Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Approximation with Szász-Chlodowsky operators employing general-Appell polynomials

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Abstract

This article explores a Chlodowsky-type extension of Szász operators using the general-Appell polynomials. The convergence properties of these operators are established by employing the universal Korovkin-type property, and the order of approximation is determined using the classical modulus of continuity. Additionally, the weighted $\mathfrak{B}$-statistical convergence and statistically weighted $\mathfrak{B}$-summability properties of the operators are derived. Theoretical results are supported by numerical and graphical examples.

Mathematics Subject Classification: 33C47; 33E20; 41A25; 41A30; 41A36

Keywords: Szász operators; General-Appell Polynomials; Order of convergence; Modulus of continuity; Weighted $\mathfrak{B}$-statistical convergence.

1 Introduction

Approximation theory is a primary field that encounters significant usage in the scientific community. The paramount problem in approximation theory is finding a sequence of functions that approximates a given function as closely as possible. Positive linear operators are pivotal among the many subfields that constitute approximation theory. Some linear positive operators, such as Bernstein operators, are defined within finite intervals, and many operators are defined in infinite intervals, such as Szász operators defined as [31]:

$$\hat{\mathcal{S}}_\eta(f,t) = e^{-\eta t}\sum_{k=0}^{\infty}\frac{(\eta t)^k}{k!}f\left(\frac{k}{\eta}\right), \tag{1.1}$$

where $t \in [0,\infty)$ and $f \in C[0,\infty)$ once the sum (1.1) converges. In 1969, Jakimovski and Leviatan [18] leveraged the Appell polynomials in constructing a generalization of Szász operators. Several operators with the right tweaks that still keep the test function intact have emerged in this area, and it has developed significantly in recent years to get a better approximation [13–15, 21, 24, 25, 27].

Recent research has focused mainly on generalizations of Szász operators utilizing special polynomials, particularly those derived using generating functions; see, for example, [1–3, 6, 7, 11, 17, 32]. These generalizations give approximation theory a wide range of new operator sequences.

Special functions play a crucial role in applied mathematics, and the hypergeometric and confluent hypergeometric functions are particularly useful for representing a diverse range of special functions in a clear and concise manner. The study of special functions is of great importance in the field of mathematical physics, and it is a fundamental aspect of its institutionalization. Several important problems in physics can be expressed as special polynomials of two variables. These polynomials are not only useful in introducing new families of special polynomials, but they are also useful in unambiguously deriving various useful identities [28].

In 2013, Khan and Raza [22] considered the family of 2-variable general polynomials $p_n(x,y)$ defined by the generating function as:

$$e^{xt}\phi(y,t)=\sum_{k=0}^{\infty}p_k(x,y)\frac{t^k}{k!},\quad \bigl(p_0(x,y)=1\bigr), \tag{1.2}$$

where $\phi(y,t)$ can be expressed as

$$\phi(y,t)=\sum_{k=0}^{\infty}\phi_k(y)\frac{t^k}{k!},\quad \phi_0(y)\neq 0. \tag{1.3}$$

Khan and Raza [22] also introduced the family of 2-variable general-Appell polynomials ${}_pA_n(x,y)$ defined by the generating function as:

$$A(t)e^{xt}\phi(y,t)=\sum_{k=0}^{\infty}{}_pA_k(x,y)\frac{t^k}{k!}, \tag{1.4}$$

where $A(t)$ can be expressed as [4]:

$$A(t)=\sum_{k=0}^{\infty}A_kt^k,\quad A_0\neq 0 \tag{1.5}$$

and $\phi(y,t)$ is given by equation (1.3).

The aim of this research article is to consider the generalization of the Szász operators associating general-Appell polynomials ${}_pA_k(x,y)$. We introduce our operators for $x\in[0,\infty)$, subject to the restrictions $h\geq 0$, $A(1)\neq 0$, ${}_pA_k(nx;h)>0$ and $\phi(h,1)\neq 0$ given by

$$\mathcal{G}^*_{n,A}(\tilde{f},x)=\frac{e^{-nx}}{A(1)\phi(h;1)}\sum_{k=0}^{\infty}\frac{{}_pA_k(nx;h)}{k!}\tilde{f}\left(\frac{k}{n}\right). \tag{1.6}$$

Remark 1.1 For $A(t)=1$ and $\phi(h,t)=1$, equation (1.6) reduces to equation (1.1). Similarly, for $\phi(h,t)=1$, equation (1.6) reduces to approximation operators involving Appell polynomials [18].

Recently, in modern research, considerable attention has been paid to Chlodowsky variants of generalized Szász-type operators (as evidenced by such works as [10, 24, 25]). These studies have aimed to elucidate certain convergence properties of these operators through the application of a weighted Korovkin-type theorem.

