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Hyperstability of Cauchy and Jensen functional equations in 2-normed spaces

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Abstract
In this article, with simple and short proofs without applying fixed point theorems, some hyperstability results corresponding to the functional equations of Cauchy and Jensen are presented in 2-normed spaces. We also obtain some results on hyperstability for the general linear functional equation $f(ax+by) = Af(x) + Bf(y) + C$.

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1 Introduction

The stability problem of functional equations has been stimulated by Ulam [28]. In fact, he was the first to raise the stability problem of homomorphisms between groups. The first answer to Ulam's question was given by Hyers [16]. The hyperstability and stability issues for many functional equations have been considered by several mathematicians. For more information and further references in this research area, we refer the reader to references [1, 2, 6, 10, 17–20, 22, 27].

Let us recall one of the classic theorems about the stability of Cauchy functional equation $\varphi(x+y) = \varphi(x) + \varphi(y)$.

Theorem 1.1 [4, 7, 15, 16, 25, 26] *Suppose that X is a normed space and Y is a Banach space. Let $\varepsilon \geq 0$ and $p \neq 1$ be a real number. If a function $f : X \to Y$ satisfies*

$$\left\|f(x+y) - f(x) - f(y)\right\| \leq \varepsilon\left(\|x\|^p + \|y\|^p\right), \quad x, y \in X \setminus \{0\},$$

then there exists a unique additive function $A : X \to Y$ such that

$$\left\|f(x) - A(x)\right\| \leq \frac{2\varepsilon}{|2-2^p|}\|x\|^p, \quad x \in X \setminus \{0\}.$$

The inequality $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon\|x\|^p\|y\|^q$, where p, q are real numbers with $p+q \in [0,1)$, was investigated by J. M. Rassias [23, 24]. Brzdęk [8] provided a complement for this result in the case $p+q < 0$ by using a fixed point theorem. A short and simple proof was provided in [21] for Brzdęk's result. Additionally, some further results concerning hyperstability of the Cauchy and Jensen functional equations can be found in [21]. Brzdęk [9] presented two possible extensions of the results in [8] to the case of n-normed spaces. The main tool in his proof [9, Theorem 3] is a fixed point theorem.

In this paper, we investigate some results of [3, 8, 9, 21] to the case of 2-normed spaces with simple proofs. In addition, we obtain more results regarding the hyperstability of functional equations in 2-normed spaces.

Now, we summarize some facts about 2-normed spaces from [12]. Concepts of 2-metric spaces and 2-normed spaces were introduced by Gähler [13, 14].

Definition 1.2 Let $\mathcal{Y}$ be a real linear space with $\dim \mathcal{Y} \geq 2$. A function $\|.,.\| : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ is said to be a 2-norm on $\mathcal{Y}$ provided $\|.,.\|$ satisfies the following four conditions:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$;
(iii) $\|x,y\| = \|y,x\|$;
(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$;

for all $x,y,z \in \mathcal{Y}$ and $\alpha \in \mathbb{R}$. In this case $(\mathcal{Y}, \|.,.\|)$ is called a 2-normed space. Obviously, it follows from $(ii) - (iv)$ that $\|x,y\| \geq 0$ for each $x,y \in \mathcal{Y}$.

Definition 1.3 A sequence $\{y_n\}$ in a 2-normed space $(\mathcal{Y}, \|.,.\|)$ is called a convergent sequence if there exists $y \in \mathcal{Y}$ such that

$$\lim_{n\to\infty} \|y_n - y, z\| = 0$$

for all $z \in \mathcal{Y}$.

We refer the readers to [5, 11], which provide a survey of the existing literature on Ulam stability concerning functional equations in 2-normed spaces.

2 Superstability results in 2-normed spaces

In this section, we present certain superstability findings related to Cauchy and Jensen functional equations within 2-normed spaces. It is worth mentioning that some stability findings regarding these equations in such spaces are documented in [5].

We start this section with the following key lemmas.

Lemma 2.1 *Suppose that $\mathcal{Y}$ is a 2-normed space and $u,v \in \mathcal{Y}$ are linearly independent elements. If $x \in \mathcal{Y}$ and $\|x,u\| = \|x,v\| = 0$, then $x = 0$.*

Proof Since $\|x,u\| = \|x,v\| = 0$, there exist scalars λ, μ such that $x = \lambda u$ and $x = \mu v$. Then $\lambda u - \mu v = 0$, and consequently, $\lambda = \mu = 0$. This yields $x = 0$. □

Lemma 2.2 *Let $\mathcal{Y}$ be a 2-normed space and $\varphi : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ be a function such that $\|\varphi(x,y),z\| = 0$ for all $x,y,z \in \mathcal{Y}$ with $\|x,z\|\,\|y,z\| \neq 0$. Then $\varphi(x,y) = 0$ for all $x,y \in \mathcal{Y} \setminus \{0\}$.*

Proof Let $x,y \in \mathcal{Y} \setminus \{0\}$. We can choose linearly independent elements $v,w \in \mathcal{Y}$ such that $\|x,v\|\,\|y,v\|\,\|x,w\|\,\|y,w\| \neq 0$. Then

$$\big\|\varphi(x,y),v\big\| = 0 \quad \text{and} \quad \big\|\varphi(x,y),w\big\| = 0.$$

Thus there exist scalars λ, μ such that $\varphi(x,y) = \lambda v$ and $\varphi(x,y) = \mu w$. So, $\lambda v - \mu w = 0$, and we conclude that $\lambda = \mu = 0$. Therefore $\varphi(x,y) = 0$, and this completes the proof. □

Now we present the main results of this section. Throughout this section $\delta, \varepsilon \geq 0$ and p, q, r, s are real numbers whose restrictions are specified in theorems.

