Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

S-Pata-type contraction: a new approach to fixed-point theory with an application

Deep Chand[1], Yumnam Rohen[1,2], Naeem Saleem[3,4*], Maggie Aphane[4] and Asima Razzaque[5,6]

*Correspondence:
naeem.saleem2@gmail.com
[3]Department of Mathematics, University of Management and Technology, Lahore, Pakistan
[4]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Garankuwa, Pretoria, Medunsa 0204, South Africa
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce new types of contraction mappings named S-Pata-type contraction mapping and Generalized S-Pata-type contraction mapping in the framework of S-metric space. Then, we prove some new fixed-point results for S-Pata-type contraction mappings and Generalized S-Pata-type contraction mappings. To support our results, we provide examples to illustrate our findings and also apply these results to the ordinary differential equation to strengthen our conclusions.

Keywords: Fixed point; S-metric space; Pata-type contraction; S-Pata-type contraction

The study of fixed-point theory stands as one of the most vibrant and evolving research areas in mathematics within the last few decades and it is the most interesting topic for the researchers because of its simplicity, easiness, and applications in different fields. There have been published many results for fixed-point theory and their applications in different areas of mathematics and sciences. Exploring new and intriguing outcomes is achievable by focusing on two primary directions:

The first involves modifying the framework and structural attributes of the space [1–6], secondly, by changing the nature of the operators by imposing and reducing some restrictions [7–11].

Numerous publications discuss the generalization of the fundamental Banach Contraction Principle [12]. Pata introduced a captivating extension of the Banach Contraction Principle in a recent publication [13]. Many other authors followed Pata's approach and improved many fixed-point results that are present in the literature [14–23].

Neugebauer in [24] proved fixed-point results in the Banach space that is symmetric, and discussed its importance, and several researchers are working on this around the globe. Recalling that symmetry is a mapping on an object X, preserving its underlying structure, Neugebaner [24] utilized this concept to derive various applications of a layered compression–expansion fixed-point theorem. These applications resulted in the derivation of solutions for a second-order difference equation with Dirichlet boundary conditions.

In this paper, we have taken the abstract space as an S-metric space and generalized the concept of Pata's approach in this space and introduced the concept of S-Pata-type contractions and Generalized S-Pata-type contractions. Then, we apply this notion to prove some results of fixed points in an S-metric space. Sufficient examples are given in support of our results.

1 Preliminaries

Let us recall a few important definitions and preliminaries. In this section, we will use the norm $\|\vartheta\| = d(\vartheta, \vartheta_0)$ in a metric space (X, d), where ϑ_0 is a fixed element of X.

In the paper [13], Pata demonstrated an advancement upon the fundamental Banach Contraction Principle as follows:

Theorem 1.1 [13] *Assume (X, d) is a metric space that is complete and fixed constants $\Lambda \geq 0$, $\alpha \geq 1$, and β that lies in the interval $[0, \alpha]$. Let us consider a function $\psi : [0, 1] \to [0, \infty)$ that is increasing and meets the condition $\psi(0) = 0$. If the mapping $\Gamma : X \to X$ fulfills the subsequent inequality for each $\varepsilon \in [0, 1]$ and all $\vartheta, \theta \in X$,*

$$d(\Gamma\vartheta, \Gamma\theta) \leq (1 - \varepsilon)d(\vartheta, \theta) + \Lambda\varepsilon^{\alpha}\big[1 + \|\vartheta\| + \|\theta\|\big]^{\beta}$$

then, Γ possesses a unique fixed point $\vartheta^* \in X$, and the sequence $\{\Gamma^n\vartheta_0\}$ exhibits convergence towards ϑ^* for any given initial element $\vartheta_0 \in X$.

Further, Chakraborty and Samnta [25], Kadelburg and Radenović [15], Jacob et al. [14] extended the Theorem 1.1 to the case Kannan-type, Chatterjea-type, and Zamfirescu-type contraction mappings, respectively, as follows.

Note For the following theorems, let (X, d) represent a complete metric space, the fixed constants are $\Lambda \geq 0$, $\alpha \geq 1$, and $\beta \in [0, \alpha]$ and consider the function $\psi : [0, 1] \to [0, \infty)$ is an increasing function, continuous and $\psi(0) = 0$.

Theorem 1.2 [25] *Let the mapping $\Gamma : X \to X$ adhere to the inequality*

$$d(\Gamma\vartheta, \Gamma\theta) \leq \frac{(1 - \varepsilon)}{2}\big[d(\vartheta, \Gamma\vartheta) + d(\theta, \Gamma\theta)\big] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|\vartheta\| + \|\theta\| + \|\Gamma\vartheta\| + \|\Gamma\theta\|\big]^{\beta}$$

for all values of $\varepsilon \in [0, 1]$ and any elements $\vartheta, \theta \in X$, then the Γ possesses a unique fixed point $\vartheta^* \in X$.

Theorem 1.3 [14] *Let the mapping $\Gamma : X \to X$ adhere to the inequality expressed as follows:*

$$d(\Gamma\vartheta, \Gamma\theta) \leq \frac{(1 - \varepsilon)}{2}\big[d(\vartheta, \Gamma\theta) + d(\theta, \Gamma\vartheta)\big] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|\vartheta\| + \|\theta\| + \|\Gamma\vartheta\| + \|\Gamma\theta\|\big]^{\beta}$$

for all values of $\varepsilon \in [0, 1]$ and any elements $\vartheta, \theta \in X$, then the Γ possesses a unique fixed point $\vartheta^* \in X$.

Theorem 1.4 [15] *Let the mapping* $\Gamma : X \to X$ *adhere to the inequality expressed as follows*:

$$d(\Gamma\vartheta,\Gamma\theta) \le (1-\varepsilon)\max\left\{d(\vartheta,\theta), \frac{1}{2}\big[d(\vartheta,\Gamma\vartheta)+d(\theta,\Gamma\theta)\big], \frac{1}{2}\big[d(\vartheta,\Gamma\theta)+d(\theta,\Gamma\vartheta)\big]\right\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+\|\vartheta\|+\|\theta\|+\|\Gamma\vartheta\|+\|\Gamma\theta\|\big]^{\beta}$$

for all values of $\varepsilon \in [0,1]$ *and any elements* $\vartheta,\theta \in X$, *then* Γ *possesses a fixed point* $\vartheta^* \in X$ *that is unique*.

One of the papers in which the authors have already proved the fixed-point theorem using the Pata-type condition is [7].

In 2012, Sedghi and colleagues [6] introduced a novel notion known as the S-metric space. This concept emerges within the broader scope of generalizing and extending the concept of metric spaces, and its definition is presented as follows.

Definition 1.5 [6] Suppose that X is a nonempty set. An S-metric is a function on X defined as $S : X^3 \to [0,\infty)$ that fulfills the subsequent conditions for all elements $\vartheta,\theta,\varrho,a \in X$:

($\mathscr{S}_1$) $S(\vartheta,\theta,\varrho) \ge 0$,

($\mathscr{S}_2$) $S(\vartheta,\theta,\varrho) = 0$ if and only if $\vartheta = \theta = \varrho$,

($\mathscr{S}_3$) $S(\vartheta,\theta,\varrho) \le S(\vartheta,\vartheta,a) + S(\theta,\theta,a) + S(\varrho,\varrho,a)$.

The pair (X,S) is referred to as an S-metric space.

Lemma 1.6 [6] *Within an S-metric space* (X,S), *the equality* $S(\vartheta,\vartheta,\theta) = S(\theta,\theta,\vartheta)$ *holds true*.