Inspired by the above research, this article conceders the following generalization of Chlodowsky-type operators [12], defined by (1.6):

$$\mathcal{G}_{n,A}(\tilde{f},x)=\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)}\sum_{k=0}^{\infty}\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\tilde{f}\left(\frac{k}{n}\beta_n\right),\quad x\in[0,\infty), \tag{1.7}$$

where $h\geq 0$, β_n is a positive increasing sequence with the characteristics

$$\lim_{n\to\infty}\beta_n=\infty,\qquad \lim_{n\to\infty}\frac{\beta_n}{n}=0 \tag{1.8}$$

and ${}_pA_k$ represents the general-Appell polynomials, as defined in equation (1.4).

In this context, it is helpful to define some terms and highlight specific results.

Definition 1.1 For any uniformly continuous function f on $[0,\infty)$ and $\sigma>0$, the modulus of continuity is $\omega(f;\sigma)$ defined by

$$\omega(f;\sigma):=\sup_{\substack{s,t\in[0,\infty)\\|s-t|\leq\sigma}}\left|f(s)-f(t)\right|. \tag{1.9}$$

Indeed, for any $\sigma>0$ and each $s,t\in[0,\infty)$, we can express the inequality as follows:

$$\left|f(s)-f(t)\right|\leq\omega(f;\sigma)\left(\frac{|s-t|}{\sigma}+1\right). \tag{1.10}$$

The remaining sections of the paper are organized as follows: In Sect. 2, we derive local approximation results using the generalized Szász operators defined by (1.7). Specifically, we examine the convergence of these operators with the help of the Korovkin theorem. We establish the order of approximation using both classical and Lipschitz class approaches. In Sect. 3, we investigate the weighted $\mathfrak{B}$-statistical convergence and statistically weighted $\mathfrak{B}$-summability properties of the operators. In Sect. 4, we present numerical examples to compute error estimation and demonstrate the efficiency of the proposed operators. The programming codes are executed using WOLFRAM MATHEMATICA v12.3.1 on the MacOSX 13.2.1 x86(64-bit) processor. In Sect. 5, we provide concluding remarks and suggest potential directions for further studies.

2 Local approximation characteristics of $\mathcal{G}_{n,A}(\tilde{f};x)$

This section discusses some lemmas about our operator that will be used in other sections to prove theorems. Then, we use the universal result established by Korovkin and also provide an estimate for the order of approximation by utilizing the modulus of continuity to present our key theorems.

Lemma 2.1 *The operators defined in equation* (1.7) *are linear and positive.*

Lemma 2.2 *With the aid of equation* (1.4), *we lead to the following equalities*:

$$\sum_{k=0}^{\infty} \frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!} = A(1)e^{\frac{n}{\beta_n}x}\phi(h;1), \tag{2.1}$$

$$\sum_{k=0}^{\infty} k\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!} = \left[\frac{n}{\beta_n}xA(1)\phi(h;1) + A(1)\phi'(h;1) + A'(1)\phi(h;1)\right]e^{\frac{n}{\beta_n}x}, \tag{2.2}$$

$$\begin{aligned}
&\sum_{k=0}^{\infty} k^2\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!} \\
&\quad = \left[\frac{n^2}{\beta_n^2}x^2A(1)\phi(h;1) + \frac{n}{\beta_n}x\bigl(A(1)\phi(h;1) + 2A'(1)\phi(h;1) + 2A(1)\phi'(h;1)\bigr)\right. \\
&\qquad + A'(1)\phi(h;1) + A(1)\phi'(h;1) + 2A'(1)\phi'(h;1) \\
&\qquad \left. + A''(1)\phi(h;1) + A(1)\phi''(h;1)\right]e^{\frac{n}{\beta_n}x}.
\end{aligned} \tag{2.3}$$

Lemma 2.3 *Let $\mu_s(\nu) = \nu^s$, $s = 0,1,2$ then the operators $\mathcal{G}_{n,A}(\tilde{f},x)$ defined in* (1.7) *satisfy the following results*:

$$\mathcal{G}_{n,A}\bigl(\mu_0(\nu);x\bigr) = 1, \tag{2.4}$$

$$\mathcal{G}_{n,A}\bigl(\mu_1(\nu);x\bigr) = x + \frac{\beta_n}{n}\left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)}\right) \tag{2.5}$$

and

$$\begin{aligned}
\mathcal{G}_{n,A}\bigl(\mu_2(\nu);x\bigr) &= x^2 + \frac{\beta_n}{n}x\left(1 + 2\frac{A'(1)}{A(1)} + 2\frac{\phi'(h;1)}{\phi(h;1)}\right) \\
&\quad + \frac{\beta_n^2}{n^2}\left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)} + 2\frac{A'(1)\phi'(h;1)}{A(1)\phi(h;1)} + \frac{A''(1)}{A(1)} + \frac{\phi''(h;1)}{\phi(h;1)}\right).
\end{aligned} \tag{2.6}$$

Proof With the aid of Lemma 2.2, the proof becomes fairly straightforward to follow. Hence, the details can be omitted. □

Now, we obtain central moments immediately.