Theorem 2.3 *Suppose that $\mathcal{Y}$ is a 2-normed space and $E \subseteq \mathcal{Y} \setminus \{0\}$ is a nonempty set. Let $p + q \neq 1$ and $\varphi : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ satisfy*

$$\|\varphi(x,y), z\| \leq \begin{cases} \delta + \varepsilon \|x,z\|^p \|y,z\|^q, & p+q<1; \\ \varepsilon \|x,z\|^p \|y,z\|^q, & p+q>1, \end{cases} \tag{2.1}$$

for all $x, y \in E$ and $z \in \mathcal{Y}$ with $\|x,z\| \|y,z\| \neq 0$. Then

$$\varphi(x,y) = 0, \quad x, y \in E.$$

Proof Two cases arise according to whether $p + q < 1$ or $p + q > 1$.

Case I: $p + q < 1$. Replacing z by nz in (2.1), we infer that

$$\|\varphi(x,y), z\| \leq \frac{\delta}{n} + \frac{n^{p+q}}{n} \varepsilon \|x,z\|^p \|y,z\|^q$$

for all $x, y \in E$ and $z \in \mathcal{Y}$ with $\|x,z\| \|y,z\| \neq 0$. Allowing n tend to infinity in the last inequality, one obtains $\|\varphi(x,y), z\| = 0$ for all $x, y \in E$ and $z \in \mathcal{Y}$ with $\|x,z\| \|y,z\| \neq 0$. By Lemma 2.2, we conclude $\varphi(x,y) = 0$, as desired.

Case II: $p + q > 1$. Replacing z by $\frac{z}{n}$ in (2.1), we infer that

$$\|\varphi(x,y), z\| \leq \frac{n}{n^{p+q}} \varepsilon \|x,z\|^p \|y,z\|^q$$

for all $x, y \in E$ and $z \in \mathcal{Y}$ with $\|x,z\| \|y,z\| \neq 0$. Now, with an argument similar to the previous case, the proof is complete. □

Corollary 2.4 *Suppose that $\mathcal{Y}$ is a 2-normed space and $E \subseteq \mathcal{Y} \setminus \{0\}$ is a nonempty set. Let $p + q \neq 1$ and $f, g, h : E \to \mathcal{Y}$ be functions satisfying*

$$\|f(x+y) - g(x) - h(y), z\| \leq \begin{cases} \delta + \varepsilon \|x,z\|^p \|y,z\|^q, & p+q<1; \\ \varepsilon \|x,z\|^p \|y,z\|^q, & p+q>1, \end{cases}$$

for every $z \in \mathcal{Y}$ and $x, y \in E$ with $x + y \in E$ and $\|x,z\| \|y,z\| \neq 0$. Then

$$f(x+y) = g(x) + h(y), \quad x, y \in E, x + y \in E.$$

Theorems 2.1 and 2.2 of [3] can be easily obtained from the following corollary.

Corollary 2.5 *Suppose that $\mathcal{Y}$ is a 2-normed space and $E \subseteq \mathcal{Y} \setminus \{0\}$ is a nonempty set. Let $p + q \neq 1$ and $f : E \to \mathcal{Y}$ satisfy*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y), z \right\| \leq \begin{cases} \delta + \varepsilon \|x,z\|^p \|y,z\|^q, & p+q<1; \\ \varepsilon \|x,z\|^p \|y,z\|^q, & p+q>1, \end{cases}$$

for every $z \in \mathcal{Y}$ *and* $x, y \in E$ *with* $\frac{x+y}{2} \in E$ *and* $\|x, z\| \|y, z\| \neq 0$. *Then*

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad x, y \in E, \frac{x+y}{2} \in E.$$

Remark 2.6 It should be noted that in the case of $p + q = 1$, none of the conditions C_1 and C_2 in Theorem 2.2 of [3] are fulfilled.

Brzdęk [9, Theorem 3] proved a version of the following theorem by using a fixed point theorem. Here we present, without using the fixed point theorem, a short and simple proof.

Theorem 2.7 *Suppose that* $(X, \|.\|_*)$ *is a normed space and* $\mathcal{Y}$ *is a* 2*-normed space. Let* $u, v \in \mathcal{Y}$ *be linearly independent elements and* $\mu : \mathcal{Y}_0 \to (0, +\infty)$, *where* $\mathcal{Y}_0 := \mathcal{Y} \setminus \{0\}$. *Assume that* $E \subseteq X \setminus \{0\}$ *is a nonempty with this property: for every* $x \in E$, *there is an integer* $m_x > 0$ *such that* $nx \in E$ *for each integer* $n \geq m_x$. *Let* $p + q < 0$. *Then a function* $f : X \to \mathcal{Y}$ *fulfills*

$$\|f(x+y) - f(x) - f(y), z\| \leq \|x\|_*^p \|y\|_*^q \mu(z), \quad x, y, x+y \in E, z \in \{u, v\}, \tag{2.2}$$

if and only if $f(x + y) = f(x) + f(y)$ *for all* $x, y \in E$ *with* $x + y \in E$.