Definition 1.7 [6] Consider (X,S) is an S-metric space,

(i) A sequence $\{\vartheta_n\}$ in X is said to converge to ϑ if and only if $S(\vartheta_n,\vartheta_n,\vartheta) \to 0$ holds as $n \to \infty$. In other words, for each given $\varepsilon > 0$, we have $n_0 \in N$ so that it holds that $S(\vartheta_n,\vartheta_n,\vartheta) < \varepsilon$ for any $n \ge n_0$, and this convergence is denoted by $\lim_{n\to\infty}\vartheta_n = \vartheta$.

(ii) A sequence ϑ_n within the set X is termed a Cauchy sequence if there exists a natural number $n_0 \in \mathbb{N}$ for any given $\varepsilon > 0$ so that the condition $S(\vartheta_n,\vartheta_n,\vartheta_m) < \varepsilon$ is fulfilled for all $n,m \ge n_0$.

(iii) If any Cauchy sequence within this space converges to a limit within X, then the space (X,S) is considered to be complete.

2 Results

In the current section, we will take into account fixed constants $\Lambda \ge 0$, $\alpha \ge 1$ and $\beta \in [0,\alpha]$, and we consider $\psi : [0,1] \to [0,\infty)$ to be a nondecreasing and continuous function vanishing at 0. In this section, we will make use of the norm defined as $\|\vartheta\| = S(\vartheta,\vartheta,\vartheta_0)$ with ϑ_0 being an arbitrary point in the space (X,S).

Definition 2.1 If (X,S) is an S-metric space. A self-map $\Gamma : X \to X$ is classified as having an S-Pata-type contraction mapping of type-I if the subsequent inequality

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho) \le (1-\varepsilon)S(\vartheta,\theta,\varrho) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+\|\vartheta\|+\|\theta\|+\|\varrho\|\big]^{\beta} \tag{1}$$

holds for any $\varepsilon \in [0,1)$ along with arbitrary $\vartheta,\theta,\varrho \in X$.

Definition 2.2 If (X,S) is an S-metric space. A self-map $\Gamma : X \to X$ is classified as having an S-Pata-type contraction mapping of type-II if the subsequent inequality

$$S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) \leq (1-\varepsilon)S(\vartheta,\vartheta,\theta) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+2\|\vartheta\|+\|\theta\|\big]^{\beta} \tag{2}$$

holds for any $\varepsilon \in [0,1)$ along with arbitrary $\vartheta,\theta \in X$.

The proof of the following lemmas in metric space can be found in [10]. Here, we extend it to the framework of an S-metric space.

Lemma 2.3 *If (X,S) is an S-metric space and the sequence $\{\vartheta_n\}$ is defined on X that is not Cauchy such that $\lim_{n\to\infty} S(\vartheta_n,\vartheta_n,\vartheta_{n+1}) = 0$, then we have two subsequences $\{\vartheta_{n_k}\}$ and $\{\vartheta_{m_k}\}$ of $\{\vartheta_n\}$ for any given $\varepsilon > 0$, such that*

$$\lim_{k\to\infty} S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k+1}) = \varepsilon^{+}, \tag{3}$$

$$\lim_{k\to\infty} S(\vartheta_{n_k},\vartheta_{n_k},\vartheta_{m_k}) = \lim_{k\to\infty} S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k}) = \lim_{k\to\infty} S(\vartheta_{n_k},\vartheta_{n_k},\vartheta_{m_k+1}) = \varepsilon. \tag{4}$$

Proof Since $\{\vartheta_n\}$ is not Cauchy and $\lim_{n\to\infty} S(\vartheta_n,\vartheta_n,\vartheta_{n+1}) = 0$, we have $\varepsilon > 0$ and $N_0 \geq 1$ so that for any $N > N_0$ and $m,n > N$ with $n \leq m$, we obtain

$$S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_{m+1}) > \varepsilon \quad \text{and} \quad S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_n) \leq \varepsilon.$$

By selecting the least $m \geq n$ for which $S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_{m+1}) > \varepsilon$ holds, we obtain the conclusion that there exists $N > N_0$ such that for any $n,m > N$,

$$S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_{m+1}) > \varepsilon \quad \text{and} \quad S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_m) \leq \varepsilon.$$

Thus, we can construct two subsequences $\{\vartheta_{n_k}\}$ and $\{\vartheta_{m_k}\}$ of $\{\vartheta_n\}$ such that

$$S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k+1}) > \varepsilon \quad \text{and} \quad S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k}) \leq \varepsilon.$$

The triangle inequality along with the above inequalities leads to the following conclusion:

$$\begin{aligned}\varepsilon < S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k+1}) &\leq S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k}) + 2S(\vartheta_{m_k},\vartheta_{m_k},\vartheta_{m_k+1}) \\ &\leq \varepsilon + 2S(\vartheta_{m_k},\vartheta_{m_k},\vartheta_{m_k+1}).\end{aligned} \tag{5}$$

We obtain (3) using the sandwich theorem. In addition, we have

$$S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k+1}) - 2S(\vartheta_{m_k+1},\vartheta_{m_k+1},\vartheta_{m_k}) \leq S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k}) \leq \varepsilon,$$

which demonstrates that the second limit in (4) is true. From the following two inequalities,

$$S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k}) - 2S(\vartheta_{n_k},\vartheta_{n_k},\vartheta_{n_k+1}) \leq S(\vartheta_{n_k},\vartheta_{n_k},\vartheta_{m_k}) \leq \varepsilon + 2S(\vartheta_{n_k},\vartheta_{n_k},\vartheta_{n_k+1}),$$

$$\varepsilon - 2S(\vartheta_{n_k},\vartheta_{n_k},\vartheta_{n_k+1}) < S(\vartheta_{n_k},\vartheta_{n_k},\vartheta_{m_k+1}) \leq S(\vartheta_{n_k+1},\vartheta_{n_k+1},\vartheta_{m_k+1}) + 2S(\vartheta_{n_k},\vartheta_{n_k},\vartheta_{n_k+1})$$

we will obtain the required limits. □

Theorem 2.4 *Let us consider a complete S-metric space* (X,S), *and if the following inequality is satisfied by a self-map* $\Gamma : X \to X$,

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho) \le (1-\varepsilon)S(\vartheta,\theta,\varrho) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+\|\vartheta\|+\|\theta\|+\|\varrho\|\big]^{\beta}$$

for any $\varepsilon \in [0,1)$ *and* $\vartheta,\theta,\varrho \in X$, *then* Γ *possesses a unique point* $\vartheta^* \in X$ *such that* $\Gamma\vartheta^* = \vartheta^*$. *If* $\vartheta_0 \in X$ *is any point,* $\vartheta_n = \Gamma\vartheta_{n-1} = \Gamma^n\vartheta_0$ *then,*

$$\lim_{n\to\infty}\vartheta_n = \vartheta^*.$$

Proof For any $\vartheta_0 \in X$, let us define a $\{\vartheta_n\}$ by $\vartheta_1 = \Gamma\vartheta_0$, $\vartheta_{n+1} = \Gamma\vartheta_n$. There is nothing to prove if n exists in N such that $\vartheta_{n+1} = \vartheta_n$. Let us suppose that $\vartheta_{n+1} \neq \Gamma\vartheta_n$ for all $n \in N$. Taking $\varepsilon = 0$, and $\vartheta = \theta = \vartheta_n$, $\varrho = \vartheta_{n-1}$, the inequality is obtained as

$$S(\vartheta_n,\vartheta_n,\vartheta_{n-1}) \le S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_{n-2}) \le \cdots \le S(\vartheta_1,\vartheta_1,\vartheta_0) = c_1.$$