Lemma 2.4 *Let $\mu_s^x(\nu) = (\nu - x)^s$, $s = 1,2$. Regarding operators* (1.7), *the following results are established*:

$$\mathcal{G}_{n,A}\bigl(\mu_1^x(\nu);x\bigr) = \frac{\beta_n}{n}\left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)}\right), \tag{2.7}$$

$$\begin{aligned}
&\mathcal{G}_{n,A}\bigl(\mu_2^x(\nu);x\bigr) \\
&\quad = \frac{\beta_n}{n}x + \frac{\beta_n^2}{n^2}\left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)} + 2\frac{A'(1)\phi'(h;1)}{A(1)\phi(h;1)} + \frac{A''(1)}{A(1)} + \frac{\phi''(h;1)}{\phi(h;1)}\right).
\end{aligned} \tag{2.8}$$

Proof By making use of Lemma 2.1 and Lemma 2.3, the proof is easy to follow. Hence, the detailed proof can be omitted. □

Let $\mathcal{C}_{\mathcal{E}}[0,\infty)$ represent the subset of continuous functions $\tilde{f}$ on the interval $[0,\infty)$ within the larger space $\mathcal{E}[0,\infty)$. $\mathcal{E}[0,\infty)$ represent the space of all functions $\tilde{f}(x)$ defined on $[0,\infty)$ and satisfying the condition $|\tilde{f}(x)| \leq c\exp(\alpha x)$, $\alpha \in \mathbb{R}$ and $c \in \mathbb{R}^+$. When $r \in \mathbb{N}$ is fixed, $\mathcal{C}^r_{\mathcal{E}} = \{\tilde{f} \in \mathcal{C}_{\mathcal{E}}[0,\infty) : \tilde{f}', \tilde{f}'', \dots, \tilde{f}^r \in \mathcal{C}_{\mathcal{E}}[0,\infty)\}$. Additionally, $\mathcal{C}_{\mathcal{B}}[0,\infty)$ denotes the space of all continuous functions defined on $[0,\infty)$ that are bounded and is equipped with the norm $\|\tilde{f}\| = \sup_{x\in[0,\infty)} |\tilde{f}(x)|$.

Theorem 2.1 *For any function $\tilde{f}$ belonging to the class $\mathcal{C}_{\mathcal{B}}[0,\infty)$, the convergence holds as follows*:

$$\lim_{n\to\infty} \mathcal{G}_{n,A}(\tilde{f}) = \tilde{f} \tag{2.9}$$

this convergence is uniform on each compact subset of $[0,\infty)$.

Proof With the aid of Lemma 2.3, it follows that

$$\lim_{n\to\infty} \mathcal{G}_{n,A}\big(\mu_s(\nu)\big) = \nu^s, \quad s = 0,1,2 \tag{2.10}$$

uniformly on each compact subset of $[0,\infty)$. Using the Korovkin theorem [23], we get assertion (2.9). □

Next, we establish the order of approximation of the operators $\mathcal{G}_{n,A}(\tilde{f};x)$.

Theorem 2.2 *Let $\tilde{f} \in \mathcal{C}_{\mathcal{B}}[0,\infty)$. Then, for $x \in [0,\mathbb{T}]$, the operators $\mathcal{G}_{n,A}(\tilde{f};x)$ satisfy the following inequality*:

$$\big|\mathcal{G}_{n,A}(\tilde{f};x) - \tilde{f}\big| \leq (1 + \sqrt{\Upsilon_n})\omega\left(\tilde{f};\sqrt{\frac{\beta_n}{n}}\right), \tag{2.11}$$

where

$$\Upsilon_n = \mathbb{T} + \frac{\beta_n}{n}\left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)} + 2\frac{A'(1)\phi'(h;1)}{A(1)\phi(h;1)} + \frac{A''(1)}{A(1)} + \frac{\phi''(h;1)}{\phi(h;1)}\right). \tag{2.12}$$

Proof Consider

$$\big|\mathcal{G}_{n,A}(\tilde{f};x) - \tilde{f}(x)\big| = \left|\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)} \sum_{k=0}^{\infty} \frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\tilde{f}\left(\frac{k}{n}\beta_n\right) - \tilde{f}(x)\right|, \tag{2.13}$$

using triangle inequality, we have

$$\big|\mathcal{G}_{n,A}(\tilde{f};x) - \tilde{f}(x)\big| \leq \frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)} \sum_{k=0}^{\infty} \frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\left|\tilde{f}\left(\frac{k}{n}\beta_n\right) - \tilde{f}(x)\right|, \tag{2.14}$$

in view of inequality (1.10), the above equation gives

$$\big|\mathcal{G}_{n,A}(\tilde{f};x) - \tilde{f}(x)\big| \leq \omega(\tilde{f},\delta)\left\{1 + \frac{1}{\delta}\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)} \sum_{k=0}^{\infty} \frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\left|\frac{k}{n}\beta_n - x\right|\right\}. \tag{2.15}$$

