Audeh et al. *Journal of Inequalities and Applications* (2024) 2024:114
https://doi.org/10.1186/s13660-024-03193-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Singular value inequalities of matrices via increasing functions

Wasim Audeh[1*], Anwar Al-Boustanji[2†], Manal Al-Labadi[1†] and Raja'a Al-Naimi[1†]

*Correspondence:
waudeh@uop.edu.jo
[1]Department of Mathematics, University of Petra, Amman, Jordan
Full list of author information is available at the end of the article
†Equal contributors

Abstract

Let A, B, X, and Y be $n \times n$ complex matrices such that A is self-adjoint, $B \geq 0$, $\pm A \leq B$, $\max(\|X\|^2, \|Y\|^2) \leq 1$, and let f be a nonnegative increasing convex function on $[0, \infty)$ satisfying $f(0) = 0$. Then

$$2s_j\left(f\left(\left|XAY^*\right|\right)\right) \leq \max\left\{\|X\|^2, \|Y\|^2\right\} s_j\left(f(B+A) \oplus f(B-A)\right)$$

for $j = 1, 2, \ldots, n$. This singular value inequality extends an inequality of Audeh and Kittaneh. Several generalizations for singular value and norm inequalities of matrices are also given.

Mathematics Subject Classification: 15A18; 15A42; 15A60; 47A63

Keywords: Singular value; Positive semidefinite matrix; Increasing function; Convex function; Unitarily invariant norm

1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, the singular values of A are denoted by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$, they are precisely the eigenvalues of the positive operator $|A| = (A^*A)^{1/2}$. Singular values have several properties: Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then

(a) $s_j(A) = s_j(A^*) = s_j(|A|)$ for $j = 1, 2, \ldots, n$.

(b) $s_j(AA^*) = s_j(A^*A)$ for $j = 1, 2, \ldots, n$.

(c) If $A, B \in \mathbb{M}_n(\mathbb{C})$, then $s_j(A) \leq s_j(B)$ if and only if $s_j(A \oplus A) \leq s_j(B \oplus B)$ for $j = 1, 2, \ldots, n$.

Bhatia and Kittaneh proved in [15] the following inequalities:

(i) If $A, B \in \mathbb{M}_n(\mathbb{C})$ such that A is self-adjoint, $B \geq 0$, and $\pm A \leq B$, then

$$s_j(A) \leq s_j(B \oplus B) \tag{1}$$

for $j = 1, 2, \ldots, n$.

(ii) If $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j\left(AB^* + BA^*\right) \leq s_j\left(\left(AA^* + BB^*\right) \oplus \left(AA^* + BB^*\right)\right) \tag{2}$$

for $j = 1, 2, \dots n$. Audeh and Kittaneh pointed out in [8] that:

(i) If $A, B \in \mathbb{M}_n(\mathbb{C})$ such that A is self-adjoint, $B \geq 0$, and $\pm A \leq B$, then

$$2s_j(A) \leq s_j\big((B + A) \oplus (B - A)\big) \tag{3}$$

for $j = 1, 2, \dots, n$.

(ii) If $A, B, C \in \mathbb{M}_n(\mathbb{C})$ such that $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$, then

$$s_j(B) \leq s_j(A \oplus C) \tag{4}$$

for $j = 1, 2, \dots, n$.

(iii) If $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(A + B) \leq s_j\big(\big(|A| + |B|\big) \oplus \big(|A^*| + |B^*|\big)\big) \tag{5}$$

for $j = 1, 2, \dots, n$.

Tao proved in [24] that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ such that $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$, then

$$2s_j(B) \leq s_j \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \tag{6}$$

for $j = 1, 2, \dots, n$. In addition, Bhatia and Kittaneh showed in [14] that if $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$2s_j\big(AB^*\big) \leq s_j\big(A^*A + B^*B\big) \tag{7}$$

for $j = 1, 2, \dots, n$. For more details and comprehensive results related to this topic, we refer to [1–7, 9, 10] and [17]. In this paper, we provide considerable generalizations of inequalities (1)–(6).

Unitarily invariant norms on $\mathbb{M}_n$ are denoted by $|||.|||$, recall that these norms satisfying $|||UAV||| = |||A|||$ for all $U, V, A \in \mathbb{M}_n$ such that U and V are unitary. Important classes of such norms are the Schatten p-norms defined by $\|A\|_p = (\sum_{j=1}^n s_j^p(A))^{1/p}$ where $p \geq 1$ and the spectral norm defined by $\|A\| = s_1(A)$. For the general theory of unitarily invariant norms, we refer the reader to [13], [16], and [23]. It follows easily from the basic properties of unitarily invariant norms that

$$\big|\big|\big|A^*A\big|\big|\big| = \big|\big|\big|AA^*\big|\big|\big|. \tag{8}$$

Bhatia and Davis proved in [13] that if $A, X, B \in \mathbb{M}_n$, then

$$2\big|\big|\big|AXB^*\big|\big|\big| \leq \big|\big|\big|A^*AX + XB^*B\big|\big|\big|. \tag{9}$$

This is a generalization of the arithmetic–geometric mean inequality for unitarily invariant norms. In this paper, we provide a considerable generalization of inequality (9). Hou and Du proved in [19] that if $A \in \mathbb{M}_n$, then

$$\|A\| \leq \big\|\big(\|A_{ij}\|\big)_{1\leq i,j\leq n}\big\|. \tag{10}$$

Popovici and Sebestyen showed in [22] the following inequalities:

1. If $A_1, A_2, \dots, A_n \in \mathbb{M}_n$ are positive, then

$$\left\| \sum_{k=1}^{n} A_k \right\| \leq \left\| \left(\left\| A_i^{1/2} A_j^{1/2} \right\| \right)_{1 \leq i,j \leq n} \right\|. \tag{11}$$

2. If $A_1, A_2, \dots, A_n \in \mathbb{M}_n$ are positive, then

$$\left\| \sum_{k=1}^{n} A_k A_k^* \right\| \leq \left\| \left(\left\| A_i^* A_j \right\| \right)_{1 \leq i,j \leq n} \right\|. \tag{12}$$

We provide inequalities that are more general and sharper than inequalities (11) and (12).

Zou in [26] demonstrated the following generalization of arithmetic–geometric mean inequality: Let $A, X, B \in \mathbb{M}_n$ such that X is positive semidefinite Then

$$2 ||| AXB^* ||| \leq ||| \left(A^*A + B^*B \right)^{1/2} X \left(A^*A + B^*B \right)^{1/2} |||. \tag{13}$$

Among our results, we obtain a generalization of inequality (13).

2 Singular value inequalities

The following lemmas are essential for supporting our conclusions. The first lemma is an immediate consequence of the min-max principle (see, e.g., [2, p. 75]). The second and third lemmas were shown in [11].

Lemma 1 *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j(AXB) \leq \|A\| \|B\| s_j(X) \tag{14}$$

for $j = 1, 2, \dots, n$.

Lemma 2 *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let f be a nonnegative increasing function on an interval I. Then*

$$f\left(s_j(A)\right) = s_j\left(f\left(|A|\right)\right)$$

for $j = 1, 2, \dots, n$. If A is Hermitian and f is increasing on an interval I, then

$$f\left(\lambda_j(A)\right) = \lambda_j\left(f(A)\right)$$

for $j = 1, 2, \dots, n$.

Lemma 3 *Let f be a monotone convex function on an interval I such that $0 \in I$ and $f(0) \leq 0$, and let $A, X \in \mathbb{M}_n(\mathbb{C})$ such that A is Hermitian and X is a contraction. Then*

$$\lambda_j\left(f\left(X^*AX\right)\right) \leq \lambda_j\left(X^* f(A) X\right)$$

for $j = 1, 2, \dots, n$.

The first result in this paper is now ready to be presented.

Theorem 1 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $\max\{\|X\|, \|Y\|\} \leq 1$, *and let f be a nonnegative increasing convex function on* $[0,\infty)$ *satisfying* $f(0) = 0$. *Then*

$$s_j\big(f\big(\left|XAB^*Y^*\right|\big)\big) \leq \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\big(f\big(AA^*\big) \oplus f\big(BB^*\big)\big) \tag{15}$$

for $j = 1, 2, \ldots, n$.

Proof Consider the operator matrix $Q = [(XA)^* \ (YB)^*] \in M_{n,2n}(\mathbb{C})$.

$$P = Q^*Q = \begin{bmatrix} XAA^*X^* & XAB^*Y^* \\ YBA^*X^* & YBB^*Y^* \end{bmatrix} \geq 0$$

for any $A, B, X, Y \in M_{2n,2n}(\mathbb{C})$.

By making use of inequality (4) and by letting

$$R = \begin{bmatrix} AA^* & 0 \\ 0 & BB^* \end{bmatrix}$$

gives

$$\begin{aligned}
s_j\big(f\big(\left|XAB^*Y^*\right|\big)\big) &= f\big(s_j\big(XAB^*Y^*\big)\big) \\
&\leq f\big(s_j\big(XAA^*X^* \oplus YBB^*Y^*\big)\big) \\
&= f\left(s_j\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} R \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix}\right)\right) \\
&= f\left(\lambda_j\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} R \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix}\right)\right) \\
&= \lambda_j\left(f\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} R \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix}\right)\right), \\
&\quad \text{(by Lemma 2)} \\
&\leq \lambda_j\left(\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} f(R) \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix}\right)\right), \\
&\quad \text{(by Lemma 3)} \\
&= s_j\left(\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} f(R) \begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix}\right)\right) \\
&\leq \left\|\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right\| \left\|\begin{bmatrix} X^* & 0 \\ 0 & Y^* \end{bmatrix}\right\| s_j\big(f(R)\big), \\
&\quad \text{(by Lemma 1)} \\
&= \left\|\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right\|^2 s_j\big(f(R)\big)
\end{aligned}$$

$$= \max\{\|X\|^2, \|Y\|^2\} s_j[f(R)]$$
$$= \max\{\|X\|^2, \|Y\|^2\} s_j\big(f(AA^*) \oplus f(BB^*)\big). \qquad \square$$

Corollary 1 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $\max\{\|X\|, \|Y\|\} \le 1$. *Then, for* $r \ge 1$,

$$s_j(XAB^*Y^*)^r \le \max\{\|X\|^2, \|Y\|^2\} s_j\big((AA^*)^r \oplus (BB^*)^r\big) \tag{16}$$

and

$$s_j\big(e^{|XAB^*Y^*|} - I\big) \le \max\{\|X\|^2, \|Y\|^2\} s_j\big[\big(e^{AA^*} - I\big) \oplus \big(e^{BB^*} - I\big)\big] \tag{17}$$

for $j = 1, 2, \ldots, n$.

