s)\,\mathrm{d}s + \frac{t^{n_2-1}}{(n_2-1)!} v^{(n_2-1)}(0). \tag{2.5}$$

By (2.4) and (2.5), combining with the conditions $u(1) = g_1(\beta_1[u], \beta_1[v])$, $v(1) = g_2(\beta_2[u], \beta_2[v])$, and letting $t = 1$, we have

$$u^{(n_1-1)}(0) = \int_0^1 (1-s)^{n_1-1} x(s)\,\mathrm{d}s + (n_1-1)! g_1\bigl(\beta_1[u], \beta_1[v]\bigr),$$
$$v^{(n_2-1)}(0) = \int_0^1 (1-s)^{n_2-1} y(s)\,\mathrm{d}s + (n_2-1)! g_2\bigl(\beta_2[u], \beta_2[v]\bigr).$$

Substituting $u^{(n_1-1)}(0)$ and $v^{(n_2-1)}(0)$ into (2.4) and (2.5), we have

$$\begin{aligned}
u(t) &= -\frac{1}{(n_1-1)!}\int_0^t (t-s)^{n_1-1} x(s)\,\mathrm{d}s + \frac{1}{(n_1-1)!}\int_0^1 t^{n_1-1}(1-s)^{n_1-1} x(s)\,\mathrm{d}s \\
&\quad + t^{n_1-1} g_1\bigl(\beta_1[u], \beta_1[v]\bigr) \\
&= \frac{1}{(n_1-1)!}\int_0^t \bigl[t^{n_1-1}(1-s)^{n_1-1} - (t-s)^{n_1-1}\bigr] x(s)\,\mathrm{d}s \\
&\quad + \frac{1}{(n_1-1)!}\int_t^1 t^{n_1-1}(1-s)^{n_1-1} x(s)\,\mathrm{d}s + t^{n_1-1} g_1\bigl(\beta_1[u], \beta_1[v]\bigr) \\
&= \int_0^1 K_1(t,s) x(s)\,\mathrm{d}s + t^{n_1-1} g_1\bigl(\beta_1[u], \beta_1[v]\bigr), \\
v(t) &= \int_0^1 K_2(t,s) y(s)\,\mathrm{d}s + t^{n_2-1} g_2\bigl(\beta_2[u], \beta_2[v]\bigr),
\end{aligned}$$

which is equivalent to the boundary value problem (2.1). □

Lemma 2.6 (see [11]) *The function $K_i(t,s)$, $i = 1,2$ defined by* (2.3) *has the following properties*:

(1) $K_i(t,s) \geq 0$, *for* $t, s \in [0,1]$;
(2) $c_i(t) G_i(s) \leq K_i(t,s) \leq G_i(s)$, *for* $t, s \in [0,1]$,

where

$$G_i(s) := \frac{\tau_i(s)^{n_i-2} s(1-s)^{n_i-1}}{(n_i-1)!}, \qquad \tau_i(s) := \frac{s}{(1-(1-s)^{\frac{n_i-1}{n_i-2}})}$$

and

$$c_i(t) := \min\biggl\{\frac{(n_i-1)^{n_i-1} t^{n_i-2}(1-t)}{(n_i-2)^{n_i-2}}, t^{n_i-1}\biggr\}.$$

Throughout this paper, we assume that the following condition is satisfied.

(H_1) $a_i(t)$ does not vanish identically on any subinterval of $(0,1)$, $0 < \int_0^1 G_i(s) a_i(s)\,\mathrm{d}s < +\infty$, $i = 1,2$, where $G_i(s)$ is defined by Lemma 2.6 and there exists $t_0 \in (0,1)$ such that $a_i(t_0) > 0$.

Remark 2.7 By (H_1), we can choose a subinterval $[\xi, \eta] \subset (0,1)$ such that $t_0 \in [\xi, \eta]$. Let $\gamma := \min\{c_i(t) : t \in [\xi, \eta], i = 1,2\}$; it is easy to see that $0 < \gamma < 1$. By Lemma 2.6, we have $\min_{t\in[\xi,\eta]} K_i(t,s) \geq \gamma G_i(s)$, $\forall s \in [0,1]$.