Using the Cauchy-Schwartz inequality, the above inequality gives

$$\begin{aligned}&\left|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right|\\&\quad\leq\omega(\tilde{f},\delta)\left\{1+\frac{1}{\delta}\left(\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)}\sum_{k=0}^{\infty}\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\left(\frac{k}{n}\beta_n-x\right)^2\right)^{\frac{1}{2}}\right\},\end{aligned}\tag{2.16}$$

which in view of equations (1.7) and (2.8) gives

$$\left|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right|\leq\omega(\tilde{f},\delta)\left\{1+\frac{1}{\delta}\sqrt{\mathcal{G}_{n,A}\left(\mu_2^x(\nu);x\right)}\right\}.\tag{2.17}$$

Taking $\delta=\sqrt{\frac{\beta_n}{n}}$ and by making use of $0\leq x\leq\mathbb{T}$ gives assertion (2.11). □

Now, we establish the degree of approximation for operators (1.7) with the aid of the Lipschitz class. For $0<\gamma_1\leq 1$ and $\tilde{f}\in\mathcal{C}[0,\infty)$, we define the Lipschitz class $Lip_{\mathcal{M}}^{(\gamma_1)}$ as follows:

$$Lip_{\mathcal{M}}^{(\gamma_1)}=\left\{\tilde{f}:\left|\tilde{f}(\nu_1)-\tilde{f}(\nu_2)\right|\leq\mathcal{M}|\nu_1-\nu_2|^{\gamma_1}\right\}.\tag{2.18}$$

Theorem 2.3 *Consider that $\tilde{f}\in Lip_{\mathcal{M}}^{(\gamma_1)}$. Then, we have*

$$\left|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right|\leq\mathcal{M}\left[\mathcal{G}_{n,A}\left(\mu_2^x(\nu);x\right)\right]^{\frac{\gamma_1}{2}}.\tag{2.19}$$

Proof Since, $\tilde{f}\in Lip_{\mathcal{M}}^{(\gamma_1)}$, in view of the definition given in (2.18), we may write

$$\begin{aligned}\left|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right|&=\left|\mathcal{G}_{n,A}\left(\tilde{f}(\nu)-\tilde{f}(x);x\right)\right|\\&\leq\mathcal{G}_{n,A}\left(\left|\tilde{f}(\nu)-\tilde{f}(x)\right|;x\right)\leq\mathcal{M}\mathcal{G}_{n,A}\left(|\nu-x|^{\gamma_1};x\right).\end{aligned}\tag{2.20}$$

Applying the Hölder inequality on the right-hand side gives

$$\mathcal{G}_{n,A}\left(|\nu-x|^{\gamma_1};x\right)=\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)}\sum_{k=0}^{\infty}\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\left|\frac{k}{n}\beta_n-x\right|^{\gamma_1}\tag{2.21}$$

$$\begin{aligned}&\mathcal{G}_{n,A}\left(|\nu-x|^{\gamma_1};x\right)\\&\quad=\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)}\sum_{k=0}^{\infty}\left\{\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\right\}^{\frac{2-\gamma_1}{2}}\left\{\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\right\}^{\frac{\gamma_1}{2}}\left|\frac{k}{n}\beta_n-x\right|^{\gamma_1}\end{aligned}\tag{2.22}$$

$$\begin{aligned}&\mathcal{G}_{n,A}\left(|\nu-x|^{\gamma_1};x\right)\\&\quad\leq\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)}\\&\qquad\times\left\{A(1)\phi(h;1)e^{\frac{n}{\beta_n}x}\right\}^{\frac{2-\gamma_1}{2}}\left\{\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)}\sum_{k=0}^{\infty}\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\right\}^{\frac{2-\gamma_1}{2}}\\&\qquad\times\left\{A(1)\phi(h;1)e^{\frac{n}{\beta_n}x}\right\}^{\frac{\gamma_1}{2}}\left\{\frac{e^{-\frac{n}{\beta_n}x}}{A(1)\phi(h;1)}\sum_{k=0}^{\infty}\frac{{}_pA_k(\frac{n}{\beta_n}x;h)}{k!}\left|\frac{k}{n}\beta_n-x\right|^2\right\}^{\frac{\gamma_1}{2}},\end{aligned}\tag{2.23}$$