We will now establish the proof for the boundedness of the sequence $\{c_n\}$:

$$\begin{aligned}
c_n &= S(\vartheta_n,\vartheta_n,\vartheta_0)\\
&\le 2S(\vartheta_n,\vartheta_n,\vartheta_{n+1}) + S(\vartheta_0,\vartheta_0,\vartheta_{n+1})\\
&\le 2c_1 + 2S(\vartheta_0,\vartheta_0,\vartheta_1) + S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_1,)\\
&\le 4c_1 + S(\Gamma\vartheta_n,\Gamma\vartheta_n,\Gamma\vartheta_0)\\
&\le 4c_1 + (1-\varepsilon)S(\vartheta_n,\vartheta_n,\vartheta_0) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+2\|\vartheta_n\|+\|\vartheta_0\|\big]^{\beta}\\
&\le 4c_1 + (1-\varepsilon)c_n + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+2c_n]^{\alpha},\\
\varepsilon c_n &\le 4c_1 + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+2c_n]^{\alpha}.
\end{aligned}$$

Let us consider the sequence $\{c_n\}$ is not bounded. There exists a subsequence $c_{n_k} \to \infty$, hence, we can write the above inequality as

$$\varepsilon_k c_{n_k} \le A + B\varepsilon_k^{\alpha}\psi(\varepsilon_k)c_{n_k}^{\alpha},$$

where $A = 4c_1$ and $B = 3\Lambda$ are fixed constant. Now, let us suppose that $\varepsilon_k = \frac{1+A}{c_{n_k}}$, then we have

$$1 \le B(1+A)^{\alpha}\psi(\varepsilon_k) \to 0,$$

which leads to a contradiction. Hence, $\{c_n\}$ is bounded.

We know that the sequence $\{S(\vartheta_n,\vartheta_n,\vartheta_{n-1})\}$ is monotonically decreasing and has a lower bound 0. Let $S(\vartheta_n,\vartheta_n,\vartheta_{n-1}) \to \delta > 0$.

By (1) we have,

$$
\begin{aligned}
S(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_n) &= S(\Gamma\vartheta_n, \Gamma\vartheta_n, \Gamma\vartheta_{n-1}) \\
&\leq (1-\varepsilon)S(\vartheta_n, \vartheta_n, \vartheta_{n-1}) + \Lambda\varepsilon^\alpha\psi(\varepsilon)\big[1 + 2\|\vartheta_n\| + \|\vartheta_{n-1}\|\big]^\beta \\
&\leq (1-\varepsilon)S(\vartheta_n, \vartheta_n, \vartheta_{n-1}) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + 2c_n + c_{n-1}]^\beta \\
\implies \quad & \delta \leq (1-\varepsilon)\delta + A\varepsilon^\alpha\psi(\varepsilon).
\end{aligned}
$$

Taking $n \to \infty$, where $A = \sup \Lambda[1 + 2c_n + c_{n-1}]^\beta$.

$$
\begin{aligned}
\implies \quad & \delta \leq A\varepsilon^{\alpha-1}\psi(\varepsilon) \\
\implies \quad & \delta = 0.
\end{aligned}
$$

Now, let us proceed to establish that $\{\vartheta_n\}$ is a Cauchy sequence. For the sake of contradiction, let us assume the opposite that $\{\vartheta_n\}$ is not Cauchy.

Using Lemma 2.3, we can identify the existence of $\delta > 0$ and two subsequences, namely ϑ_{n_k} and ϑ_{m_k}, derived from the original sequence ϑ_n, such that:

$$
\lim_{k\to\infty} S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k}) = \delta^+ \quad \text{and} \quad \lim_{k\to\infty} S(\vartheta_{n_k-1}, \vartheta_{n_k-1}, \vartheta_{m_k-1}) = \delta.
$$

This would make it clear that the inequality (1) implies the following statement. It follows that

$$
\begin{aligned}
S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k}) &= S(\Gamma\vartheta_{n_k-1}, \Gamma\vartheta_{n_k-1}, \Gamma\vartheta_{m_k-1}) \\
&\leq (1-\varepsilon)S(\vartheta_{n_k-1}, \vartheta_{n_k-1}, \vartheta_{m_k-1}) + \Lambda\varepsilon^\alpha\psi(\varepsilon)\big[1 + 2\|\vartheta_{n_k-1}\| + \|\vartheta_{m_k-1}\|\big]^\beta \\
&\leq (1-\varepsilon)S(\vartheta_{n_k-1}, \vartheta_{n_k-1}, \vartheta_{m_k-1}) + A\varepsilon^\alpha\psi(\varepsilon),
\end{aligned}
$$

where, $A = \sup \Lambda[1 + 2C_{n_k-1} + C_{m_k-1}]^\beta$. Now, as $k \to \infty$ we obtain the following inequality

$$
\begin{aligned}
&\delta \leq (1-\varepsilon)\delta + A\varepsilon^\alpha\psi(\varepsilon), \\
&\delta \leq A\varepsilon^{\alpha-1}\psi(\varepsilon),
\end{aligned}
$$

which implies $\delta = 0$, which is a contradiction. Hence, it can be concluded that $\{\vartheta_n\}$ is, in fact, a Cauchy sequence. Taking into consideration the completeness property of (X, S), the existence of an element ϑ^* in X such that the sequence ϑ_n converges to ϑ^* can be asserted.

Now, for any $n \in \mathbb{N}$, it follows that

$$
\begin{aligned}
S\big(\Gamma\vartheta^*, \Gamma\vartheta^*, \vartheta^*\big) &\leq 2S\big(\Gamma\vartheta^*, \Gamma\vartheta^*, \vartheta_{n+1}\big) + S\big(\vartheta^*, \vartheta^*, \vartheta_{n+1}\big) \\
&\leq 2\Big\{(1-\varepsilon)S\big(\vartheta^*, \vartheta^*, \vartheta_n\big) + \Lambda\varepsilon^\alpha\psi(\varepsilon)\big[1 + 2\big\|\vartheta^*\big\| + \|\vartheta_n\|\big]^\beta\Big\} \\
&\quad + S\big(\vartheta^*, \vartheta^*, \vartheta_{n+1}\big)
\end{aligned}
$$

taking $\varepsilon = 0$

$$
S\big(\Gamma\vartheta^*, \Gamma\vartheta^*, \vartheta^*\big) \leq 2S\big(\Gamma\vartheta^*, \Gamma\vartheta^*, \vartheta_n\big) + S\big(\vartheta^*, \vartheta^*, \vartheta_{n+1}\big).
$$

Hence, letting $n \to \infty$, $S(\Gamma\vartheta^*, \Gamma\vartheta^*, \vartheta^*) = 0$, that is $\Gamma\vartheta^* = \vartheta^*$.

Now, the task remaining is to establish that the fixed point is unique. For the sake of contradiction, let us assume that there are ϑ^* and θ^* two distinct fixed points for the mapping Γ:

$$\begin{aligned}
S(\vartheta^*, \vartheta^*, \theta^*) &= S(\Gamma\vartheta^*, \Gamma\vartheta^*, \Gamma\theta^*) \\
&\leq (1-\varepsilon)S(\vartheta^*, \vartheta^*, \theta^*) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + 2\|\vartheta^*\| + \|\theta^*\|\big]^{\beta}, \\
S(\vartheta^*, \vartheta^*, \theta^*) &\leq \Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)\big[1 + 2\|\vartheta^*\| + \|\theta^*\|\big]^{\beta} \\
\implies \quad S(\vartheta^*, \vartheta^*, \theta^*) &= 0 \quad \Leftrightarrow \quad \vartheta^* = \theta^*.
\end{aligned}$$

This concludes the proof of the fixed point's uniqueness. □

Remark 1 [14] The Bernaulli's inequality is $(1 + pq) \leq (1 + q)^p$, for $p \geq 1$ and $q \in [-1, +\infty)$.