By Lemma 2.5, it is easy to prove that $(u,v) \in C^{n_1}[0,1] \times C^{n_2}[0,1]$ is a positive solution of the system (1.1) if and only if $(u,v) \in C[0,1] \times C[0,1]$ is a positive solution of the following integral system:

$$\begin{cases} u(t) = \int_0^1 K_1(t,s)a_1(s)f_1(s,u(s),v(s))\,\mathrm{d}s + t^{n_1-1}g_1(\beta_1[u],\beta_1[v]), \\ v(t) = \int_0^1 K_2(t,s)a_2(s)f_2(s,u(s),v(s))\,\mathrm{d}s + t^{n_2-1}g_2(\beta_2[u],\beta_2[v]). \end{cases}$$

Let $E = C([0,1],R) \times C([0,1],R)$ be a Banach space endowed with the norm $\|(u,v)\| := \|u\| + \|v\|$, where $\|u\| = \max_{0\le t\le 1}|u(t)|$, $\|v\| = \max_{0\le t\le 1}|v(t)|$ and define the cone $K \subset E$ by

$$K := \Big\{(u,v) \in E : u(t) \ge 0, v(t) \ge 0, t \in [0,1], \min_{t\in[\xi,\eta]}\big(u(t)+v(t)\big) \ge \gamma\big\|(u,v)\big\|\Big\}.$$

It is easy to prove that E is a Banach space and K is a cone in E.

Define the operator $T : K \to E$ by

$$T(u,v)(t) = \big(T_1(u,v)(t), T_2(u,v)(t)\big), \quad \forall t \in [0,1],$$

where

$$T_1(u,v)(t) = \int_0^1 K_1(t,s)a_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + t^{n_1-1}g_1\big(\beta_1[u],\beta_1[v]\big), \tag{2.6}$$

$$T_2(u,v)(t) = \int_0^1 K_2(t,s)a_2(s)f_2\big(s,u(s),v(s)\big)\,\mathrm{d}s + t^{n_2-1}g_2\big(\beta_2[u],\beta_2[v]\big). \tag{2.7}$$

Lemma 2.8 *The operator $T : K \to K$.*

Proof For any $(u,v) \in K$, considering $K_i(t,s) \ge 0$, $i = 1,2$, we have $T_1(u,v)(t) \ge 0$, $T_2(u,v)(t) \ge 0$, for $\forall t \in [0,1]$. From (2.6) and Lemma 2.6, we have

$$\big\|T_1(u,v)\big\| \le \int_0^1 G_1(s)a_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + g_1\big(\beta_1[u],\beta_1[v]\big). \tag{2.8}$$

It follows from (2.8) and Lemma 2.6 that we have

$$\begin{aligned} &\min_{t\in[\xi,\eta]} T_1(u,v)(t) \\ &\quad = \min_{t\in[\xi,\eta]}\bigg[\int_0^1 K_1(t,s)a_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + t^{n_1-1}g_1\big(\beta_1[u],\beta_1[v]\big)\bigg] \\ &\quad \ge \gamma\int_0^1 G_1(s)a_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + \gamma g_1\big(\beta_1[u],\beta_1[v]\big) \\ &\quad \ge \gamma\big\|T_1(u,v)\big\|. \end{aligned}$$

Similarly, it follows from (2.7) and Lemma 2.6 that we have

$$\min_{t\in[\xi,\eta]} T_2(u,v)(t) \ge \gamma\big\|T_2(u,v)\big\|.$$

Therefore,

$$\begin{aligned}
&\min_{t\in[\xi,\eta]}\big(T_1(u,v)(t)+T_2(u,v)(t)\big)\\
&\quad\geq \min_{t\in[\xi,\eta]}T_1(u,v)(t)+\min_{t\in[\xi,\eta]}T_2(u,v)(t)\\
&\quad\geq \gamma\big\|T_1(u,v)\big\|+\gamma\big\|T_2(u,v)\big\|\\
&\quad= \gamma\big\|\big(T_1(u,v),T_2(u,v)\big)\big\|.
\end{aligned}$$