$$\mathcal{G}_{n,A}\left(|\nu - x|^{\gamma_1}; x\right) \leq \left[\mathcal{G}_{n,A}\left(\mu_0(\nu); x\right)\right]^{\frac{2-\gamma_1}{2}} \left[\mathcal{G}_{n,A}\left(\mu_2^x(\nu); x\right)\right]^{\frac{\gamma_1}{2}} \tag{2.24}$$

using inequality (2.24) in inequality (2.20), we prove assertion (2.19). □

3 Statistical convergence properties

In this section, we will establish the weighted $\mathfrak{B}$-statistical convergence and statistical weighted $\mathfrak{B}$-summability properties of the operators $\mathcal{G}_{n,A}$ using a sequence of infinite matrices denoted by $\mathfrak{B} = (\mathfrak{B}_i)$, where $\mathfrak{B}_i = (\beta_{nk}(i))_{i\in\mathbb{N}}$ (refer to [8, 19]). Furthermore, we will estimate the rate of weighted $\mathfrak{B}$-statistical convergence for the proposed operator.

Definition 3.1 [16] Consider $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathfrak{E} \subseteq \mathbb{N}_0$ and $\mathfrak{E}_n = \{r : r \leq n \text{ and } r \in \mathfrak{E}\}$. Assume that the cardinality of the $\mathfrak{E}_n$ is denoted by the symbol $|\mathfrak{A}_n|$. Then,

$$\delta(\mathfrak{E}) = \lim_{n\to\infty} \frac{|\mathfrak{E}_n|}{n} = \lim_{n\to\infty} \frac{1}{n}\left|\{r : r \leq n \text{ and } r \in \mathfrak{E}\}\right| \tag{3.1}$$

is referred to as the natural density of $\mathfrak{E}$.

Remark 3.1 It is worth noting that every sequence that converges in the classical sense will also converge statistically, but there are sequences that converge statistically but not classically.

Karakaya *et al.* [20] initially suggested the idea of weighted statistical convergence. A revised form of weighted statistical convergence was proposed by Mursaleen *et al.* [26].

Definition 3.2 Suppose that $l = (l_k)$ is a sequence of nonnegative numbers with $l_0 > 0$, and $L_n = \sum_{k=0}^{\infty} l_k \to \infty$ as $n \to \infty$. Then, the sequence $x = (x_k)$ is considered weighted statistically convergent to L if, for each $\nu > 0$, the following condition holds:

$$\lim_{n\to\infty} \frac{1}{L_n}\left|\left\{k \leq L_n : l_k|x_k - L| \geq \nu\right\}\right| = 0. \tag{3.2}$$

Remark 3.2 Indeed, if we set $l_k = 1$ for all k, then the above definition is simplified to classical statistical convergence.

Kolk [30] proposed a novel matrix method called $\mathfrak{B}$-summability, originally attributed to Steiglitz.

Definition 3.3 Consider a sequence of infinite matrices $\mathfrak{B} = (\mathfrak{B}_i)$, where $\mathfrak{B}_i = (\beta_{nk}(i))$. Given a bounded sequence $x = (x_n)$, we say that x is $\mathfrak{B}$-summable to the value $\mathfrak{B} - \lim x$ if $\lim_{n\to\infty}(\mathfrak{B}_i x)_n = \mathfrak{B} - \lim x$ uniformly for $i = 0, 1, 2, \ldots$. The matrix $\mathfrak{B} = (\mathfrak{B}_i)$ is considered regular if and only if the following conditions are satisfied [9, 30]:

1. $\|\mathfrak{B}\| = \sup_{n,i} \sum_k |\beta_{nk}(i)| < \infty$,
2. $\lim_{n\to\infty} \beta_{nk} = 0$ uniformly in i for each $k \in \mathbb{N}$,
3. $\lim_{n\to\infty} \beta_{nk} = 1$ uniformly in i.

The set of all regular matrices $\mathfrak{B}$ with $\beta_{nk}(i) \geq 0$ for all n, k, and i is denoted by $\mathfrak{R}^+$. For a regular nonnegative summability matrix $\mathfrak{B} \in \mathfrak{R}^+$ and a bounded sequence $x = (x_n)$, x is considered $\mathfrak{B}$-statistically convergent to the number l if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k:|x_k-l|\geq\epsilon} \beta_{nk}(i) = 0 \quad \text{uniformly in } i. \tag{3.3}$$

Definition 3.4 [19] Consider a sequence of regular infinite nonnegative matrices $\mathfrak{B} = (\mathfrak{B}_i)_{i\in\mathbb{N}}$. Additionally, let $p = (p_n)$ be a sequence of nonnegative numbers with $p_0 > 0$, and $P_n = \sum_{k=0}^{n} p_k \to \infty$, as $n \to \infty$. A sequence $x = (x_n)$ is said to be weighted $\mathfrak{B}$-statistically convergent to the number l if, for every $\epsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{P_m} \sum_{n=0}^{m} p_n \sum_{k:|x_k-l|\geq\epsilon} \beta_{nk}(i) = 0 \quad \text{uniformly in } i. \tag{3.4}$$

In this scenario, we will refer to it as $[stat_{\mathfrak{B}}, p_n] - \lim x = l$.