Corollary 2.5 *Suppose that (X, S) is a complete S-metric space and that $\Gamma : X \to X$ satisfies the following inequality*

$$S(\Gamma\vartheta, \Gamma\theta, \Gamma\varrho) \leq \lambda S(\vartheta, \theta, \varrho), \quad \text{where } \lambda \in [0, 1),$$

then there is an unique point ϑ in X, where $\Gamma\vartheta = \vartheta$.

Proof We are given that

$$\begin{aligned}
S(\Gamma\vartheta, \Gamma\theta, \Gamma\varrho) &\leq \lambda S(\vartheta, \theta, \varrho) \\
&= (1-\varepsilon)S(\vartheta, \theta, \varrho) + (\lambda + \varepsilon - 1)S(\vartheta, \theta, \varrho) \\
&\leq (1-\varepsilon)S(\vartheta, \theta, \varrho) + \lambda\left(1 + \frac{\varepsilon - 1}{\lambda}\right)\big[S(\vartheta, \vartheta, \vartheta_0) + S(\theta, \theta, \vartheta_0) + S(\varrho, \varrho, \vartheta_0)\big] \\
&\leq (1-\varepsilon)S(\vartheta, \theta, \varrho) + \lambda\varepsilon^{\frac{1}{\lambda}}\big[1 + \|\vartheta\| + \|\theta\| + \|\varrho\|\big],
\end{aligned}$$

using Remark 1. Since $\lambda \in [0, 1)$ implies $\frac{1}{\lambda} \geq 1$.

$$S(\Gamma\vartheta, \Gamma\theta, \Gamma\varrho) \leq (1-\varepsilon)S(\vartheta, \theta, \varrho) + \lambda\varepsilon\varepsilon^{\frac{1-\lambda}{\lambda}}\big[1 + \|\vartheta\| + \|\theta\| + \|\varrho\|\big],$$

which is the inequality given in Theorem 2.4 for $\Lambda = \lambda$, $\alpha = 1$, $\psi(\varepsilon) = \varepsilon^{\frac{1-\lambda}{\lambda}}$, and $\beta = 1$. Hence, the Γ possesses a unique fixed point. □

Example 2.6 Consider $\mathbb{X} = [1, 100]$ with an S-metric defined as $S(\vartheta, \theta, \varrho) = |\vartheta - \theta| + |\theta - \varrho| + |\varrho - \vartheta|$, then it can be proved that the space $(\mathbb{X}, S)$ is a complete S-metric space. Let us consider a self-map $\Gamma : \mathbb{X} \to \mathbb{X}$ defined as $\Gamma\vartheta = 1 - \sqrt{\vartheta} + \vartheta$. Then, for any $\vartheta, \theta \in \mathbb{X}$, we have

$$\begin{aligned}
|\Gamma\vartheta - \Gamma\theta| &= \big|(1 - \sqrt{\vartheta} + \vartheta) - (1 - \sqrt{\theta} + \theta)\big| = \big|(\vartheta - \theta) - (\sqrt{\vartheta} - \sqrt{\theta})\big| \\
&= \left|(\vartheta - \theta) - \frac{\vartheta - \theta}{\sqrt{\vartheta} + \sqrt{\theta}}\right| = \left|1 - \frac{1}{\sqrt{\vartheta} + \sqrt{\theta}}\right||\vartheta - \theta|
\end{aligned}$$

$$\leq \left|1-\frac{1}{\sqrt{100}+\sqrt{100}}\right||\vartheta-\theta|$$
$$=\frac{19}{20}|\vartheta-\theta|.$$

Then, for some fixed $\vartheta_0 \in X$ and any $\vartheta, \theta, \varrho \in X$, we obtain

$$\begin{aligned}
&S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho)\\
&\quad=|\Gamma\theta-\Gamma\varrho|+|\Gamma\varrho-\Gamma\vartheta|=\frac{19}{20}\big[|\vartheta-\theta|+|\theta-\varrho|+|\varrho-\vartheta|\big]=\frac{19}{20}S(\vartheta,\theta,\varrho)\\
&\quad=(1-\varepsilon)S(\vartheta,\theta,\varrho)+\left\{\frac{19}{20}-(1-\varepsilon)\right\}S(\vartheta,\theta,\varrho)\\
&\quad\leq(1-\varepsilon)S(\vartheta,\theta,\varrho)+\frac{19}{20}\left\{1-\frac{(1-\varepsilon)}{\frac{19}{20}}\right\}\big[S(\vartheta,\vartheta,\vartheta_0)+S(\theta,\theta,\vartheta_0)+S(\varrho,\varrho,\vartheta_0)\big]\\
&\quad\leq(1-\varepsilon)S(\vartheta,\theta,\varrho)+\frac{19}{20}\varepsilon^{\frac{20}{19}}\big[1+S(\vartheta,\vartheta,\vartheta_0)+S(\theta,\theta,\vartheta_0)+S(\varrho,\varrho,\vartheta_0)\big]\\
&\quad\leq(1-\varepsilon)S(\vartheta,\theta,\varrho)+\frac{19}{20}\varepsilon^{1}.\varepsilon^{\frac{1}{19}}\big[1+\|\vartheta\|+\|\theta\|+\|\varrho\|\big],
\end{aligned}$$

which is the inequality (1) for all $\varepsilon\in[0,1]$, $\Lambda=\frac{19}{20}$, $\alpha=\beta=1$ and $\psi(\varepsilon)=\varepsilon^{\frac{1}{20}}$. By Theorem 2.4, the map Γ has a unique fixed point. It is worth noting that 1 is the only fixed point of the mapping Γ.

Remark 2 The function satisfying the inequality given in equation (1), must be a continuous function.

Definition 2.7 Let us consider an S-metric space (X,S). A self-map $\Gamma: X\to X$ is said to have a generalized S-Pata-type contraction mapping of type-I if it satisfies the following inequality

$$\begin{aligned}
S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho)\leq{}&\frac{(1-\varepsilon)}{3}\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\}\\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+\|\vartheta\|+\|\theta\|+\|\varrho\|+\|\Gamma\vartheta\|+\|\Gamma\theta\|+\|\Gamma\varrho\|\big]^{\beta}
\end{aligned}\tag{6}$$

for each $\varepsilon\in[0,1)$ and $\vartheta,\theta,\varrho\in X$.

Definition 2.8 Let us consider an S-metric space (X,S). A self-map $\Gamma: X\to X$ is said to have a generalized S-Pata-type contraction mapping of type-II if it satisfies the following inequality

$$\begin{aligned}
S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta)\leq{}&\frac{(1-\varepsilon)}{3}\big\{2S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big\}\\
&+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+2\|\vartheta\|+\|\theta\|+2\|\Gamma\vartheta\|+\|\Gamma\theta\|\big]^{\beta}
\end{aligned}\tag{7}$$

for each $\varepsilon\in[0,1)$ and $\vartheta,\theta\in X$.