Proof Letting $f(t) = t^r$, $r \ge 1$, and $f(t) = e^t - 1$ in Theorem 1 gives inequalities (16) and (17), respectively. $\square$

By using Theorem 1, we here by present the following theorem.

Theorem 2 *Let* $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $P = \left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \ge 0$, $\max\{\|X\|^2, \|Y\|^2\} \le 1$, *and let f be a nonnegative increasing convex function on* $[0, \infty)$ *satisfying* $f(0) = 0$. *Then*

$$s_j\big(f\big(|XBY^*|\big)\big) \le \max\{\|X\|^2, \|Y\|^2\} s_j\big(f(A) \oplus f(C)\big) \tag{18}$$

for $j = 1, 2, \ldots, n$.

Proof Let $P = \left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \ge 0$. Then there exists a matrix $Q = [(H)^*(L)^*] \in M_{n,2n}(\mathbb{C})$ such that $P = Q^*Q$ for some $H, L \in M_n(\mathbb{C})$. Then $A = HH^*$, $C = LL^*$ and $B = HL^*$. Applying inequality (4) gives

$$\begin{aligned} s_j\big(f\big(|XBY^*|\big)\big) &= s_j\big(f\big(|XHL^*Y^*|\big)\big) \\ &\le \max\{\|X\|^2, \|Y\|^2\} s_j\big(f\big(HH^*\big) \oplus f\big(LL^*\big)\big), \quad \text{(by Theorem 1)} \\ &= \max\{\|X\|^2, \|Y\|^2\} s_j\big(f(A) \oplus f(C)\big). \end{aligned}$$

Inequality (18) has thus been proved. $\square$

Remark 1 Letting $X = Y = I$ and $f(t) = t$ in inequality (18) gives inequality (4). In that sense, inequality (18) is certainly a generalization of inequality (4).

Corollary 2 *Let* $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $P = \left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \ge 0$ *and* $\max\{\|X\|^2, \|Y\|^2\} \le 1$. *Then, for* $r \ge 1$,

$$s_j\big((XBY^*)^r\big) \le \max\{\|X\|^2, \|Y\|^2\} s_j\big(A^r \oplus C^r\big) \tag{19}$$

and

$$s_j\big(e^{|XBY^*|} - I\big) \le \max\{\|X\|^2, \|Y\|^2\} s_j\big(\big(e^A - I\big) \oplus \big(e^C - I\big)\big) \tag{20}$$

for $j = 1, 2, \ldots, n$.

Proof Letting $f(t) = t^r$, $r \geq 1$, and $f(t) = e^t - 1$ in Theorem 2 gives inequalities (19) and (20), respectively. □

Using the proper incites of Theorem 2 gives the following inequality, which is a generalization of inequality (1).

Theorem 3 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* A *is self-adjoint,* $B \geq 0$, $\pm A \leq B$, $\max\{\|X\|^2, \|Y\|^2\} \leq 1$, *and let* f *be a nonnegative increasing convex function on* $[0,\infty)$ *satisfying* $f(0) = 0$. *Then*

$$s_j\big(f\big(\big|XAY^*\big|\big)\big) \leq \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\big(f(B) \oplus f(B)\big) \tag{21}$$

for $j = 1, 2, \ldots, n$.

Proof Let $P = \left[\begin{smallmatrix} B & A \\ A & B \end{smallmatrix}\right]$. Since $\left[\begin{smallmatrix} B & A \\ A & B \end{smallmatrix}\right]$ is unitarily equivalent to $\left[\begin{smallmatrix} B+A & 0 \\ 0 & B-A \end{smallmatrix}\right]$ and since $\pm A \leq B$, it follows that P is a positive matrix. Applying inequality (18) to the operator matrix P gives inequality (21). □

Remark 2 Letting $f(t) = t$ and $X = Y = I$ in inequality (21) gives inequality (1). In that sense, inequality (21) is a generalization of inequality (1).

Corollary 3 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* A *is self-adjoint,* $B \geq 0$, $\pm A \leq B$, *and* $\max\{\|X\|^2, \|Y\|^2\} \leq 1$. *Then*

$$s_j\big(XAY^*\big)^r\big) \leq \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\big(B^r \oplus B^r\big) \tag{22}$$

and

$$s_j\big(e^{|XAY^*|} - I\big) \leq \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\big(\big(e^B - I\big) \oplus \big(e^B - I\big)\big) \tag{23}$$

for $j = 1, 2, \ldots, n$.

Proof Letting $f(t) = t^r$, $r \geq 1$, and $f(t) = e^t - 1$ in Theorem 3 gives inequalities (22) and (23), respectively. □

The following lemma, which was proved in [12], is necessary to prove the next result.

Lemma 4 *Let* $A \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$\begin{bmatrix} |A| & \pm A^* \\ \pm A & |A^*| \end{bmatrix} \geq 0. \tag{24}$$

Theorem 4 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $\max\{\|X\|^2, \|Y\|^2\} \leq 1$, *and let* f *be a nonnegative increasing convex function on* $[0,\infty)$ *satisfying* $f(0) = 0$ *and*

$$P = A|C|A^* + B\big|C^*\big|B^*.$$

Then

$$s_j\big(f\big(\big|X\big(AC^*B^* + BCA^*\big)Y^*\big|\big)\big) \le \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\big(f(P) \oplus f(P)\big) \tag{25}$$

for $j = 1, 2, \ldots, n$.