Theorem 3.1 [19] *Consider $\mathfrak{B} = (\mathfrak{B}_i)_{i\in\mathbb{N}} \in \mathfrak{R}^+$. Consider a sequence of positive linear operators, denoted as $(\mathfrak{T}_n)_{n\in\mathbb{N}}$, which operate on the space $\mathcal{C}[0,1]$ and map it back into itself. If the following property holds*:

$$[stat_{\mathfrak{B}}, p_n] - \lim_{m\to\infty} \left\| \mathfrak{T}_n(e_j) - e_j \right\|_{\mathcal{C}[0,1]} = 0, \quad j = 0,1,2. \tag{3.5}$$

Then, for each $f \in \mathcal{C}[0,1]$,

$$[stat_{\mathfrak{B}}, p_n] - \lim_{m\to\infty} \left\| \mathfrak{T}_n(f) - f \right\|_{\mathcal{C}[0,1]} = 0. \tag{3.6}$$

Theorem 3.2 *Consider $\mathfrak{B} = (\mathfrak{B}_i)_{i\in\mathbb{N}} \in \mathfrak{R}^+$ and $f \in \mathcal{C}_{\mathcal{E}}[0,\infty)$. Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence satisfying conditions in equation (1.8). Additionally, assume that $p = (p_n)$ is a sequence of positive numbers, including zero with $p_0 > 0$ and $P_m = \sum_{n=0}^{m} p_n \to \infty$, as $m \to \infty$. Then, for each $\tilde{f} \in C[0,T]$, we have*:

$$[stat_{\mathfrak{B}}, p_n] - \lim_{m\to\infty} \left\| \mathcal{G}_{n,A}(\tilde{f}) - \tilde{f} \right\| = 0. \tag{3.7}$$

Proof Suppose $\tilde{f} \in \mathcal{C}_{\mathcal{E}}[0,\infty)$ and $x \in [0,T]$, where $T \in \mathbb{R}^+$ is a fixed constant. With the aid of Theorem 3.1, it suffices to demonstrate

$$[stat_{\mathfrak{B}}, p_n] - \lim_{m\to\infty} \left\| \mathcal{G}_{n,A}(\mu_s) - \mu_s \right\| = 0, \quad s = 0,1,2. \tag{3.8}$$

Using Lemma 2.3, we have

$$[stat_{\mathfrak{B}}, p_n] - \lim_{m\to\infty} \left\| \mathcal{G}_{n,A}(\mu_0) - \mu_0 \right\| = 0. \tag{3.9}$$

Further,

$$\sup_{x\in[0,T]} \left| \mathcal{G}_{n,A}(\mu_1; x) - \mu_1(x) \right| = \frac{\beta_n}{n} \left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)} \right). \tag{3.10}$$

Now, for a given $\epsilon' > 0$, let us choose a number $\epsilon > 0$ such that $\epsilon' < \epsilon$. We will then set:

$$\mathscr{A} := \left\{ k \in \mathbb{N} : k \leq n \text{ and } \left\| \mathcal{G}_{n,A}(\mu_1; x) - \mu_1(x) \right\| \geq \epsilon' \right\}, \tag{3.11}$$

$$\mathscr{A}_1 := \left\{k \in \mathbb{N} : k \leq n \text{ and } \frac{\beta_n}{n}\left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)}\right) \geq \epsilon' - \epsilon\right\}. \tag{3.12}$$

We observe that $\mathscr{A} \subset \mathscr{A}_1$, which implies that, for $m \in \mathbb{N}$, we have:

$$\frac{1}{P_m}\sum_{n=0}^{m} p_n \sum_{k\in\mathscr{A}} \beta_{nk}(i) \leq \frac{1}{P_m}\sum_{n=0}^{m} p_n \sum_{k\in\mathscr{A}_1} \beta_{nk}(i) \quad i \in \mathbb{N}. \tag{3.13}$$

As $m \to \infty$ in the above inequality, we get

$$[stat_{\mathfrak{B}}, p_n] - \lim_{m\to\infty} \left\|\mathcal{G}_{n,A}(\mu_1) - \mu_1\right\| = 0. \tag{3.14}$$