From the above, we conclude that $T(u,v)=(T_1(u,v),T_2(u,v))\in K$, that is, $T:K\to K$. $\square$

3 Main result

For convenience, we use the following notation:

$$\begin{aligned}
&M_i=\max_{t\in[0,1]}\int_0^1 K_i(t,s)a_i(s)\,\mathrm{d}s,\qquad m_i=\min_{t\in[\xi,\eta]}\int_\xi^\eta K_i(t,s)a_i(s)\,\mathrm{d}s,\\
&L_i=\frac{1}{\beta_i[1]},\quad i=1,2.
\end{aligned}$$

Then $0\leq m_i\leq M_i$, $i=1,2$.

Theorem 3.1 *Suppose that the condition* (H_1) *is satisfied and there exist non-negative numbers* h_i, k_i, d_i, e_i, $i=1,2$ *such that for all* $t\in[0,1]$ *and* $(u,v),(\overline{u},\overline{v})\in K$:

$$\big|f_i(t,u,v)-f_i(t,\overline{u},\overline{v})\big|\leq h_i|u-\overline{u}|+k_i|v-\overline{v}|,\quad i=1,2,\tag{3.1}$$

$$\big|g_i(u,v)-g_i(\overline{u},\overline{v})\big|\leq d_i|u-\overline{u}|+e_i|v-\overline{v}|,\quad i=1,2\tag{3.2}$$

and

$$A_1+A_2<1,\tag{3.3}$$

where $A_i=(h_i+k_i)M_i+(d_i+e_i)\beta_i[1]$, $i=1,2$. *Then the system* (1.1) *has a unique positive solution in* K.

Proof By Lemma 2.5, the system (1.1) has a unique positive solution if and only if the operator T has a unique fixed point in K.

Define $\sup_{t\in[0,1]}f_i(t,0,0)=N_i<\infty$, $i=1,2$ and $g_i(0,0)=G_i<\infty$, $i=1,2$ such that

$$r\geq\frac{M_1N_1+G_1+M_2N_2+G_2}{1-A_1-A_2}.$$

First we show that $TB_r\subset B_r$, where $B_r=\{(u,v)|(u,v)\in K,\|(u,v)\|\leq r\}$. For $(u,v)\in B_r$, we have

$$\begin{aligned}
\big|T_1(u,v)(t)\big|\leq{}&\int_0^1 K_1(t,s)a_1(s)\big[\big|f_1\big(s,u(s),v(s)\big)-f_1(s,0,0)\big|+\big|f_1(s,0,0)\big|\big]\,\mathrm{d}s\\
&+t^{n_1-1}\big[\big|g_1\big(\beta_1[u],\beta_1[v]\big)-g_1(0,0)\big|+\big|g_1(0,0)\big|\big]
\end{aligned}$$

$$\begin{aligned}&\le M_1\big(h_1\|u\|+k_1\|v\|+N_1\big)+\beta_1[1]\big(d_1\|u\|+e_1\|v\|\big)+G_1\\&\le A_1r+M_1N_1+G_1,\end{aligned}$$

hence

$$\big\|T_1(u,v)\big\|\le A_1r+M_1N_1+G_1.$$

In the same way, we obtain

$$\big\|T_2(u,v)\big\|\le A_2r+M_2N_2+G_2.$$

Consequently, $\|T(u,v)\|=\|T_1(u,v)\|+\|T_1(u,v)\|\le r$.