Proof Let $Z = \left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$ and $Y = \left[\begin{smallmatrix} |C| & \pm C^* \\ \pm C & |C^*| \end{smallmatrix}\right]$. Then $Y \ge 0$ and

$$ZYZ^* = \begin{bmatrix} A|C|A^* \pm BCA^* \pm AC^*B^* + B|C^*|B^* & 0 \\ 0 & 0 \end{bmatrix} \ge 0,$$

this implies that

$$A|C|A^* + B\big|C^*\big|B^* \ge \pm\big(AC^*B^* + BCA^*\big). \tag{26}$$

Applying the conclusion of inequality (21) to the operator matrix ZYZ^* gives

$$s_j\big(f\big(\big|X\big(AC^*B^* + BCA^*\big)Y^*\big|\big)\big) \le \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\big(f(P) \oplus f(P)\big)$$

which is precisely (25). □

Remark 3 Substituting $f(t) = t$, $X, Y, C = I$, and $R = (AA^* + BB^*)$ in Theorem 4 gives the following inequality, which is a generalization of inequality (2):

$$s_j\big(f\big(\big|AB^* + BA^*\big|\big)\big) \le s_j\big(f(R) \oplus f(R)\big) \tag{27}$$

for $j = 1, 2, \ldots, n$.

Remark 4 Letting $f(t) = t$ in inequality (27), we give inequality (2). In that sense, inequality (27) is certainly a generalization of inequality (2).

The following result is a direct consequence of Theorem 2.

Corollary 4 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *where* $\max\{\|X\|^2, \|Y\|^2\} \le 1$, *and let* f *be a nonnegative increasing convex function on* $[0, \infty)$ *satisfying* $f(0) = 0$ *and* $K = M \oplus N$, *where*

$$M = |A| + |B| \quad \textit{and} \quad N = \big|A^*\big| + \big|B^*\big|.$$

Then

$$s_j\big(f\big(\big|X(A + B)Y^*\big|\big)\big) \le \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\big(f(M) \oplus f(N)\big) \tag{28}$$

for $j = 1, 2, \ldots, n$.

Proof It was shown in [15] that the matrix

$$K = \begin{bmatrix} |A| + |B| & A + B \\ A^* + B^* & |A^*| + |B^*| \end{bmatrix}$$

is positive semidefinite. Now, inequality (28) is a direct consequence of Theorem 2. □

Remark 5 Substituting $X = Y = I$ in inequality (28) leads to inequality (5). In that sense, inequality (28) is certainly a generalization of inequality (5).

The following inequality is a generalization of inequality (6).

Theorem 5 *Let* $A, B, C, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$, $\max\{\|X\|^2, \|Y\|^2\} \leq 1$, *and let* f *be a nonnegative increasing convex function on* $[0,\infty)$ *satisfying* $f(0) = 0$. *Then*

$$2s_j\big(f\big(\big|XBY^*\big|\big)\big) \leq \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\left(f\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right)\right) \tag{29}$$

for $j = 1, 2, \dots, n$.

Proof Let $P = \left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] \geq 0$. Then there exist $H, L \in \mathbb{M}_n(\mathbb{C})$ such that $P = Q^*Q$, where $Q = [(H)^*(L)^*]$. This means that $A = HH^*$, $C = LL^*$, and $B = HL^*$.

$$\begin{aligned} 2s_j\big(f\big(\big|XBY^*\big|\big)\big) &= 2s_j\big(f\big(\big|XHL^*Y^*\big|\big)\big) \\ &= 2f\big(s_j\big(XHL^*Y^*\big)\big) \\ &\leq f\big(s_j(W)\big) \end{aligned}$$

for $j = 1, 2, \dots, n$, where $W = H^*X^*XH + L^*Y^*YL$. Now, letting $Z = \left[\begin{smallmatrix} X^*X & 0 \\ 0 & Y^*Y \end{smallmatrix}\right]$ gives

$$\begin{aligned} f\big(s_j(W)\big) &= f\left(s_j\left(\begin{bmatrix} H^* & L^* \\ 0 & 0 \end{bmatrix} Z \begin{bmatrix} H & 0 \\ L & 0 \end{bmatrix}\right)\right) \\ &= f\left(\lambda_j\left(\begin{bmatrix} H^* & L^* \\ 0 & 0 \end{bmatrix} Z \begin{bmatrix} H & 0 \\ L & 0 \end{bmatrix}\right)\right) \\ &= f\left(\lambda_j\left(Z^{1/2}\begin{bmatrix} H & 0 \\ L & 0 \end{bmatrix}\begin{bmatrix} H^* & L^* \\ 0 & 0 \end{bmatrix} Z^{1/2}\right)\right) \\ &= f\left(\lambda_j\left(Z^{1/2}\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} Z^{1/2}\right)\right) \\ &= \lambda_j\left(f\left(Z^{1/2}\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} Z^{1/2}\right)\right) \\ &\leq \lambda_j\left(Z^{1/2} f\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) Z^{1/2}\right) \\ &= s_j\left(Z^{1/2} f\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) Z^{1/2}\right) \\ &\leq \|Z\| s_j\left(f\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right)\right) \\ &= \max\big\{\|X\|^2, \|Y\|^2\big\} s_j\left(f\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right)\right). \end{aligned}$$

Inequality (29) has thus been substantiated. □

Remark 6 Letting $f(t) = t$ and $X = Y = I$ in Theorem 5 gives inequality (6). In that sense inequality (29) is a generalization of inequality (6).