Theorem 2.9 *A self-map $\Gamma : X \to X$ is defined on a complete S-metric space (X,S) and holds the subsequent inequality*

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho) \leq \frac{(1-\varepsilon)}{3}\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+\|\vartheta\|+\|\theta\|+\|\varrho\|+\|\Gamma\vartheta\|+\|\Gamma\theta\|+\|\Gamma\varrho\|\big]^{\beta}$$

for each $\varepsilon \in [0,1)$ and all $\vartheta,\theta,\varrho \in X$. Then, Γ possess a unique fixed point ϑ^* and if $\vartheta_n = \Gamma\vartheta_{n-1}$ for any $\vartheta_0 \in X$ then $\lim_{n\to\infty}\vartheta_n = \vartheta^*$

Proof Let us choose a point ϑ_0 from X, and let $\vartheta_n = \Gamma\vartheta_{n-1}$ and $c_n = S(\vartheta_n,\vartheta_n,\vartheta_0) = \|\vartheta_n\|$. In the inequality (7) taking $\varepsilon = 0$ and using Lemma 1.6 we have

$$\begin{aligned}
&S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_n) \leq \frac{1}{3}\big\{2S(\vartheta_n,\vartheta_n,\vartheta_{n+1})+S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)\big\},\\
&S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_n) \leq S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n) = S(\vartheta_n,\vartheta_n,\vartheta_{n-1})\\
&\quad\Longrightarrow\quad S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_n) \leq S(\vartheta_n,\vartheta_n,\vartheta_{n-1}) \leq \cdots \leq S(\vartheta_1,\vartheta_1,\vartheta_0) = C_1.
\end{aligned}$$

Now, we will prove that c_n is a bounded sequence:

$$\begin{aligned}
c_n = S(\vartheta_n,\vartheta_n,\vartheta_0) &\leq 2S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_n)+2S(\vartheta_1,\vartheta_1,\vartheta_0)+S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_1)\\
&\leq 2c_1+2c_1+S(\Gamma\vartheta_n,\Gamma\vartheta_n,\Gamma\vartheta_0)\\
&\leq 4c_1+\frac{(1-\varepsilon)}{3}\big\{2S(\vartheta_n,\vartheta_n,\Gamma\vartheta_n)+S(\vartheta_0,\vartheta_0,\Gamma\vartheta_0)\big\}\\
&\quad+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+2\|\vartheta_n\|+\|\vartheta_0\|+\|\Gamma\vartheta_n\|+\|\Gamma\vartheta_0\|\big]^{\beta}\\
&\leq 4c_1+\frac{(1-\varepsilon)}{3}\{2c_1+c_1\}+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\big[1+2\|\vartheta_n\|+\|\vartheta_0\|+\|\Gamma\vartheta_n\|+\|\Gamma\vartheta_0\|\big]^{\beta}\\
&\leq 5c_1,\quad \text{for } \varepsilon = 0,
\end{aligned}$$

which proves that $\{c_n\}$ is bounded. Since the sequence $S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_n)$ is monotonically decreasing having a bound 0, it follows that there exists $\delta \geq 0$ such that $S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_n) \to \delta$, then

$$\begin{aligned}
S(\vartheta_{n+1},\vartheta_{n+1},\vartheta_n) &\leq S(\Gamma\vartheta_n,\Gamma\vartheta_n,\Gamma\vartheta_{n-1})\\
&\leq \frac{(1-\varepsilon)}{3}\big\{2S(\vartheta_n,\vartheta_n,\Gamma\vartheta_n)+S(\vartheta_{n-1},\vartheta_{n-1},\Gamma\vartheta_{n-1})\big\}+K\varepsilon^{\alpha}\psi(\varepsilon)\\
&\leq \frac{(1-\varepsilon)}{3}\big\{2S(\vartheta_n,\vartheta_n,\vartheta_{n+1})+S(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)\big\}+K\varepsilon^{\alpha}\psi(\varepsilon),
\end{aligned}$$

where, $K > 0$ and letting $n \to \infty$ we obtain

$$\begin{aligned}
&\delta \leq \frac{(1-\varepsilon)}{3}\{2\delta+\delta\}+K\varepsilon^{\alpha}\psi(\varepsilon),\\
&\delta \leq K\varepsilon^{\alpha-1}\psi(\varepsilon) \quad\Longrightarrow\quad \delta = 0.
\end{aligned}$$

To prove $\{\vartheta_n\}$ is a Cauchy sequence, on the contrary let us suppose that $\{\vartheta_n\}$ is not a Cauchy sequence. By Lemma 2.3 there exist two subsequences $\{\vartheta_{n_k}\}$ and $\{\vartheta_{m_k}\}$ of sequence $\{\vartheta_n\}$ for any $\varepsilon > 0$, such that

$$\lim_{k\to\infty} S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k}) = \varepsilon \quad \text{and} \quad \lim_{k\to\infty} S(\vartheta_{n_k-1}, \vartheta_{n_k-1}, \vartheta_{m_k-1}) = \varepsilon.$$

Hence,

$$\begin{aligned} S(\vartheta_{n_k}, \vartheta_{n_k}, \vartheta_{m_k}) &= S(\Gamma\vartheta_{n_k-1}, \Gamma\vartheta_{n_k-1}, \Gamma\vartheta_{m_k-1}) \\ &\leq \frac{(1-\varepsilon)}{3}\big\{2S(\vartheta_{n_k-1}, \vartheta_{n_k-1}, \vartheta_{n_k}) + S(\vartheta_{m_k-1}, \vartheta_{m_k-1}, \vartheta_{m_k})\big\} + K\varepsilon^\alpha\psi(\varepsilon). \end{aligned}$$

Considering the above limit and letting $n \to \infty$, we obtain

$$\begin{aligned} &\varepsilon \leq K\varepsilon^\alpha\psi(\varepsilon) \\ &\quad\implies \quad \varepsilon = 0, \end{aligned}$$

which leads to a contradiction. Hence, sequence $\{\vartheta_n\}$ is a Cauchy sequence. Taking into account the completeness of X we have $\vartheta^* \in X$ such that $\vartheta_n \to \vartheta^*$. Now, we verify that ϑ^* is a fixed point for the mapping Γ. We observe that, for all $n \in \mathbb{N}$,

$$\begin{aligned} S(\vartheta^*, \vartheta^*, \Gamma\vartheta^*) &\leq 2S(\vartheta^*, \vartheta^*, \vartheta_{n+1}) + S(\Gamma\vartheta^*, \Gamma\vartheta^*, \Gamma\vartheta_n) \\ &\leq 2S(\vartheta^*, \vartheta^*, \vartheta_{n+1}) \\ &\quad + \frac{(1-\varepsilon)}{3}\big\{2S(\vartheta^*, \vartheta^*, \Gamma\vartheta^*) + S(\vartheta_n, \vartheta_n, \Gamma\vartheta_n)\big\} + K\varepsilon^\alpha\psi(\varepsilon) \\ &\leq 2S(\vartheta^*, \vartheta^*, \vartheta_{n+1}) \\ &\quad + \frac{(1-\varepsilon)}{3}\big\{2S(\vartheta^*, \vartheta^*, \Gamma\vartheta^*) + S(\vartheta_n, \vartheta_n, \vartheta_{n+1})\big\} + K\varepsilon^\alpha\psi(\varepsilon). \end{aligned}$$

Letting $n \to \infty$, and taking $\varepsilon = 0$, we have

$$\begin{aligned} &S(\vartheta^*, \vartheta^*, \Gamma\vartheta^*) \leq \frac{2}{3}S(\vartheta^*, \vartheta^*, \Gamma\vartheta^*) \\ &\quad\implies \quad S(\vartheta^*, \vartheta^*, \Gamma\vartheta^*) = 0 \\ &\quad\implies \quad \Gamma\vartheta^* = \vartheta^*. \end{aligned}$$