Now we shall prove that T is a contraction. Let $(u,v),(\overline{u},\overline{v})\in K$; applying (2.6) we get

$$\begin{aligned}T_1(u,v)(t)-T_1(\overline{u},\overline{v})(t)&=\int_0^1K_1(t,s)a_1(s)\big[f_1\big(s,u(s),v(s)\big)-f_1\big(s,\overline{u}(s),\overline{v}(s)\big)\big]\,\mathrm{d}s\\&\quad+t^{n_1-1}\big[g_1\big(\beta_1[u],\beta_1[v]\big)-g_1\big(\beta_1[\overline{u}],\beta_1[\overline{v}]\big)\big].\end{aligned}$$

With the help of (3.1) and (3.2) we obtain

$$\begin{aligned}\big|T_1(u,v)(t)-T_1(\overline{u},\overline{v})(t)\big|&\le h_1M_1\max_{t\in[0,1]}\big|u(t)-\overline{u}(t)\big|+k_1M_1\max_{t\in[0,1]}\big|v(t)-\overline{v}(t)\big|\\&\quad+d_1\beta_1[1]\max_{t\in[0,1]}\big|u(t)-\overline{u}(t)\big|+e_1\beta_1[1]\max_{t\in[0,1]}\big|v(t)-\overline{v}(t)\big|\\&=\big(h_1M_1+d_1\beta_1[1]\big)\|u-\overline{u}\|+\big(k_1M_1+e_1\beta_1[1]\big)\|v-\overline{v}\|,\end{aligned}$$

this together with (3.3) implies

$$\big\|T_1(u,v)-T_1(\overline{u},\overline{v})\big\|\le A_1\big(\|u-\overline{u}\|+\|v-\overline{v}\|\big).\tag{3.4}$$

Similarly, applying (2.7), with the help of (3.1) and (3.2) we have

$$\big\|T_2(u,v)-T_2(\overline{u},\overline{v})\big\|\le A_2\big(\|u-\overline{u}\|+\|v-\overline{v}\|\big).\tag{3.5}$$

Taking (3.4) and (3.5) into account we have

$$\begin{aligned}\big\|T(u,v)-T(\overline{u},\overline{v})\big\|&=\big\|T_1(u,v)-T_1(\overline{u},\overline{v})\big\|+\big\|T_2(u,v)-T_2(\overline{u},\overline{v})\big\|\\&\le(A_1+A_2)\big(\|u-\overline{u}\|+\|v-\overline{v}\|\big),\end{aligned}$$

where $A_1+A_2<1$. So, T is a contraction, hence it has a unique point fixed in K which is the unique positive solution of the system (1.1). The proof is completed. □

Define the non-negative continuous concave functional on K by

$$\beta(u,v)=\min_{t\in[\xi,\eta]}\big(u(t)+v(t)\big).$$

We observe here that $\beta(u,v)\le\|(u,v)\|$, for each $(u,v)\in K$.

Throughout this section, we assume that p_i, q_i, $i = 1,2$ are four positive numbers satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q_1} + \frac{1}{q_2} \leq 1$.

Theorem 3.2 *Suppose that the condition* (H_1) *is satisfied and there exist non-negative numbers*: a, b, c *such that* $0 < a < b \leq \min\{\gamma, \frac{m_1}{p_1 M_1}, \frac{m_2}{p_2 M_2}\}c$ *and* $f_i(t,u,v)$, $g_i(x,y)$ *satisfy the following growth conditions*:

(H_2) $g_i(x,y) \leq \frac{1}{q_i} L_i(x+y)$, $\forall x+y \in [0, c\beta_i[1]]$, $i = 1,2$;
(H_3) $f_i(t,u,v) \leq \frac{1}{p_i} \cdot \frac{c}{M_i}$, $\forall t \in [0,1]$, $u+v \in [0,c]$, $i = 1,2$;
(H_4) $f_i(t,u,v) > \frac{b}{m_i}$, $\forall t \in [\xi,\eta]$, $u+v \in [b, \frac{b}{\gamma}]$, $i = 1,2$;
(H_5) $f_i(t,u,v) < \frac{1}{p_i} \cdot \frac{a}{M_i}$, $\forall t \in [0,1]$, $u+v \in [0,a]$, $i = 1,2$.