At this stage of our discussion, we provide a considerable generalization of inequality (3).

Theorem 6 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* A *is self-adjoint,* $B \geq 0$, $\pm A \leq B$, $\max\{\|X\|^2, \|Y\|^2\} \leq 1$, *and let* f *be a nonnegative increasing convex function on* $[0,\infty)$ *satisfying* $f(0) = 0$. *Then*

$$2s_j\bigl(f\bigl(|XAY^*|\bigr)\bigr) \leq \max\bigl\{\|X\|^2, \|Y\|^2\bigr\} s_j\bigl(f\bigl((B+A) \oplus f(B-A)\bigr)\bigr) \tag{30}$$

for $j = 1, 2, \ldots, n$.

Proof Since $\pm A \leq B$, it follows that

$$P = \begin{bmatrix} B+A & 0 \\ 0 & B-A \end{bmatrix} \geq 0.$$

If $U = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} I & -I \\ I & I \end{smallmatrix}\right]$, then

$$Q = \begin{bmatrix} B & A \\ A & B \end{bmatrix} = UPU^*,$$

which is equivalent to stating that $Q \geq 0$. Now applying Theorem 5 to the operator matrix Q leads to

$$\begin{aligned} 2s_j\bigl(f\bigl(|XAY^*|\bigr)\bigr) &\leq \max\bigl\{\|X\|^2, \|Y\|^2\bigr\} s_j\left(f\left(\begin{bmatrix} B & A \\ A & B \end{bmatrix}\right)\right) \\ &= \max\bigl\{\|X\|^2, \|Y\|^2\bigr\} s_j\left(f\left(\begin{bmatrix} B+A & 0 \\ 0 & B-A \end{bmatrix}\right)\right) \\ &= \max\bigl\{\|X\|^2, \|Y\|^2\bigr\} s_j\bigl(f\bigl((B+A) \oplus f(B-A)\bigr)\bigr). \end{aligned}$$

Inequality (30) has thus been substantiated. □

Corollary 5 *Let* $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ *such that* A *is self-adjoint,* $B \geq 0$, $\pm A \leq B$, *and* $\max\{\|X\|^2, \|Y\|^2\} \leq 1$. *Then*

$$2s_j\bigl(XAY^*\bigr)^r \leq \max\bigl\{\|X\|^2, \|Y\|^2\bigr\} s_j\bigl((B+A)^r \oplus (B-A)^r\bigr) \tag{31}$$

and

$$2s_j\bigl(e^{|XAY^*|} - I\bigr) \leq \max\bigl\{\|X\|^2, \|Y\|^2\bigr\} s_j\bigl(\bigl(e^{B+A} - I\bigr) \oplus \bigl(e^{B-A} - I\bigr)\bigr) \tag{32}$$

for $j = 1, 2, \ldots, n$.

Proof Letting $f(t) = t^r$, $r \geq 1$, and $f(t) = e^t - 1$ in Theorem 6 gives inequalities (31) and (32), respectively. □

3 Norm inequalities

We begin this section by the following lemmas that are essential in our study. The first lemma is folk lemma, the second lemma was introduced in [22], and the third lemma was obtained in [3].

Lemma 5 *Let $A, B \in \mathbb{M}_n$, then*

$$\left\|\left|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right|\right\| = \left\|\left|\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}\right|\right\|. \tag{33}$$

Lemma 6 *Let $A, B \in \mathbb{M}_n$, where AB is Hermitian. Then*

$$|||AB||| \leq |||\operatorname{Re}(BA)|||. \tag{34}$$

Let $A, B, X \in B(H)$ such that X is positive. Then

$$2|||AXB^*||| \leq |||X^{1/2}|A|^2X^{1/2} + X^{1/2}|B|^2X^{1/2}|||. \tag{35}$$

The following theorem can be obtained from norm inequality (13).

Theorem 7 *Let $A_i, B_i, X_i \in \mathbb{M}_n$, $i = 1, 2, \ldots, n$, such that every X_i is positive,*

$$C = \begin{bmatrix} A_1^*A_1 + B_1^*B_1 & \cdots & A_1^*A_n + B_1^*B_n \\ A_2^*A_1 + B_2^*B_1 & \cdots & A_2^*A_n + B_2^*B_n \\ \vdots & \ddots & \vdots \\ A_n^*A_1 + B_n^*B_1 & \cdots & A_n^*A_n + B_n^*B_n \end{bmatrix},$$

and

$$X = \begin{bmatrix} X_1 & 0 & \ldots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix},$$

then

$$2\left\|\left|\left(\sum_{i=1}^{n} A_iX_iB_i^*\right)\right|\right\| \leq |||C^{1/2}XC^{1/2}|||. \tag{36}$$