Proof It is clear that if f is additive on E, then f satisfies (2.2). For the converse implication, we can assume that $p < 0$. Let $x, y \in E$ with $x + y \in E$. By hypothesis, we can find an integer $m > 0$ such that $nx, ny, n(x + y) \in E$ for each $n \geq m$. By (2.2), we have

$$\begin{aligned}
&\|f(nx + x) - f(nx) - f(x), z\| \leq n^p \|x\|_*^{p+q} \mu(z), \\
&\|f(ny + y) - f(ny) - f(y), z\| \leq n^p \|y\|_*^{p+q} \mu(z), \\
&\big\|f\big(n(x + y) + (x + y)\big) - f\big(n(x + y)\big) - f(x + y), z\big\| \leq n^p \|x + y\|_*^{p+q} \mu(z)
\end{aligned}$$

for all $z \in \{u, v\}$. Allowing n tend to infinity in the above three inequalities, we get

$$\begin{aligned}
&\lim_{n\to\infty} \|f(nx + x) - f(nx) - f(x), z\| = 0, \\
&\lim_{n\to\infty} \|f(ny + y) - f(ny) - f(y), z\| = 0, \\
&\lim_{n\to\infty} \big\|f\big(n(x + y) + (x + y)\big) - f\big(n(x + y)\big) - f(x + y), z\big\| = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\|f(x + y) - f(x) - f(y), z\| \\
&\quad \leq \lim_{n\to\infty} \big\|f(x + y) + f\big(n(x + y)\big) - f\big(n(x + y) + (x + y)\big), z\big\| \\
&\qquad + \lim_{n\to\infty} \|f(nx + x) - f(nx) - f(x), z\| \\
&\qquad + \lim_{n\to\infty} \|f(ny + y) - f(ny) - f(y), z\| \\
&\qquad + \lim_{n\to\infty} \big\|f\big(n(x + y) + (x + y)\big) - f(nx + x) - f(ny + y), z\big\|
\end{aligned}$$

$$+ \lim_{n\to\infty} \left\| f(nx) + f(ny) - f\big(n(x+y)\big), z \right\|$$
$$\le \lim_{n\to\infty} \big[(n+1)^{p+q} + n^{p+q}\big] \|x\|_*^p \|y\|_*^q \mu(z) = 0$$

for all $z \in \{u, v\}$. This ends the proof. □

Theorem 2.8 *Let* $(X, \|.\|_*)$ *be a normed space,* $\mathcal{Y}$ *be a 2-normed space,* $X_0 := X \setminus \{0\}$, $\mathcal{Y}_0 := \mathcal{Y} \setminus \{0\}$, *and* $\mu : \mathcal{Y}_0 :\to (0, +\infty)$. *Suppose that* $u, v \in \mathcal{Y}$ *are linearly independent elements and a function* $f : X \to \mathcal{Y}$ *fulfills*

$$\left\| rf\left(\frac{x+y}{r}\right) - f(x) - f(y), z \right\| \le \varepsilon \|x\|_*^p \|y\|_*^q \mu(z) \tag{2.3}$$

for all $z \in \{u, v\}$ *and* $x, y \in X_0$, *where* $\min\{p, q\} < 0$ *and* $r \neq 0, 1$. *Then*

$$rf\left(\frac{x+y}{r}\right) = f(x) + (r-1)f\left(\frac{y}{r-1}\right), \quad x, y \in X_0, x+y \in X_0.$$

Proof We may assume that $p < 0$. Let $x, y \in X_0$ and $x + y \in X_0$. Then there is an integer $m > 0$ such that for all $n \ge m$, (2.3) yields

$$\left\| rf\left(\frac{nx+x}{r}\right) - f(nx) - f(x), z \right\| \le \varepsilon n^p \|x\|_*^{p+q} \mu(z),$$
$$\left\| rf\left(\frac{\frac{y}{r-1} + nx + (x+y)}{r^2}\right) - f\left(\frac{\frac{y}{r-1} + nx}{r}\right) - f\left(\frac{x+y}{r}\right), z \right\| \le \varepsilon \left\| \frac{\frac{y}{r-1} + nx}{r} \right\|_*^p \left\| \frac{x+y}{r} \right\|_*^q \mu(z)$$

for all $z \in \{u, v\}$. Allowing $n \to \infty$ in the above inequalities, one obtains

$$f(x) = \lim_{n\to\infty} \left[rf\left(\frac{nx+x}{r}\right) - f(nx) \right],$$
$$f\left(\frac{x+y}{r}\right) = \lim_{n\to\infty} \left[rf\left(\frac{\frac{y}{r-1} + nx + (x+y)}{r^2}\right) - f\left(\frac{\frac{y}{r-1} + nx}{r}\right) \right].$$