Then the system (1.1) *has at least three positive solutions* (u_1,v_1), (u_2,v_2), (u_3,v_3) *such that* $\|(u_1,v_1)\| < a$, $b < \min_{t\in[\xi,\eta]}(u_2(t)+v_2(t))$ *and* $\|(u_3,v_3)\| > a$ *with* $\min_{t\in[\xi,\eta]}(u_3(t)+v_3(t)) < b$.

Proof It is clear that the existence of positive solutions for the system (1.1) is equivalent to the existence of fixed points of T in K.

We first prove that $T : \overline{K}_c \to \overline{K}_c$ is a completely continuous operator. In fact, if $(u,v) \in \overline{K}_c$, then $\|(u,v)\| \leq c$ and by condition (H_2), we have

$$g_i\bigl(\beta_i[u], \beta_i[v]\bigr) \leq \frac{1}{q_i} L_i \beta_i[u+v] \leq \frac{1}{q_i} L_i c \beta_i[1] = \frac{1}{q_i} c, \quad i = 1,2.$$

Thus, by condition (H_3), we have

$$\begin{aligned}
\|T(u,v)\| &= \max_{t\in[0,1]} \bigl|T_1(u,v)(t)\bigr| + \max_{t\in[0,1]} \bigl|T_2(u,v)(t)\bigr| \\
&= \max_{t\in[0,1]} \biggl[\int_0^1 K_1(t,s) a_1(s) f_1\bigl(s,u(s),v(s)\bigr)\,\mathrm{d}s + t^{n_1-1} g_1\bigl(\beta_1[u],\beta_1[v]\bigr)\biggr] \\
&\quad + \max_{t\in[0,1]} \biggl[\int_0^1 K_2(t,s) a_2(s) f_2\bigl(s,u(s),v(s)\bigr)\,\mathrm{d}s + t^{n_2-1} g_2\bigl(\beta_2[u],\beta_2[v]\bigr)\biggr] \\
&\leq \max_{t\in[0,1]} \int_0^1 K_1(t,s) a_1(s) f_1\bigl(s,u(s),v(s)\bigr)\,\mathrm{d}s + \frac{1}{q_1} c \\
&\quad + \max_{t\in[0,1]} \int_0^1 K_2(t,s) a_2(s) f_2\bigl(s,u(s),v(s)\bigr)\,\mathrm{d}s + \frac{1}{q_2} c \\
&\leq \frac{1}{p_1} \cdot \frac{c}{M_1} M_1 + \frac{1}{q_1} c + \frac{1}{p_2} \cdot \frac{c}{M_2} M_2 + \frac{1}{q_2} c \leq c.
\end{aligned}$$

Therefore, $\|T(u,v)\| \leq c$, that is, $T : \overline{K}_c \to \overline{K}_c$. Standard applications of the Arzelà-Ascoli theorem imply that T is a completely continuous operator.

Now, we show that conditions (A_1)-(A_3) of Lemma 2.3 are satisfied.

Firstly, let $u(t) = \frac{b}{2}$, $v(t) = \frac{b}{2\gamma}$, it follows that $\beta(u,v) > b$, $\|(u,v)\| < \frac{b}{\gamma}$, which shows that $\{(u,v) \in P(\beta, b, \frac{b}{\gamma}) : \beta(u,v) > b\} \neq \varnothing$, and, for $(u,v) \in P(\beta, b, \frac{b}{\gamma})$, we have $b \leq u(s) + v(s) \leq \frac{b}{\gamma}$, $s \in [\xi,\eta]$. By condition (H_4) of Theorem 3.2, we obtain

$$\begin{aligned}
\beta\bigl(T(u,v)(t)\bigr) &= \min_{t\in[\xi,\eta]} \bigl(T_1(u,v)(t) + T_2(u,v)(t)\bigr) \\
&\geq \min_{t\in[\xi,\eta]} \int_\xi^\eta K_1(t,s) a_1(s) f_1\bigl(s,u(s),v(s)\bigr)\,\mathrm{d}s + \gamma g_1\bigl(\beta_1[u],\beta_1[v]\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ \min_{t\in[\xi,\eta]} \int_\xi^\eta K_2(t,s)a_2(s)f_2\bigl(s,u(s),v(s)\bigr)\,\mathrm{d}s + \gamma g_2\bigl(\beta_2[u],\beta_2[v]\bigr)\\
&> \frac{b}{m_1}\min_{t\in[\xi,\eta]}\int_\xi^\eta K_1(t,s)a_1(s)\,\mathrm{d}s = \frac{b}{m_1}\cdot m_1 = b.
\end{aligned}
$$