Proof On $\oplus\mathbb{M}_n$, define

$$A = \begin{bmatrix} A_1 & A_2 & \ldots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 & \ldots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then

$$AXB^* = \begin{bmatrix} \sum_{i=1}^n A_iX_iB_i^* & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$\begin{aligned} X^{1/2}|A|^2X^{1/2} + X^{1/2}|B|^2X^{1/2} &= X^{1/2}\big(|A|^2 + |B|^2\big)X^{1/2} \\ &= X^{1/2}CX^{1/2} \\ &= \begin{bmatrix} V_{11} & V_{12} & \dots & V_{1n} \\ V_{21} & V_{22} & \cdots & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & V_{n2} & \cdots & V_{nn} \end{bmatrix}, \end{aligned}$$

where

$$V_{i,j} = X_i^{1/2}A_i^*A_jX_j^{1/2} + X_i^{1/2}B_i^*B_jX_j^{1/2}.$$

Applying inequality (35) gives

$$\begin{aligned} 2\left|\left|\left|\left(\sum_{i=1}^n A_iX_iB_i^*\right)\right|\right|\right| &\le \big|\big|\big|X^{1/2}CX^{1/2}\big|\big|\big| \\ &= \big|\big|\big|X^{1/2}C^{1/2}C^{1/2}X^{1/2}\big|\big|\big| \\ &= \big|\big|\big|C^{1/2}XC^{1/2}\big|\big|\big|, \\ &\quad \big(\text{by applying inequality (8)}\big). \end{aligned}$$

Thus, inequality (36) has been substantiated. □

Remark 7 Inequality (36) is a general norm inequality that involves inequality (13). To see this, substitute $A_i = X_i = B_i = 0$ for $i = 2, \dots, n$ in inequality (36), which leads to inequality (13).

Theorem 8 *Let $A_i, X_i, B_i \in \mathbb{M}_n$, $i = 1, 2, \dots, n$. Then*

$$2\left|\left|\left|\sum_{i=1}^n A_iX_iB_i^*\right|\right|\right| \le \big|\big|\big|A^*AX + XB^*B\big|\big|\big|, \tag{37}$$

where

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 & \dots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & X_n \end{bmatrix}.$$

Proof Replacing the operator matrices A, B, and X in inequality (9) gives inequality (37). □

The next special case of Theorem 8 was proved by Bhatia and Davis in [13].

Corollary 6 *Let $A, X, B \in \mathbb{M}_n$. Then*

$$2|||AXB^*||| \leq |||A^*AX + XB^*B|||. \tag{38}$$

Proof Inequality (38) follows from inequality (37) by substituting $A_i = B_i = X_i = 0$ for $i = 2, 3, \ldots, n$. □

Another conclusion of Theorem 8 is a generalization of arithmetic–geometric mean inequality for unitarily invariant norms.

Corollary 7 *Let $A_i, B_i \in \mathbb{M}_n$, $i = 1, 2, \ldots, n$, such that*

$$A = \begin{bmatrix} A_1 & A_2 & \ldots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & B_2 & \ldots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then

$$2\left|\left|\left|\left(\sum_{i=1}^{n} A_i B_i^*\right)\right|\right|\right| \leq |||A^*A + B^*B|||. \tag{39}$$

Proof Substituting $X = I$ in inequality (37) gives inequality (39). □

Remark 8 Letting $A_i = B_i = 0$ for $i = 2, 3, \ldots, n$ in inequality (39) gives

$$2|||A_1B_1^*||| \leq |||A_1^*A_1 + B_1^*B_1|||,$$

which is the arithmetic–geometric mean inequality for unitarily invariant norms.

The following inequality is a generalization and more general than inequality (24).

Corollary 8 *Let* $A_i \in \mathbb{M}_n$, $i = 1,2,\dots,n$, *where*

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then

$$\left|\left|\left| \left(\sum_{i=1}^{n} A_i A_i^* \right) \right|\right|\right| \le |||A^*A|||. \tag{40}$$

Proof Substituting $B_i = A_i$ in inequality (39) gives inequality (40). □

Remark 9 By making use of inequality (10), we note that when we specify inequality (40) to the usual spectral norm, it is sharper than inequality (12).

The following inequality is a generalization and more general than inequality (11).

Corollary 9 *Let* $A_i \in \mathbb{M}_n$, $i = 1,2,\dots,n$. *Then*

$$\left|\left|\left| \sum_{i=1}^{n} A_i \right|\right|\right| \le \left|\left|\left| \left(A_i^{1/2} A_j^{1/2} \right)_{1\le i,j\le n} \right|\right|\right|. \tag{41}$$

Proof Replacing A by $A^{1/2}$ in inequality (40) gives inequality (41). □

Remark 10 By using inequality (27), we note that when we specify inequality (41) to the spectral norm, it is sharper than inequality (21).

Because of the large number of papers that discuss unitarily invariant norms for 2×2 operator matrices, we specialize inequality (37) for $n = 2$. This special case contains several remarkable inequalities.

Corollary 10 *Let* $A_i, X_i, B_i \in \mathbb{M}_n$, $i = 1,2$. *Then*

$$2\left|\left|\left| A_1X_1B_1^* + A_2X_2B_2^* \right|\right|\right| \le |||Z|||, \tag{42}$$

where

$$Z = \begin{bmatrix} A_1^*A_1X_1 + X_1B_1^*B_1 & A_1^*A_2X_2 + X_1B_1^*B_2 \\ A_2^*A_1X_1 + X_2B_2^*B_1 & A_2^*A_2X_2 + X_2B_2^*B_2 \end{bmatrix}.$$

Proof Substituting $A_i = X_i = B_i = 0$ for $i = 3,4,\dots,n$ in inequality (37) gives inequality (42). □

By making use of inequality (42), we provide the following inequality.