Now, we are left to prove that the fixed point is unique. On the contrary, let us assume two fixed points ϑ^* and θ^* for Γ, it follows that

$$\begin{aligned} S(\vartheta^*, \vartheta^*, \theta^*) &= S(\Gamma\vartheta^*, \Gamma\vartheta^*, \Gamma\theta^*) \\ &\leq \frac{(1-\varepsilon)}{3}\big\{2S(\vartheta^*, \vartheta^*, \Gamma\vartheta^*) + S(\theta^*, \theta^*, \Gamma\theta^*)\big\} + K\varepsilon^\alpha\psi(\varepsilon), \quad \text{where } K > 0 \\ &\leq K\varepsilon^\alpha\psi(\varepsilon) \\ \implies \quad & S(\vartheta^*, \vartheta^*, \theta^*) = 0 \end{aligned}$$

$$\implies \quad \vartheta^* = \theta^*.$$

This illustrates the fixed point's uniqueness. □

Example 2.10 Let us consider an S-metric space $(\mathbb{X}, S)$, where $\mathbb{X} = [0,1]$ and the metric is defined by $S(\vartheta,\theta,\varrho) = |\vartheta-\theta| + |\theta-\varrho| + |\varrho-\vartheta|$. Let us define a self-map on $\Gamma : \mathbb{X} \to \mathbb{X}$ by

$$\Gamma\vartheta = \begin{cases} \frac{\vartheta}{8}, & \text{if } \vartheta \in [0,\frac{1}{2}); \\ \frac{\vartheta}{10}, & \text{if } \vartheta \in [\frac{1}{2},1]. \end{cases}$$

Here, Γ is discontinuous at $\vartheta = \frac{1}{2}$; consequently, Theorem 2.4 can not be applied. Further, to apply Theorem 2.9, let us discuss the following cases:

Case 1. For $\vartheta,\theta \in [0,\frac{1}{2})$;

$$\begin{aligned} S(\vartheta,\vartheta,\Gamma\vartheta) + S(\vartheta,\vartheta,\Gamma\vartheta) + S(\theta,\theta,\Gamma\theta) &= 4\left(\frac{7\vartheta}{8}\right) + 2\left(\frac{7\theta}{8}\right) = \frac{7}{4}[2\vartheta+\theta], \\ S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) = \frac{1}{4}|\vartheta-\theta| &\le \frac{2}{9} \times \frac{7}{4}[2\vartheta+\theta] \\ &\le \frac{2}{9}\big[S(\vartheta,\vartheta,\Gamma\vartheta) + S(\vartheta,\vartheta,\Gamma\vartheta) + S(\theta,\theta,\Gamma\theta)\big]. \end{aligned}$$

Case 2. For $\vartheta,\theta \in [\frac{1}{2},1]$;

$$\begin{aligned} S(\vartheta,\vartheta,\Gamma\vartheta) + S(\vartheta,\vartheta,\Gamma\vartheta) + S(\theta,\theta,\Gamma\theta) &= 4\left(\frac{9\vartheta}{10}\right) + 2\left(\frac{9\theta}{10}\right) = \frac{9}{5}[2\vartheta+\theta], \\ S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) = \frac{1}{5}|\vartheta-\theta| &\le \frac{2}{9} \times \frac{9}{5}[2\vartheta+\theta] \\ &\le \frac{2}{9}\big[S(\vartheta,\vartheta,\Gamma\vartheta) + S(\vartheta,\vartheta,\Gamma\vartheta) + S(\theta,\theta,\Gamma\theta)\big]. \end{aligned}$$

Case 3. For $\vartheta \in [0,\frac{1}{2})$, $\theta \in [\frac{1}{2},\frac{5}{8})$;

$$\begin{aligned} S(\vartheta,\vartheta,\Gamma\vartheta) + S(\vartheta,\vartheta,\Gamma\vartheta) + S(\theta,\theta,\Gamma\theta) &= 4\left(\frac{7\vartheta}{8}\right) + 2\left(\frac{9\theta}{10}\right) \\ &\ge 0 + \frac{9}{5} \times \frac{1}{2} = \frac{9}{10}, \\ S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) = 2\left|\frac{\vartheta}{8}\right| - \frac{\theta}{10} &\le 2 \times \frac{1}{16} = \frac{1}{8} \le \frac{2}{9} \times \frac{9}{10} \\ &\le \frac{2}{9}\big[S(\vartheta,\vartheta,\Gamma\vartheta) + S(\vartheta,\vartheta,\Gamma\vartheta) + S(\theta,\theta,\Gamma\theta)\big]. \end{aligned}$$

Case 4. For $\vartheta \in [0,\frac{1}{2})$, $\theta \in [\frac{5}{8},1]$;

$$\begin{aligned} S(\vartheta,\vartheta,\Gamma\vartheta) + S(\vartheta,\vartheta,\Gamma\vartheta) + S(\theta,\theta,\Gamma\theta) &= 4\left(\frac{7\vartheta}{8}\right) + 2\left(\frac{9\theta}{10}\right) \\ &\ge 0 + \frac{9}{5} \times \frac{5}{8} = \frac{9}{8}, \end{aligned}$$

$$\begin{aligned} S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) &= 2\left|\frac{\vartheta}{8}-\frac{\theta}{10}\right| \le 2\times\frac{1}{10} = \frac{2}{9}\times\frac{9}{8} \\ &\le \frac{2}{9}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big]. \end{aligned}$$

Therefore, for all $\vartheta,\theta \in \mathbb{X}$ and $\varepsilon \in [0,1]$, we have

$$\begin{aligned} S(\Gamma\vartheta,\Gamma\vartheta,\Gamma\theta) &\le \frac{2}{9}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &= \frac{(1-\varepsilon)}{3}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &\quad+\left\{\frac{2}{9}-\frac{(1-\varepsilon)}{3}\right\}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &= \frac{(1-\varepsilon)}{3}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &\quad+\frac{2}{9}\left\{1-\frac{(1-\varepsilon)}{\frac{2}{3}}\right\}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &\le \frac{(1-\varepsilon)}{3}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &\quad+\frac{2}{9}\varepsilon^{\frac{3}{2}}\big[(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &\le \frac{(1-\varepsilon)}{3}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &\quad+\frac{2}{9}\varepsilon\varepsilon^{\frac{1}{2}}\big[2\|\vartheta\|+\|\Gamma\vartheta\|+2\|\vartheta\|+\|\Gamma\vartheta\|+2\|\theta\|+\|\Gamma\theta\|\big] \\ &\le \frac{(1-\varepsilon)}{3}\big[S(\vartheta,\vartheta,\Gamma\vartheta)+S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)\big] \\ &\quad+\frac{4}{9}\varepsilon\varepsilon^{\frac{1}{2}}\big[1+2\|\vartheta\|+\|\theta\|+2\|\Gamma\vartheta\|+\|\Gamma\theta\|\big], \end{aligned}$$

which indicates that the map Γ meets all the requirements outlined in Theorem 2.9 when considering $\alpha=\beta=1$, $\Lambda=\frac{4}{9}$, and $\psi(\varepsilon)=\varepsilon^{\frac{1}{2}}$. Consequently, it possesses a unique fixed point.

Corollary 2.11 *If (X,S) is a complete S-metric space. If for any nonnegative numbers $\alpha,\gamma,\delta \in [0,1)$, the self-map $\Gamma: X \to X$ satisfies the following inequality*:

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho) \le \frac{1}{3}\big\{\phi S(\vartheta,\vartheta,\Gamma\vartheta)+\gamma S(\theta,\theta,\Gamma\theta)+\delta S(\varrho,\varrho,\Gamma\varrho)\big\}.$$

Then, there exists a fixed point for Γ.