Similarly, by condition (H_4) of Theorem 3.2, we can obtain

$$
\begin{aligned}
\beta\bigl(T(u,v)(t)\bigr) &= \min_{t\in[\xi,\eta]}\bigl(T_1(u,v)(t)+T_2(u,v)(t)\bigr)\\
&> \frac{b}{m_2}\min_{t\in[\xi,\eta]}\int_\xi^\eta K_2(t,s)a_2(s)\,\mathrm{d}s = \frac{b}{m_2}\cdot m_2 = b.
\end{aligned}
$$

Therefore, condition (A_1) of Lemma 2.3 is satisfied.

Secondly, in a completely analogous argument to the proof of $T:\overline{K}_c\to\overline{K}_c$, by condition (H_5) of Theorem 3.2, condition (A_2) of Lemma 2.3 is satisfied.

Finally, we show that condition (A_3) of Lemma 2.3 is satisfied. If $(u,v)\in P(\beta,b,\frac{b}{\gamma})$ and $\|T(u,v)(t)\|>\frac{b}{\gamma}$, then

$$
\beta\bigl(T(u,v)(t)\bigr) = \min_{t\in[\xi,\eta]}\bigl(T_1(u,v)(t)+T_2(u,v)(t)\bigr)\ge\gamma\bigl\|T(u,v)(t)\bigr\|>b.
$$

Therefore, condition (A_3) of Lemma 2.3 is satisfied.

Thus, all conditions of Lemma 2.3 are satisfied. By Lemma 2.3, the system (1.1) has at least three positive solutions (u_1,v_1), (u_2,v_2), (u_3,v_3) such that $\|(u_1,v_1)\|<a$, $b<\min_{t\in[\xi,\eta]}(u_2(t)+v_2(t))$ and $\|(u_3,v_3)\|>a$, with $\min_{t\in[\xi,\eta]}(u_3(t)+v_3(t))<b$. The proof is completed. □

4 Example

Example 4.1 Consider the following system of nonlinear mixed-order ordinary differential equations:

$$
\begin{cases}
u^{(3)}(t)+a_1(t)f_1(t,u(t),v(t))=0, & 0<t<1,\\
v^{(4)}(t)+a_2(t)f_2(t,u(t),v(t))=0, & 0<t<1,\\
u(0)=u'(0)=0, & u(1)=g_1(\beta_1[u],\beta_1[v]),\\
v(0)=v'(0)=v''(0)=0, & v(1)=g_2(\beta_2[u],\beta_2[v]).
\end{cases}
\tag{4.1}
$$

Then the system (4.1) is equivalent to the following system of nonlinear integral equations:

$$
\begin{cases}
u(t)=\int_0^1 K_1(t,s)a_1(s)f_1(s,u(s),v(s))\,\mathrm{d}s+t^2g_1(\beta_1[u],\beta_1[v]),\\
v(t)=\int_0^1 K_2(t,s)a_2(s)f_2(s,u(s),v(s))\,\mathrm{d}s+t^3g_2(\beta_2[u],\beta_2[v]),
\end{cases}
$$

where

$$
K_1(t,s)=\frac{1}{2}\begin{cases}
t^2(1-s)^2-(t-s)^2, & 0\le s\le t\le 1,\\
t^2(1-s)^2, & 0\le t\le s\le 1,
\end{cases}
$$

$$
K_2(t,s)=\frac{1}{6}\begin{cases}
t^3(1-s)^3-(t-s)^3, & 0\le s\le t\le 1,\\
t^3(1-s)^3, & 0\le t\le s\le 1.
\end{cases}
$$