Similarly, using Definition 3.4 and Lemma 2.3, we obtain

$$\begin{aligned} &\sup_{x\in[0,T]} \left|\mathcal{G}_{n,A}(\mu_2;x) - \mu_2(x)\right| \\ &\quad = \left| T\frac{\beta_n}{n}\left(1 + 2\frac{A'(1)}{A(1)} + 2\frac{\phi'(h;1)}{\phi(h;1)}\right)\right. \\ &\qquad \left. + \frac{\beta_n^2}{n^2}\left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)} + 2\frac{A'(1)\phi'(h;1)}{A(1)\phi(h;1)} + \frac{A''(1)}{A(1)}\frac{\phi''(h;1)}{\phi(h;1)}\right)\right|. \end{aligned} \tag{3.15}$$

Given $r > 0$, we choose a number ϵ_0 such that $\epsilon_0 < r$. Then, we set:

$$\tilde{\mathcal{S}} := \left\{k \in \mathbb{N} : k \leq n \text{ and } \left\|\mathcal{G}_{n,A}(\mu_2) - \mu_2\right\| \geq r\right\}, \tag{3.16}$$

$$\tilde{\mathcal{S}}_1 := \left\{k \in \mathbb{N} : k \leq n \text{ and } T\frac{\beta_k}{k}\left(1 + 2\frac{A'(1)}{A(1)} + 2\frac{\phi'(h;1)}{\phi(h;1)}\right) \geq \frac{r-\epsilon_0}{2}\right\}, \tag{3.17}$$

$$\begin{aligned} \tilde{\mathcal{S}}_2 := \Bigg\{&k \in \mathbb{N} : k \leq n \text{ and } \\ &\frac{\beta_k^2}{k^2}\left(\frac{A'(1)}{A(1)} + \frac{\phi'(h;1)}{\phi(h;1)} + 2\frac{A'(1)\phi'(h;1)}{A(1)\phi(h;1)} + \frac{A''(1)}{A(1)}\frac{\phi''(h;1)}{\phi(h;1)}\right) \geq \frac{r-\epsilon_0}{2}\Bigg\}, \end{aligned} \tag{3.18}$$

then it can be concluded that the inclusion $\tilde{\mathcal{S}} \subset \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2$ is valid, and this implication suggests that

$$\frac{1}{P_m}\sum_{n=0}^{m} p_n \sum_{k\in\tilde{\mathcal{S}}} \beta_{nk}(i) \leq \frac{1}{P_m}\sum_{n=0}^{m} p_n \sum_{k\in\tilde{\mathcal{S}}_1} \beta_{nk}(i) + \frac{1}{P_m}\sum_{n=0}^{m} p_n \sum_{k\in\tilde{\mathcal{S}}_2} \beta_{nk}(i), \quad i \in \mathbb{N}. \tag{3.19}$$

By taking the limit as m approaches infinity in the above inequality, we obtain:

$$[stat_{\mathfrak{B}}, p_n] - \lim_{m\to\infty} \left\|\mathcal{G}_{n,A}(\mu_2) - \mu_2\right\| = 0. \tag{3.20}$$

Conclusively, in view of equations (3.9), (3.14), and (3.20), we obtain

$$[stat_{\mathfrak{B}}, p_n] - \lim_{m\to\infty} \left\|\mathcal{G}_{n,A}(\tilde{f}) - \tilde{f}\right\| = 0, \tag{3.21}$$

which completes the proof. □

Definition 3.5 [19] Assume that $\mathfrak{B} = (\mathfrak{B}_i)_{i\in\mathbb{N}}$ is a sequence of infinite nonnegative regular matrices. A sequence $x = (x_n)$ is statistically weighted $\mathfrak{B}$-summable to $\mathcal{L}$ if, for each $\epsilon > 0$,

$$\lim_{j\to\infty} \frac{1}{j} \left| \left\{ m \le j : \left| \frac{1}{P_m} \sum_{n=0}^{m} p_n \sum_{k=0}^{\infty} x_k \beta_{nk}(i) - \mathcal{L} \right| \ge 0 \right\} \right| = 0 \quad \text{uniformly in } i. \tag{3.22}$$

In this case, it is denoted by $\bar{N}_{\mathfrak{B}}(stat) - \lim x = \mathcal{L}$.

Theorem 3.3 [19] *Let $x = (x_k)$ be a bounded sequence. If x is weighted $\mathfrak{B}$-statistically convergent to $\mathcal{L}$, then it is statistically weighted $\mathfrak{B}$-summable to the same $\mathcal{L}$, but not conversely.*

Corollary 3.1 *Suppose that $\mathfrak{B} \in \mathcal{R}^+$ and $\tilde{f} \in \mathcal{C}_{\mathcal{E}}[0,\infty)$. Then, for every $\tilde{f} \in [0,T]$,*

$$\bar{N}_{\mathfrak{B}}(stat) - \lim \left\| \mathcal{G}_{n,A}(\tilde{f}) - \tilde{f} \right\| = 0. \tag{3.23}$$

Proof The proof immediately follows with the aid of Theorem 3.2 and Theorem 3.3. Hence, the details can be omitted. □

Moreover, we can estimate the rate of weighted $\mathfrak{B}$-statistical convergence of $\mathcal{G}_{n,A}(\tilde{f};x)$ to $\tilde{f} \in \mathcal{C}[0,T]$ using the modulus of continuity defined as (1.9).