Then

$$\left\| rf\left(\frac{x+y}{r}\right) - f(x) - (r-1)f\left(\frac{y}{r-1}\right), z \right\|$$
$$= \lim_{n\to\infty} \left\| \left[r^2 f\left(\frac{\frac{y}{r-1} + nx + (x+y)}{r^2}\right) - rf\left(\frac{\frac{y}{r-1} + nx}{r}\right) \right] - \left[rf\left(\frac{nx+x}{r}\right) - f(nx) \right] \right.$$
$$\left. - (r-1)f\left(\frac{y}{r-1}\right), z \right\|$$
$$\le |r| \limsup_{n\to\infty} \left\| rf\left(\frac{\frac{y}{r-1} + nx + x + y}{r^2}\right) - f\left(\frac{nx+x}{r}\right) - f\left(\frac{y}{r-1}\right), z \right\|$$
$$+ \limsup_{n\to\infty} \left\| rf\left(\frac{\frac{y}{r-1} + nx}{r}\right) - f(nx) - f\left(\frac{y}{r-1}\right), z \right\| \quad \text{(by (2.3))}$$
$$\le \limsup_{n\to\infty} \varepsilon \left[|r| \left(\frac{n+1}{r}\right)^p + n^p \right] \|x\|_*^p \left\| \frac{y}{r-1} \right\|_*^q \mu(z) = 0$$

for all $z \in \{u, v\}$. Therefore

$$rf\left(\frac{x+y}{r}\right) = f(x) + (r-1)f\left(\frac{y}{r-1}\right)$$

for all $x, y \in X_0$ with $x + y \in X_0$. □

By using the idea of proving Theorem 2.8, one can obtain the following result.

Theorem 2.9 *Let $(X, \|.\|_*)$ be a normed space, $\mathcal{Y}$ be a 2-normed space, $X_0 := X \setminus \{0\}$, $\mathcal{Y}_0 := \mathcal{Y} \setminus \{0\}$, and $\mu : \mathcal{Y}_0 :\to (0, +\infty)$. Suppose that $u, v \in \mathcal{Y}$ are linearly independent elements and a function $f : X \to \mathcal{Y}$ fulfills*

$$\left\| rf\left(\frac{x+y}{r}\right) - f(x) - f(y), z \right\| \leq \|x\|_*^p \|y\|_*^q \big(\varepsilon \|x+y\|_*^s + \delta \|x-y\|_*^s\big)\mu(z) \tag{2.4}$$

for all $z \in \{u, v\}$ and $x, y \in X_0$ with $x \pm y \in X_0$, where $\min\{p+s, q+s\} < 0$ and $r \neq 0, 1$. Then

$$rf\left(\frac{x+y}{r}\right) = f(x) + (r-1)f\left(\frac{y}{r-1}\right), \quad x, y \in X_0, x+y \in X_0.$$

The following gives a result on hyperstability of the Jensen function defined on a restricted domain.

Corollary 2.10 *Let $(X, \|.\|_*)$ be a normed space, $\mathcal{Y}$ be a 2-normed space, $E \subseteq X \setminus \{0\}$ be a nonempty set with $E + E \subseteq E \cap 2E$, and $\mu : \mathcal{Y}_0 := \mathcal{Y} \setminus \{0\} \to (0, +\infty)$. Let $u, v \in \mathcal{Y}$ be linearly independent elements. Assume that a function $f : E \to \mathcal{Y}$ fulfills one of the following inequalities*:

(i) $\min\{p, q\} < 0$ *and* $\|Jf(x, y), z\| \leq \varepsilon \|x\|_*^p \|y\|_*^q \mu(z), z \in \{u, v\}$ *and* $x, y \in E$;

(ii) $\min\{p+s, q+s\} < 0$ *and* $\|Jf(x, y), z\| \leq \|x\|_*^p \|y\|_*^q (\varepsilon \|x+y\|_*^s + \delta \|x-y\|_*^s)\mu(z)$

for all $z \in \{u, v\}$ and $x, y \in E$ with $x \pm y \in X_0$, where $Jf(x, y) = 2f(\frac{x+y}{2}) - f(x) - f(y)$. Then f is Jensen on E, i.e.,

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad x, y \in E.$$

Proof Applying the proof of Theorem 2.8 for the case $r = 2$, one obtains the desired result. □

For the case $r = 1$, we have the following.

Theorem 2.11 *Let X be a linear space, $\mathcal{Y}$ be a 2-normed space, $E \subseteq X \setminus \{0\}$ be a nonempty set with $E + E \subseteq E$, and $\mu : \mathcal{Y} \setminus \{0\} \to (0, +\infty)$. Let $u, v \in \mathcal{Y}$ be linearly independent elements and $\varphi : X \times X \to [0, +\infty)$ be a function satisfying*

$$\lim_{m\to\infty} \varphi(mx, y) = 0 \quad \Big(\textit{resp.} \lim_{m\to\infty} \varphi(x, my) = 0\Big), x, y \in E.$$