We choose $a_1(t) = \frac{5}{\sqrt{t}}$, $a_2(t) = 50$, $\beta_1[u] = \int_0^1 u(s)\,\mathrm{d}s$, $\beta_2[u] = 2\int_0^1 u(s)\,\mathrm{d}s$, $\beta_1[v] = \int_0^1 v(s)\,\mathrm{d}s$, $\beta_2[v] = 2\int_0^1 v(s)\,\mathrm{d}s$, and

$$f_1(t,u,v) = \begin{cases} 0.1t + 0.01(u+v)^2, & t \in [0,1], 0 \le u+v \le 2, \\ 0.1t + 15[(u+v)^2 - 2(u+v)] + 0.04, & t \in [0,1], 2 < u+v < 4, \\ 0.1t + 15[3\log_2(u+v) + (u+v)/2] + 0.04, & t \in [0,1], 4 \le u+v \le 16, \\ 0.1t + 300.04, & t \in [0,1], u+v > 16 \end{cases}$$

and

$$f_2(t,u,v) = \begin{cases} 0.01t + 0.02(u+v)^2, & t \in [0,1], 0 \le u+v \le 2, \\ 0.01t + 17[(u+v)^2 - 2(u+v)] + 0.08, & t \in [0,1], 2 < u+v < 4, \\ 0.01t + 17[3\log_2(u+v) + (u+v)/2] + 0.08, & t \in [0,1], 4 \le u+v \le 16, \\ 0.01t + 340.08, & t \in [0,1], 16 < u+v < +\infty, \end{cases}$$

and

$$g_1\bigl(\beta_1[u], \beta_1[v]\bigr) = \begin{cases} 0.2\ln(\beta_1[u] + \beta_1[v] + 1), & 0 \le u+v \le 800, \\ 0.2\ln 801, & 800 < u+v < +\infty, \end{cases}$$

$$g_1\bigl(\beta_1[u], \beta_1[v]\bigr) = \begin{cases} 0.125\ln(\beta_1[u] + \beta_1[v] + 1), & 0 \le u+v \le 1{,}600, \\ 0.125\ln 1{,}601, & 1{,}600 < u+v < +\infty. \end{cases}$$

By Lemma 2.6, we have

$$c_1(t) = \begin{cases} t^2, & 0 \le t \le 0.8, \\ 4t(1-t), & 0.8 \le t \le 1, \end{cases} \qquad c_2(t) = \begin{cases} t^3, & 0 \le t \le 0.87, \\ 27t^2(1-t)/4, & 0.87 \le t \le 1. \end{cases}$$

Choose $[\xi, \eta] = [0.6, 0.8]$; by Remark 2.7, we obtain $\gamma = 0.216$. Then by direct calculation we obtain

$$M_1 \approx 0.248, \qquad m_1 \approx 0.054, \qquad M_2 \approx 0.220, \qquad m_2 \approx 0.048,$$
$$L_1 = 1, \qquad L_2 = 0.5, \qquad \beta_1[1] = 1, \qquad \beta_2[1] = 2.$$

It is easy to verify that the condition (H_1) holds. Let $q_1 = 5$, $q_2 = 4$, $p_1 = 5$, $p_2 = 7$, $a = 1$, $b = 4$, $c = 800$. Also, it is easy to verify that f_1, f_2, g_1, g_2 satisfy conditions (H_2)-(H_5).

Thus, by Theorem 3.2, the system (4.1) has at least three positive solutions (u_1, v_1), (u_2, v_2), (u_3, v_3) such that $\|(u_1, v_1)\| < 1$, $4 < \min_{t\in[0.6,0.8]}(u_2, v_2)$ and $\|(u_3, v_3)\| > 1$ with $\min_{t\in[0.6,0.8]}(u_3, v_3) < 4$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The work presented here was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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