Proof Let us suppose that $\lambda=\max\{\phi,\gamma,\delta\}$. Using Remark 1, we can write the above inequality as

$$\begin{aligned} S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho) &\le \frac{1}{3}\big\{\phi S(\vartheta,\vartheta,\Gamma\vartheta)+\gamma S(\theta,\theta,\Gamma\theta)+\delta S(\varrho,\varrho,\Gamma\varrho)\big\} \\ &\le \frac{\lambda}{3}\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1-\varepsilon)}{3}\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\}\\
&\quad+\left(\frac{\lambda}{3}+\frac{\varepsilon-1}{3}\right)\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\}\\
&\leq \frac{(1-\varepsilon)}{3}\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\}\\
&\quad+\frac{\lambda}{3}\varepsilon^{\frac{1}{\lambda}}\begin{bmatrix}2S(\vartheta,\vartheta,\vartheta_0)+S(\Gamma\vartheta,\Gamma\vartheta,\vartheta_0)+2S(\theta,\theta,\vartheta_0)+S(\Gamma\theta,\Gamma\theta,\vartheta_0)\\ +2S(\varrho,\varrho,\vartheta_0)+S(\Gamma\varrho,\Gamma\varrho,\vartheta_0)\end{bmatrix}\\
&\leq \frac{(1-\varepsilon)}{3}\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\}\\
&\quad+\frac{2\lambda}{3}\varepsilon\varepsilon^{\frac{1-\lambda}{\lambda}}\big[1+\|\vartheta\|+\|\theta\|+\|\varrho\|+\|\Gamma\vartheta\|+\|\Gamma\theta\|+\|\Gamma\varrho\|\big],
\end{aligned}$$

which is the inequality given in Theorem 2.9 for $\Lambda = \frac{2\lambda}{3}$, $\alpha = 1$, $\psi(\varepsilon) = \varepsilon^{\frac{1-\lambda}{\lambda}}$, and $\beta = 1$. Hence, the map Γ has a unique fixed point. □

Corollary 2.12 *If (X,S) is a complete S-metric space and γ is a nonnegative number in $[0,\frac{1}{3})$ and if the self-map $\Gamma : X \to X$ satisfies the subsequent inequality*

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho) \leq \gamma\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\}, \tag{8}$$

then there exists a fixed point for Γ.

Proof Let us consider $\lambda = 3\gamma$. Since $\gamma \in [0,\frac{1}{3})$ implies that $\lambda \in [0,1)$, and the above inequality (8) can be written as

$$S(\Gamma\vartheta,\Gamma\theta,\Gamma\varrho) \leq \frac{\lambda}{3}\big\{S(\vartheta,\vartheta,\Gamma\vartheta)+S(\theta,\theta,\Gamma\theta)+S(\varrho,\varrho,\Gamma\varrho)\big\}.$$

Now, the rest of the proof is followed by Corollary 2.11. □

3 Application to ordinary differential equations

The objective of this section is to determine the presence of a solution for the boundary value problems by utilizing the outcomes established in the preceding section.

We examine the subsequent BVP concerning a second-order differential equation:

$$\begin{cases} -\frac{d^2u}{dt^2} = f(t,u(t)), & t \in [0,1];\\ u(0) = u(1) = 0, \end{cases} \tag{9}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ represents a continuous function.

The corresponding Green function to equation (9) is

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1;\\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

The integral equation associated with the boundary value problem (9) can be formulated as:

$$u(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds \quad \text{for all } t \in [0,1]. \tag{10}$$

Let us suppose a set $C([0,1])$, which comprises every continuous function defined on the interval $[0,1]$. It is a commonly recognized fact that the set $C([0,1])$ with the metric defined as follows:

$$S(u,v,w) = \|u-v\|_\infty + \|v-w\|_\infty + \|w-u\|_\infty = \max_{t\in[0,1]}\bigl(|u-v| + |v-w| + |w-u|\bigr),$$

is a complete S-metric space. Let us consider a self-map $T : C([0,1]) \to C([0,1])$ defined by

$$Tu(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,ds \quad \text{for } t \in [0,1]. \tag{11}$$

Theorem 3.1 *The given BVP* (9) *has at least one solution* $u^* \in C([0,1])$ *if the subsequent inequality is satisfied for all* $t \in [0,1]$ *and* $a,b \in \mathbb{R}$:

$$\bigl|f(t,a) - f(t,b)\bigr| \le 2|a - Tb| + |b - Tb|,$$

where $T : C([0,1]) \to C([0,1])$ *is a function defined in* (11).

Proof The equivalence between a solution of equation (9) and a solution of the integral equation (10) is a well-established fact. In other words, solving problem (9) is essentially the same as finding a function $u^* \in C([0,1])$ that satisfies $Tu^* = u^*$, where T is defined in equation (11). This means that the problem (9) is effectively reduced to identifying a fixed point of the operator T in the function space $C([0,1])$.

Now, let $u,v \in C([0,1])$ such that $\xi(u(t),u(t),v(t)) \ge 0$ for all $t \in [0,1]$:

$$\begin{aligned}
\bigl|Tu(t) - Tv(t)\bigr| &= \left|\int_0^1 G(t,s)\bigl[f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr]\,ds\right| \\
&\le \int_0^1 G(t,s)\bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|\,ds \\
&\le \left(\sup_{t\in I}\int_0^1 G(t,s)\,ds\right)\bigl[2\bigl|Tu(t) - u(t)\bigr| + \bigl|Tv(t) - v(t)\bigr|\bigr] \\
&= \frac{1}{8}\bigl[2\bigl|Tu(t) - u(t)\bigr| + \bigl|Tv(t) - v(t)\bigr|\bigr].
\end{aligned}$$

Note that for all $t \in [0,1]$, $\int_0^1 G(t,s)\,ds = -\frac{t^2}{2} + \frac{t}{2}$, which implies $\sup_{t\in[0,1]}\int_0^1 G(t,s)\,ds = \frac{1}{8}$:

$$2\max_{t\in[0,1]}\bigl|Tu(t) - Tv(t)\bigr| \le \frac{2}{8}\Bigl[2\max_{t\in[0,1]}\bigl(\bigl|Tu(t) - u(t)\bigr| + \bigl|Tv(t) - v(t)\bigr|\bigr)\Bigr],$$

$$\begin{aligned}
S\big(Tu(t),Tu(t),Tv(t)\big) &\leq \frac{1}{8}\big[2S\big(u(t),u(t),Tu(t)\big)+S\big(v(t),v(t),Tv(t)\big)\big] \\
&= \frac{1-\epsilon}{3}\big[2S\big(u(t),u(t),Tu(t)\big)+S\big(v(t),v(t),Tv(t)\big)\big] \\
&\quad + \left(\frac{1}{8}-\frac{1-\epsilon}{3}\right)\big[2S\big(u(t),u(t),Tu(t)\big)+S\big(v(t),v(t),Tv(t)\big)\big] \\
&\leq \frac{1-\epsilon}{3}\big[2S\big(u(t),u(t),Tu(t)\big)+S\big(v(t),v(t),Tv(t)\big)\big] \\
&\quad + \frac{1}{8}\left(1-\frac{1-\epsilon}{\frac{3}{8}}\right)\big[4\|u\|+2\|v\|+2\|Tu\|+\|Tv\|\big] \\
&\leq \frac{1-\epsilon}{3}\big[2S\big(u(t),u(t),Tu(t)\big)+S\big(v(t),v(t),Tv(t)\big)\big] \\
&\quad + \frac{1}{4}\varepsilon^{\frac{8}{3}}\big[2\|u\|+\|v\|+2\|Tu\|+\|Tv\|\big] \quad \{\text{Using Remark 1}\} \\
&\leq \frac{1-\epsilon}{3}\big[2S\big(u(t),u(t),Tu(t)\big)+S\big(v(t),v(t),Tv(t)\big)\big] \\
&\quad + \frac{1}{4}\varepsilon^{2}\varepsilon^{\frac{2}{3}}\big[1+2\|u\|+\|v\|+2\|Tu\|+\|Tv\|\big].
\end{aligned}$$