Assume that a function $f : X \to \mathcal{Y}$ fulfills the inequality

$$\big\| f(x+y) - f(x) - f(y), z \big\| \leq \varphi(x, y)\mu(z) \tag{2.5}$$

for all $z \in \{u,v\}$ *and* $x,y \in E$. *Then*

$$f(x+y) = f(x) + f(y), \quad x,y \in E.$$

Proof We may assume that $\lim_{m\to\infty} \varphi(mx,y) = 0$ for all $x,y \in E$. Let $x,y \in E$. Then (2.5) yields

$$\begin{aligned}
&\|f(nx+x) - f(nx) - f(x), z\| \le \varphi(nx,x)\mu(z),\\
&\|f\big(nx+(x+y)\big) - f(nx) - f(x+y), z\| \le \varphi(nx,x+y)\mu(z)
\end{aligned}$$

for all $z \in \{u,v\}$ and positive integers n. Allowing $n \to \infty$ in the above inequalities, one obtains

$$\begin{aligned}
&\lim_{n\to\infty} \|f(nx+x) - f(nx) - f(x), z\| = 0,\\
&\lim_{n\to\infty} \|f(x+y) - f(nx+x+y) + f(nx), z\| = 0
\end{aligned}$$

for all $z \in \{u,v\}$. Then

$$\begin{aligned}
\|f(x+y) - f(x) - f(y), z\| &\le \lim_{n\to\infty} \|f(x+y) - f(nx+x+y) + f(nx), z\|\\
&\quad + \lim_{n\to\infty} \|f(nx+x) - f(nx) - f(x), z\|\\
&\quad + \lim_{n\to\infty} \|f(nx+x+y) - f(nx+x) - f(y), z\|\\
&\le \lim_{n\to\infty} \varphi\big((n+1)x, y\big)\mu(z) = 0
\end{aligned}$$

for all $z \in \{u,v\}$. Therefore $f(x+y) = f(x) + f(y)$ for all $x,y \in E$. □

Corollary 2.12 *Let* $(X, \|.\|_*)$ *be a normed space*, $\mathcal{Y}$ *be a* 2-*normed space*, $E \subseteq X \setminus \{0\}$ *be a nonempty set with* $E + E \subseteq E$, *and* $\mu : \mathcal{Y} \setminus \{0\} \to (0, +\infty)$. *Take* $\delta, \varepsilon \ge 0$ *and let* p,q,s *be real numbers. Assume that* $u,v \in \mathcal{Y}$ *are linearly independent elements and a function* $f : X \to \mathcal{Y}$ *fulfills one of the following inequalities*:

(i) $\|Df(x,y), z\| \le \varepsilon \|x\|_*^p \|y\|_*^q \mu(z), \min\{p,q\} < 0$,

(ii) $\|Df(x,y), z\| \le \|x\|_*^p \|y\|_*^q (\varepsilon \|x+y\|^s + \delta \|x-y\|^s)\mu(z), \min\{p+s, q+s\} < 0$

for all $z \in \{u,v\}$ *and* $x,y \in E$, *where* $Df(x,y) = f(x+y) - f(x) - f(y)$. *Then* f *is additive on* E.

Theorem 2.13 *Let* $\mathcal{Y}$ *be a* 2-*normed space and* $f : \mathcal{Y} \to \mathcal{Y}$. *Suppose that* $r \ne 0$ *and*

$$\left\| rf\left(\frac{x+y}{r}\right) - f(x) - f(y), z \right\| \le \varepsilon \|x,z\|^p \|y,z\|^q \tag{2.6}$$

for all $x,y,z \in \mathcal{Y}$ *with* $\|x,z\|\|y,z\| \ne 0$. *Then we have the following assertions*:

(i) *if* $\min\{p,q\} < 0$ *and* $r \ne 1$, *then*

$$rf\left(\frac{x+y}{r}\right) = f(x) + (r-1)f\left(\frac{y}{r-1}\right), \quad x,y,x+y \ne 0;$$

(ii) *if* $p+q\neq 1$, *then*

$$rf\left(\frac{x+y}{r}\right)=f(x)+f(y),\quad x,y\neq 0;$$

$$2f\left(\frac{x+y}{2}\right)=f(x)+f(y),\quad x,y,x+y\neq 0.$$

Proof We may assume that $p<0$. Let $x,y,z\in\mathcal{Y}$ with $\|x,z\|\|y,z\|\|x+y,z\|\neq 0$. We can choose m such that $\|\frac{y}{r-1}+nx,z\|\neq 0$ for all $n\geq m$. Then (2.6) yields

$$\left\|rf\left(\frac{nx+x}{r}\right)-f(nx)-f(x),z\right\|\leq \varepsilon n^p\|x,z\|^{p+q},$$

$$\left\|rf\left(\frac{\frac{y}{r-1}+nx+(x+y)}{r^2}\right)-f\left(\frac{\frac{y}{r-1}+nx}{r}\right)-f\left(\frac{x+y}{r}\right),z\right\|\leq\varepsilon\left\|\frac{\frac{y}{r-1}+nx}{r},z\right\|^p\left\|\frac{x+y}{r},z\right\|^q.$$

Allowing n tend to infinity in the above inequalities, one obtains

$$\lim_{n\to\infty}\left\|rf\left(\frac{nx+x}{r}\right)-f(nx)-f(x),z\right\|=0,$$

$$\lim_{n\to\infty}\left\|rf\left(\frac{\frac{y}{r-1}+nx+(x+y)}{r^2}\right)-f\left(\frac{\frac{y}{r-1}+nx}{r}\right)-f\left(\frac{x+y}{r}\right),z\right\|=0.$$