Corollary 11 *Let $A_i, B_i \in \mathbb{M}_n$, $i = 1, 2$. Then*

$$2\left|\left|\left|A_1B_1^* + A_2B_2^*\right|\right|\right| \leq |||L \oplus M|||, \tag{43}$$

where

$$L = A_1^*A_1 + B_1^*B_1 + \left|A_2^*A_1 + B_2^*B_1\right|$$

and

$$M = A_2^*A_2 + B_2^*B_2 + \left|A_1^*A_2 + B_1^*B_2\right|.$$

Proof Throughout the proof of this theorem, let

$$S = \begin{bmatrix} A_1^*A_1 + B_1^*B_1 & 0 \\ 0 & A_2^*A_2 + B_2^*B_2 \end{bmatrix},$$
$$T = \begin{bmatrix} 0 & A_1^*A_2 + B_1^*B_2 \\ A_2^*A_1 + B_2^*B_1 & 0 \end{bmatrix}.$$

Substituting $X_1 = X_2 = I$ in inequality (42) gives

$$\begin{aligned} 2\left|\left|\left|A_1B_1^* + A_2B_2^*\right|\right|\right| &\leq \left|\left|\left|\begin{bmatrix} A_1^*A_1 + B_1^*B_1 & A_1^*A_2 + B_1^*B_2 \\ A_2^*A_1 + B_2^*B_1 & A_2^*A_2 + B_2^*B_2 \end{bmatrix}\right|\right|\right| \\ &= \left|\left|\left|(S+T)\right|\right|\right| \\ &= \left|\left|\left|\left(|S+T|\right)\right|\right|\right| \\ &\leq \left|\left|\left|\left(|S| + |T|\right)\right|\right|\right| \\ &= \left|\left|\left|S + \begin{bmatrix} |A_2^*A_1 + B_2^*B_1| & 0 \\ 0 & |A_1^*A_2 + B_1^*B_2| \end{bmatrix}\right|\right|\right| \\ &= \left|\left|\left|\begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}\right|\right|\right|. \end{aligned}$$

Inequality (43) has thus been substantiated. □

In turn, inequality (43) gives us the following finding.

Corollary 12 *Let $A, B \in \mathbb{M}_n$. Then*

$$2\left|\left|\left|AB^* + BA^*\right|\right|\right| \leq |||X \oplus X|||, \tag{44}$$

where

$$X = A^*A + B^*B + \left|B^*A + A^*B\right|.$$

Proof Substituting $A_1 = B_2 = A$ and $A_2 = B_1 = B$ in inequality (43) gives inequality (44). □

Another attractive special case of inequality (43) is the following result, which was shown in [21].

Corollary 13 *Let $A, B \in \mathbb{M}_n$ be positive. Then*

$$|||A + B||| \leq \left|\left|\left| \left(A + \left|B^{1/2}A^{1/2}\right|\right) \oplus \left(B + \left|A^{1/2}B^{1/2}\right|\right) \right|\right|\right|. \tag{45}$$

Proof Substituting $A_1 = B_1 = A^{1/2}$ and $A_2 = B_2 = B^{1/2}$ in inequality (43) gives inequality (45). □

If we look at inequality (42) from another side, we obtain the following inequality, which was proven in [20].

Corollary 14 *Let $A, B \in \mathbb{M}_n$. Then*

$$2\left|\left|\left| AB^* + BA^* \right|\right|\right| \leq \left|\left|\left| (A+B)^*(A+B) \oplus (A-B)^*(A-B) \right|\right|\right|. \tag{46}$$

Proof Throughout this proof, let

$$P = A^*A + B^*B + A^*B + B^*A$$

and

$$Q = A^*A + B^*B - A^*B - B^*A.$$

Substituting $A_1 = B_2 = A$ and $A_2 = B_1 = B$ in inequality (42) gives

$$\begin{aligned}
2\left|\left|\left| AB^* + BA^* \right|\right|\right| &\leq \left|\left|\left| \begin{bmatrix} A^*A + B^*B & A^*B + B^*A \\ B^*A + A^*B & A^*A + B^*B \end{bmatrix} \right|\right|\right| \\
&= \left|\left|\left| \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \right|\right|\right| \\
&\quad \left(\text{by applying equation (33)}\right) \\
&= \left|\left|\left| \begin{bmatrix} (A+B)^*(A+B) & 0 \\ 0 & (A-B)^*(A-B) \end{bmatrix} \right|\right|\right|,
\end{aligned}$$

which is precisely inequality (46). □

The next inequality is a special case of inequality (42), which was shown in [18].

Corollary 15 *Let $X_1, X_2 \in \mathbb{M}_n$. Then*

$$|||X_1 + X_2||| \leq 2|||X_1 \oplus X_2|||. \tag{47}$$

Proof Substituting $A_1 = A_2 = B_1 = B_2 = I$ in inequality (42) gives

$$2|||X_1 + X_2||| \leq \left|\left|\left| \begin{bmatrix} 2X_1 & X_1 + X_2 \\ X_1 + X_2 & 2X_2 \end{bmatrix} \right|\right|\right|$$

$$\le 2\left|\left|\left|\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\right|\right|\right| + \left|\left|\left|\begin{bmatrix} 0 & X_1+X_2 \\ X_1+X_2 & 0 \end{bmatrix}\right|\right|\right|$$
$$= 2|||X_1 \oplus X_2||| + |||X_1 + X_2|||.$$

This is equivalent to saying that

$$|||X_1 + X_2||| \le 2|||X_1 \oplus X_2|||.$$ □

Another special case of inequality (42) has been established by using a completely different technique in [25].