Thus, the criteria outlined in Theorem 2.9 are met for the mapping T, where $\alpha = 2$, $\beta = 1$, $\Lambda = \frac{1}{4}$, and $\psi(\varepsilon) = \varepsilon^{\frac{2}{3}}$. Consequently, this indicates that the mapping T possesses a unique fixed point $u^* \in C([0,1])$ that is $Tu^* = u^*$. As a result, it can be inferred that u^* represents the unique solution to the BVP (9). □

4 Conclusions

In conclusion, following Pata-type contraction we defined new types of contraction mappings and studied fixed-point results within the framework of S-metric spaces for these mappings. These new ideas of contractions generalized for the self-mappings generalize a large number of results present in the literature related the existence and uniqueness of fixed points in S-metric space. We demonstrated that the self-operators, on a complete metric space, that fulfill this contraction need to have only one fixed point. Also, we proved the Banach contraction theorem in S-metric space as a corollary of our obtained results. Additionally, we present examples that explain the obtained results, and an application to the ordinary differential equation demonstrates how significant they are in the literature.

Acknowledgements

The first author received financial assistance from the University Grants Commission (UGC) India, under a Senior Research Fellowship (SRF) award.

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No.6186].

Author contributions

Conceptualization, D.C., N.S. and Y.R.; methodology, D.C., A.R.; validation, D.C., Y.R., N.S. and M.A.; formal analysis, N.S., A.R. and M.A.; writing—original draft preparation, D.C.; writing—review and editing, D.C., Y.R. and N.S.; All authors have read and agreed to the published version of the manuscript.

Funding

Not applicable.

Data availability

No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, National Institute of Technology Manipur, Imphal, 795004, Manipur, India. [2]Department of Mathematics, Manipur University, Canchipur, Imphal, 795003, Manipur, India. [3]Department of Mathematics, University of Management and Technology, Lahore, Pakistan. [4]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Garankuwa, Pretoria, Medunsa 0204, South Africa. [5]Department of Basic Sciences, Preparatory Year, King Faisal University Al Ahsa, Al Hofuf 31982, Saudi Arabia. [6]Department of Mathematics, College of Science, King Faisal University, Al-Ahsa 31982, Saudi Arabia.

Received: 3 March 2024 Accepted: 8 April 2024 Published online: 18 April 2024

References
1. Chand, D., Rohen, Y.: Fixed points of $(\alpha_s - \beta_s - \psi)$-contractive mappings in S-metric space. Nonlinear Funct. Anal. Appl. **28**, 571–587 (2023)
2. Gangwar, A., Rawat, S., Dimri, R.C.: Solution of differential inclusion problem in controlled S-metric spaces via new multivalued fixed point theorem. J. Anal. **31**(4), 2459–2472 (2023)
3. Khomdram, B., Rohen, Y.: Some common coupled fixed point theorems in S b-metric spaces. Fasc. Math. (2018)
4. Priyobarta, N., Rohen, Y., Thounaojam, S., Radenovic, S.: Some remarks on α-admissibility in S-metric spaces. J. Inequal. Appl. **2022**(1), 34 (2022)
5. Priyobarta, N., Rohen, Y., Mlaiki, N.: Complex valued S_b-metric spaces. J. Math. Anal. **8**(3), 13–24 (2017)
6. Sedghi, S., Shobe, N., Aliouche, A.: A generalization of fixed point theorems in S-metric spaces. Mat. Vesn. **64**(249), 258–266 (2012)
7. Saleem, N., Abbas, M., Bin-Mohsin, B., Radenovic, S.: Pata type best proximity point results in metric spaces. Miskolc Math. Notes **21**(1), 367–386 (2020)
8. Saleem, N., Vujakovìc, J., Baloch, W.U., Radenović, S.: Coincidence point results for multivalued Suzuki type mappings using θ-contraction in b-metric spaces. Mathematics **7**(11), 1017 (2019)
9. Saleem, N., Raazzia, M.T., Hussain, N., Asiri, A.: Geraghty–Pata–Suzuki-type proximal contractions and related coincidence best proximity point results. Symmetry **15**(8), 1572 (2023)
10. Alghamdi, M.A., Gulyaz-Ozyurt, S., Fulga, A.: Fixed points of Proinov E-contractions. Symmetry **13**(6), 962 (2021)
11. Nazam, M.U., Ameer, E.S., Mursaleen, M.O., Acar, O.: Nonlinear inequalities and related fixed-point problems. J. Math. Inequal. **15**(3), 941–967 (2021)
12. Banach, S.: Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales. Fundam. Math. **3**, 133–181 (1922)
13. Pata, V.: A fixed point theorem in metric spaces. J. Fixed Point Theory Appl. **10**(2), 299–305 (2011)
14. Jacob, G.K., Khan, M.S., Park, C., Yun, S.: On generalized Pata type contractions. Mathematics **6**(2), 25 (2018)
15. Kadelburg, Z., Radenović, S.: Fixed point theorems under Pata-type conditions in metric spaces. J. Egypt. Math. Soc. **24**(1), 77–82 (2016)
16. Kadelburg, Z., Radenović, S.: Fixed point and tripled fixed point theorems under Pata-type conditions in ordered metric spaces. Int. J. Anal. Appl. **6**(1), 113–122 (2014)
17. Kadelburg, Z., Radenović, S.: Fixed point theorems for Pata-type maps in metric spaces. J. Egypt. Math. Soc. **24**(1), 77–82 (2016)
18. Balasubramanian, S.: A Pata-type fixed point theorem. Math. Sci. **8**, 65–69 (2014)
19. Eshaghi, M., Mohseni, S., Delavar, M.R., De La Sen, M., Kim, G.H., Arian, A.: Pata contractions and coupled type fixed points. Fixed Point Theory Appl. **2014**(1), 130 (2014)
20. Paknazar, M., Eshaghi, M., Cho, Y.J., Vaezpour, S.M.: A Pata-type fixed point theorem in modular spaces with application. Fixed Point Theory Appl. **2013**(1), 239 (2013)
21. Roy, S., Chakraborty, P., Ghosh, S., Saha, P., Choudhury, B.S.: Investigation of a fixed point problem for Pata-type contractions with respect to w-distance. J. Anal. **32**(1), 125–136 (2024)
22. Aktay, M., Özdemir, M.: On Pata convex-type contractive mappings. J. Funct. Spaces **2022**, Article ID 6963446 (2022)
23. Aktay, M., Özdemir, M.: On (α, φ)-weak Pata contractions. Manas J. Eng. **10**(2), 228–240 (2022)
24. Neugebbauer, J.T.: The role of symmetry and concavity in the existence of solutions of a difference equation with Dirichlet boundary conditions. Int. J. Difference Equ. **15**, 483–491 (2020)
25. Chakraborty, M., Samanta, S.K.: A fixed point theorem for Kannan-type maps in metric spaces (2012). arXiv:1211.7331

Publisher's Note