Then

$$\begin{aligned}
&\left\|rf\left(\frac{x+y}{r}\right)-f(x)-(r-1)f\left(\frac{y}{r-1}\right),z\right\|\\
&\leq\lim_{n\to\infty}|r|\left\|-rf\left(\frac{\frac{y}{r-1}+nx+(x+y)}{r^2}\right)+f\left(\frac{\frac{y}{r-1}+nx}{r}\right)+f\left(\frac{x+y}{r}\right),z\right\|\\
&\quad+\lim_{n\to\infty}\left\|rf\left(\frac{nx+x}{r}\right)-f(nx)-f(x),z\right\|\\
&\quad+|r|\limsup_{n\to\infty}\left\|rf\left(\frac{\frac{y}{r-1}+nx+x+y}{r^2}\right)-f\left(\frac{nx+x}{r}\right)-f\left(\frac{y}{r-1}\right),z\right\|\\
&\quad+\limsup_{n\to\infty}\left\|-rf\left(\frac{\frac{y}{r-1}+nx}{r}\right)+f(nx)+f\left(\frac{y}{r-1}\right),z\right\|\quad\text{(by (2.6))}\\
&\leq\limsup_{n\to\infty}\varepsilon\left[|r|\left(\frac{n+1}{r}\right)^p+n^p\right]\|x,z\|^p\left\|\frac{y}{r-1},z\right\|^q=0.
\end{aligned}$$

Therefore

$$\left\|rf\left(\frac{x+y}{r}\right)-f(x)-(r-1)f\left(\frac{y}{r-1}\right),z\right\|=0$$

for all $x,y,z\in\mathcal{Y}$ with $\|x,z\|\|y,z\|\|x+y,z\|\neq 0$.

Let $x,y\in\mathcal{Y}$ such that $x,y,x+y\neq 0$. Since $\dim\mathcal{Y}\geq 2$, there exist linearly independent elements $z,w\in\mathcal{Y}$ such that $\|x,z\|\|y,z\|\|x+y,z\|\|x,w\|\|y,w\|\|x+y,w\|\neq 0$. Indeed, if x,y are linearly independent, we let $z=2x+y$ and $w=x+2y$. If x,y are not linearly independent, there exists $z\in\mathcal{Y}$ such that x,z are linearly independent, and we put $w=x+z$.

Consequently,

$$\left\| rf\left(\frac{x+y}{r}\right) - f(x) - (r-1)f\left(\frac{y}{r-1}\right), z \right\| = 0,$$
$$\left\| rf\left(\frac{x+y}{r}\right) - f(x) - (r-1)f\left(\frac{y}{r-1}\right), w \right\| = 0.$$

Thus, we infer

$$rf\left(\frac{x+y}{r}\right) = f(x) + (r-1)f\left(\frac{y}{r-1}\right)$$

for all $x, y \in \mathcal{Y}$ such that $x, y, x+y \neq 0$. This proves (i).

The claims in (ii) are derived from Theorem 2.3. □

In Theorem 2.13, the case $r = 1$ is left out. So we now consider the case $r = 1$.

Theorem 2.14 *Let $\mathcal{Y}$ be a 2-normed space and $f : \mathcal{Y} \to \mathcal{Y}$. Suppose that $\min\{p, q\} < 0$ and*

$$\|f(x+y) - f(x) - f(y), z\| \leq \varepsilon \|x, z\|^p \|y, z\|^q \tag{2.7}$$

for all $x, y, z \in \mathcal{Y}$ with $\|x, z\| \|y, z\| \neq 0$. Then f is additive on $\mathcal{Y}$.

Proof We may assume that $p < 0$. Let $x, y, z \in \mathcal{Y}$ with $\|x, z\| \|y, z\| \|x+y, z\| \neq 0$. Then (2.7) yields

$$\|f(nx+x) - f(nx) - f(x), z\| \leq n^p \varepsilon \|x, z\|^p \|y, z\|^q,$$
$$\|f(nx + (x+y)) - f(nx) - f(x+y), z\| \leq n^p \varepsilon \|x, z\|^p \|x+y, z\|^q.$$

Allowing n tending to infinity in the above inequalities, one gets

$$\lim_{n\to\infty} \|f(nx+x) - f(nx) - f(x), z\| = 0,$$
$$\lim_{n\to\infty} \|f(nx + (x+y)) - f(nx) - f(x+y), z\| = 0.$$

Then

$$\begin{aligned}
\|f(x+y) - f(x) - f(y), z\| \leq{} & \lim_{n\to\infty} \|f(x+y) - f(nx + (x+y)) + f(nx), z\| \\
& + \lim_{n\to\infty} \|f(nx+x) - f(nx) - f(x), z\| \\
& + \lim_{n\to\infty} \|f(nx + (x+y)) - f(nx+x) - f(y), z\| \\
\leq{} & \lim_{n\to\infty} (n+1)^p \varepsilon \|x, z\|^p \|y, z\|^q = 0.
\end{aligned}$$

Therefore

$$\|f(x+y) - f(x) - f(y), z\| = 0$$

for all $x,y,z \in \mathcal{Y}$ with $\|x,z\|\|y,z\|\|x+y,z\| \neq 0$. With a proof similar to the argument of Theorem 2.13, we conclude that f is additive on $\mathcal{Y}$. So the proof is complete. □

3 Hyperstability of the general linear functional equation

In this section, $(X, \|.\|_*)$ denotes a normed space over the field $\mathbb{F}$, $\mathcal{Y}$ is a 2-normed space over the field $\mathbb{K}$, and $\mu : \mathcal{Y} \setminus \{0\} \to (0,+\infty)$. We also assume that $A,B \in \mathbb{K}$, $a,b \in \mathbb{F} \setminus \{0\}$, and $C \in \mathcal{Y}$ are fixed.