Corollary 16 *Let $A, B \in \mathbb{M}_n$ be positive. Then*

$$|||A - B||| \le |||A \oplus B|||. \tag{48}$$

Proof Letting $A_1 = B_1 = A^{1/2}$, $A_2 = -B_2 = B^{1/2}$ in inequality (42) gives

$$2|||A - B||| \le \left|\left|\left|\begin{bmatrix} 2A & 0 \\ 0 & 2B \end{bmatrix}\right|\right|\right|$$
$$= 2|||A \oplus B|||,$$

which is inequality (48). □

Acknowledgements
The authors are grateful to the referees for their valuable suggestions. The authors are indebted to University of Petra for its support.

Author contributions
Wasim Audeh and Manal Al-Labadi wrote section two, Raja'a Al-Naimi and Anwar Boustanji wrote section three, all authors reviewed the paper.

Funding
Not applicable.

Data availability
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, University of Petra, Amman, Jordan. [2]Department of Basic Sciences, German Jordanian University, Amman, Jordan.

Received: 16 October 2023 Accepted: 26 August 2024 Published online: 03 September 2024

References
1. Audeh, W.: Singular value inequalities and applications. Positivity **25**, 843–852 (2020)
2. Audeh, W.: Generalizations for singular value and arithmetic-geometric mean inequalities of operators. J. Math. Anal. Appl. **489**, 1–8 (2020)
3. Audeh, W.: Generalizations for singular value inequalities of operators. Adv. Oper. Theory **5**, 371–381 (2020)
4. Audeh, W.: Singular value and norm inequalities of Davidson-power type. J. Math. Inequal. **15**, 1311–1320 (2021)

5. Audeh, W.: Some generalizations for singular value inequalities of compact operators. Adv. Oper. Theory **6** (2021)
6. Audeh, W.: Singular value inequalities for operators and matrices. Ann. Funct. Anal. **13** (2022)
7. Audeh, W.: Singular value inequalities for accretive-dissipative normal operators. J. Math. Inequal. **16**, 729–737 (2022)
8. Audeh, W., Kittaneh, F.: Singular value inequalities for compact operators. Linear Algebra Appl. **437**, 2516–2522 (2012)
9. Audeh, W., Moradi, H., Sababheh, M.: Singular values inequalities via matrix monotone functions. Anal. Math. Phys. **13** (2023). https://doi.org/10.1007/s13324-023-00832-8
10. Audeh, W., Moradi, H., Sababheh, M.: Davidson-Power type and singular value inequalities. Hokkaido Math. J. To appear
11. Aujla, J.S., Silva, F.C.: Weak majorization inequalities and convex functions. Linear Algebra Appl. **369**, 217–233 (2003)
12. Bhatia, R.: Matrix Analysis, GTM169. Springer, New York (1997)
13. Bhatia, R., Davis, C.: More matrix forms of the arithmetic-geometric mean inequality. SIAM J. Matrix Anal. Appl. **14**, 132–136 (1993)
14. Bhatia, R., Kittaneh, F.: On the singular values of a product of operators. SIAM J. Matrix Anal. Appl. **11**, 272–277 (1990)
15. Bhatia, R., Kittaneh, F.: The matrix arithmetic-geometric mean inequality revisited. Linear Algebra Appl. **428**, 2177–2191 (2008)
16. Gohberg, I.C., Krein, M.G.: Introduction to the Theory of Linear Nonselfadjoint Operators. Am. Math. Soc., Providence (1969)
17. Hirzallah, O.: Inequalities for sums and products of operators. Linear Algebra Appl. **407**, 32–42 (2005)
18. Hirzallah, O., Kittaneh, F.: Inequalities for sums and direct sums of Hilbert space operators. Linear Algebra Appl. **424**, 71–82 (2007)
19. Hou, J.C., Du, H.K.: Norm inequalities of positive operator matrices. Integral Equ. Oper. Theory **22**, 281–294 (1995)
20. Kittaneh, F.: Some norm inequalities for operators. Can. Math. Bull. **42**, 87–96 (1999)
21. Kittaneh, F.: Norm inequalities for sums of positive operators. II. Positivity **10**, 251–260 (2006)
22. Popovici, D., Sebestyen, Z.: Norm estimations for finite sums of positive operators. J. Oper. Theory **56**, 3–15 (2006)
23. Simon, B.: Trace Ideals and Their Applications. Cambridge University Press, Cambridge (1979)
24. Tao, Y.: More results on singular value inequalities of matrices. Linear Algebra Appl. **416**, 724–729 (2006)
25. Zhan, X.: Singular values of differences of positive semidefinite matrices. SIAM J. Matrix Anal. Appl. **22**(3), 819–823 (2000)
26. Zou, L.: An arithmetic-geometric mean inequality for singular values and its applications. Linear Algebra Appl. **528**, 25–32 (2017)

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