Theorem 3.1 *Let $d > 0$ and $u,v \in \mathcal{Y}$ be linearly independent elements. Let $\varphi : X \times X \to [0,+\infty)$ and $f : X \to \mathcal{Y}$ be functions such that*

$$\lim_{m\to\infty} \varphi(mx,my) = 0, \qquad \lim_{m\to\infty} \varphi\bigl(a^{-1}(m+1)x, -b^{-1}mx\bigr) = 0, \quad x,y \in X \setminus \{0\}, \tag{3.1}$$

and

$$\bigl\|f(ax+by) - Af(x) - Bf(y) - C, z\bigr\| \le \varphi(x,y)\mu(z) \tag{3.2}$$

for all $x,y \in E_d := \{w \in X : \|w\|_* \ge d\}$ and $z \in \{u,v\}$. Then f satisfies

$$\begin{aligned} &Af(x) + Bf\bigl(-ab^{-1}x\bigr) = (A+B)f(0), \quad x \in X, \\ &f(ax+by) = Af(x) + Bf(y) + C, \qquad (1-A-B)f(0) = C, \quad x,y \in X \setminus \{0\}. \end{aligned} \tag{3.3}$$

Proof Replacing y by $-b^{-1}mx$ and x by $a^{-1}(m+1)x$ in (3.2), we have

$$\bigl\|f(x) - Af\bigl(a^{-1}(m+1)x\bigr) - Bf\bigl(-b^{-1}mx\bigr) - C, z\bigr\| \le \varphi\bigl(a^{-1}(m+1)x, -b^{-1}mx\bigr)\mu(z) \tag{3.4}$$

for all $x \in X \setminus \{0\}, z \in \{u,v\}$ and positive integer $m \ge n$, where $a^{-1}(n+1)x, b^{-1}nx \in E_d$. Allowing $m \to \infty$ in (3.4) and using (3.1), one obtains

$$\lim_{m\to\infty} \bigl\|f(x) - Af\bigl(a^{-1}(m+1)x\bigr) - Bf\bigl(-b^{-1}mx\bigr) - C, z\bigr\| = 0 \tag{3.5}$$

for all $x \in X \setminus \{0\}$ and $z \in \{u,v\}$. Let $x \in X \setminus \{0\}$, then (3.1) and (3.5) yield

$$\begin{aligned} &\bigl\|Af(x) + Bf\bigl(-ab^{-1}x\bigr) - (A+B)f(0), z\bigr\| \\ &\quad\le \lim_{m\to\infty} \bigl\|Af(x) - A^2 f\bigl(a^{-1}(m+1)x\bigr) - ABf\bigl(-b^{-1}mx\bigr) - AC, z\bigr\| \\ &\qquad + \lim_{m\to\infty} \bigl\|Bf\bigl(-ab^{-1}x\bigr) - ABf\bigl(-b^{-1}(m+1)x\bigr) - B^2 f\bigl(ab^{-2}mx\bigr) - BC, z\bigr\| \\ &\qquad + \lim_{m\to\infty} \bigl\|A^2 f\bigl(a^{-1}(m+1)x\bigr) + ABf\bigl(-b^{-1}(m+1)x\bigr) + AC - Af(0), z\bigr\| \\ &\qquad + \lim_{m\to\infty} \bigl\|ABf\bigl(-b^{-1}mx\bigr) + B^2 f\bigl(ab^{-2}mx\bigr) + BC - Bf(0), z\bigr\| \\ &\quad\le |A| \lim_{m\to\infty} \varphi\bigl(a^{-1}(m+1)x, -b^{-1}(m+1)x\bigr)\mu(z) \\ &\qquad + |B| \lim_{m\to\infty} \varphi\bigl(-b^{-1}mx, ab^{-2}mx\bigr)\mu(z) = 0 \end{aligned}$$

for all $z \in \{u, v\}$. Then

$$Af(x) + Bf\left(-ab^{-1}x\right) = (A + B)f(0), \quad x \in X. \tag{3.6}$$

Replace y by $-amx$ and x by bmx in (3.2) to obtain

$$\left\|f(0) - Af(bmx) - Bf(-amx) - C, z\right\| \leq \varphi(bmx, -amx)\mu(z) \tag{3.7}$$

for all $x \in X \setminus \{0\}$, $z \in \{u, v\}$ and positive integer $m \geq n$, where $anx, bnx \in E_d$. On the other hand, replacing x by bmx in (3.6), we get

$$(A + B)f(0) = Af(bmx) + Bf(-amx), \quad x \in X.$$

Therefore, (3.7) becomes as follows:

$$\left\|f(0) - (A + B)f(0) - C, z\right\| \leq \varphi(bmx, -amx)\mu(z), \quad z \in \{u, v\}.$$

Letting $m \to \infty$ in the inequality above, we get

$$(1 - A - B)f(0) = C.$$

This shows that (3.5) holds true for all $x \in X$. To prove (3.3), let $x, y \in X \setminus \{0\}$. By applying (3.5), we have

$$\begin{aligned}
&\left\|f(ax + by) - Af(x) - Bf(y) - C, z\right\| \\
&\leq \lim_{m\to\infty} \left\|f(ax + by) - Af\left(a^{-1}(m + 1)(ax + by)\right) - Bf\left(-b^{-1}m(ax + by)\right) - C, z\right\| \\
&\quad + \lim_{m\to\infty} \left\|-Af(x) + A^2 f\left(a^{-1}(m + 1)x\right) + ABf\left(-b^{-1}mx\right) + AC, z\right\| \\
&\quad + \lim_{m\to\infty} \left\|-Bf(y) + ABf\left(a^{-1}(m + 1)y\right) + B^2 f\left(-b^{-1}my\right) + BC, z\right\| \\
&\quad + \lim_{m\to\infty} \left\|Af\left(a^{-1}(m + 1)(ax + by)\right) - A^2 f\left(a^{-1}(m + 1)x\right)\right. \\
&\quad \left. - ABf\left(a^{-1}(m + 1)y\right) - AC, z\right\| \\
&\quad + \lim_{m\to\infty} \left\|Bf\left(-b^{-1}m(ax + by)\right) - ABf\left(-b^{-1}mx\right) - B^2 f\left(-b^{-1}my\right) - BC, z\right\| \\
&\leq |A| \lim_{m\to\infty} \varphi\left(a^{-1}(m + 1)x, -a^{-1}(m + 1)y\right)\mu(z) \\
&\quad + |B| \lim_{m\to\infty} \varphi\left(-b^{-1}mx, -b^{-1}my\right)\mu(z) = 0, z \in \{u, v\}.
\end{aligned}$$

Thus f satisfies (3.3) for all $x, y \in X \setminus \{0\}$. □

Corollary 3.2 *Let $d > 0, \varepsilon, \delta \geq 0$ and p, q, s be real numbers. Put $Gf(x, y) := f(ax + by) - Af(x) - Bf(y) - C$ for a given function $f : X \to \mathcal{Y}$. Suppose that $u, v \in \mathcal{Y}$ are linearly independent elements. Then we have the following assertions*:

(i) *if $p, q < 0$ and $\|Gf(x, y), z\| \leq \varepsilon(\|x\|_*^p + \|y\|_*^q)\mu(z)$ for all $x, y \in E_d$ and $z \in \{u, v\}$, then*

$$f(ax + by) = Af(x) + Bf(y) + C, x, y \in X \setminus \{0\},$$

$$Af(x) + Bf\left(-ab^{-1}x\right) = (A+B)f(0), x \in X.$$

If $a \pm b \neq 0$ *and* f *satisfies one of the following inequalities*:

(ii) $p+s, q+s < 0$ *and*

$$\left\| Gf(x,y), z \right\| \leq \left(\|x\|_*^p + \|y\|_*^q\right)\left(\varepsilon \|x+y\|_*^s + \delta \|x-y\|_*^s\right)\mu(z),$$
$$x, y \in E_d, x \pm y \neq 0, z \in \{u, v\};$$

(iii) $p+q+s < 0$ *and*

$$\left\| Gf(x,y), z \right\| \leq \|x\|_*^p \|y\|_*^q \left(\varepsilon \|x+y\|_*^s + \delta \|x-y\|_*^s\right)\mu(z), \quad x, y \in E_d, x \pm y \neq 0, z \in \{u, v\},$$

then f *satisfies*

$$f(ax+by) = Af(x) + Bf(y) + C, \quad x, y \in X \setminus \{0\}, x \pm y \neq 0.$$

Corollary 3.3 *Let* $u, v \in \mathcal{Y}$ *be linearly independent elements. Then each function* $f : X \to \mathcal{Y}$ *fulfills one of the following statements*:

(i) $f(ax+by) = Af(x) + Bf(y) + C, x, y \in X \setminus \{0\}$.

(ii) $\limsup_{\min\{\|x\|_*, \|y\|_*\} \to \infty} \|f(ax+by) - Af(x) - Bf(y) - C, u\| \, \|x\|_*^p \|y\|_*^q = +\infty$, *or* $\limsup_{\min\{\|x\|_*, \|y\|_*\} \to \infty} \|f(ax+by) - Af(x) - Bf(y) - C, v\| \, \|x\|_*^p \|y\|_*^q = +\infty$ *for all real numbers* p, q *with* $p+q > 0$.

Corollary 3.4 *Let* $u, v \in \mathcal{Y}$ *be linearly independent elements. Then each function* $f : X \to \mathcal{Y}$ *fulfills one of the following statements*:

(i) $f(ax+by) = Af(x) + Bf(y) + C, x, y \in X \setminus \{0\}$;

(ii) $\limsup_{\min\{\|x\|_*, \|y\|_*\} \to \infty} \frac{\|x\|_*^p \|y\|_*^q}{\|x\|_*^p + \|y\|_*^q} \|f(ax+by) - Af(x) - Bf(y) - C, u\| = +\infty$, *or* $\limsup_{\min\{\|x\|_*, \|y\|_*\} \to \infty} \frac{\|x\|_*^p \|y\|_*^q}{\|x\|_*^p + \|y\|_*^q} \|f(ax+by) - Af(x) - Bf(y) - C, v\| = +\infty$, *for all real positive numbers* p, q.

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AN and YKY prepared the first draft of the manuscript, and all authors provided comments on earlier versions of the manuscript. AN and YKY contributed to the concept and design of the study. The final version of the manuscript was approved by all authors. AN supervised this project.

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