Definition 3.6 [19] Assume that $\mathfrak{B} = (\mathfrak{B}_i)_{i\in\mathbb{N}} \in \mathcal{R}^+$, and let (q_m) be a sequence of positive nondecreasing sequence of real numbers. A sequence $x = (x_n)$ is weighted $\mathfrak{B}$-statistically convergent to l with the rate $o(q_m)$ if for any $\epsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{q_m P_m} \sum_{n=0}^{m} p_n \sum_{k:|x_k - l| \ge \epsilon} x_k \beta_{nk}(i) = 0 \quad \text{uniformly in } i. \tag{3.24}$$

In this case, it is denoted by $x_n - l = [stat_{\mathfrak{B}}, p_n] - o(q_m)$.

Theorem 3.4 *Suppose that $(c_s)_{s\in\mathbb{N}}$ and $(d_s)_{s\in\mathbb{N}}$ are positive nondecreasing sequences and consider $\mathfrak{B} \in \mathcal{R}^+$. Assuming the following assumptions are correct*:

1. $\|\mathcal{G}_{n,A}(\mu_0) - \mu_0\| = [stat_{\mathfrak{B}}, p_n] - o(c_s)$,
2. $\omega(f;\delta_n) = [stat_{\mathfrak{B}}, p_n] - o(d_s)$ *on* $[0,T]$, *where*

$$\delta_n := \left\| \mathcal{G}_{n,A}\left(\mu_2^x\right) \right\| \quad \textit{with } \mu_2^x(\nu) = (\nu - x)^2, \nu \in [0,T]. \tag{3.25}$$

Then,

$$\left\| \mathcal{G}_{n,A}(\tilde{f}) - \tilde{f} \right\| = [stat_{\mathfrak{B}}, p_n] - o(h_s) \quad \tilde{f} \in C[0,T], \tag{3.26}$$

where $h_s = \max\{c_s, d_s\}$.

Proof Let us assume that $\tilde{f} \in \mathcal{C}_{\mathcal{E}}[0,\infty)$ and $x \in [0,T]$, where $T \in \mathbb{R}$ is a fixed value. Due to the linearity and positivity of $\mathcal{G}_{n,A}(\tilde{f};x)$, we can express it as follows:

$$\left\| \mathcal{G}_{n,A}(\tilde{f};x) - \tilde{f}(x) \right\| \le \mathcal{G}_{n,A}\left(\left| \tilde{f}(t) - \tilde{f}(x) \right|; x \right) + \left| \tilde{f}(x) \right| \left| \mathcal{G}_{n,A}(\mu_0;x) - \mu_0 \right| \tag{3.27}$$

using inequality (1.10), we obtain

$$\left\|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right\| \leq \omega(\tilde{f},s)\mathcal{G}_{n,A}\left(\frac{|t-x|}{s}+1;x\right)+\left|\tilde{f}(x)\right|\left|\mathcal{G}_{n,A}(\mu_0;x)-\mu_0\right| \tag{3.28}$$

or, equivalently

$$\begin{aligned}\left\|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right\| &\leq \omega(\tilde{f},s)\left\{\mathcal{G}_{n,A}(\mu_0;x)+\frac{1}{s^2}\mathcal{G}_{n,A}\left(\mu_2^x;x\right)\right\} \\ &\quad+\left|\tilde{f}(x)\right|\left|\mathcal{G}_{n,A}(\mu_0;x)-\mu_0\right|.\end{aligned} \tag{3.29}$$

By considering the supremum over $x \in [0,T]$ on both sides of inequality (3.29), we deduce the following:

$$\begin{aligned}\left\|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right\| &\leq \omega(\tilde{f},s)\left\{\frac{1}{s^2}\left\|\mathcal{G}_{n,A}\left(\mu_2^x\right)\right\|\right. \\ &\quad\left.+\left\|\mathcal{G}_{n,A}(\mu_0)-\mu_0\right\|+1\right\}+\mathcal{N}\left\|\mathcal{G}_{n,A}(\mu_0;x)-\mu_0\right\|\end{aligned} \tag{3.30}$$

where $\mathcal{N}=\|\tilde{f}\|$.

Now, if we choose $s=\delta_n=\|\mathcal{G}_{n,A}(\mu_2^x)\|^{\frac{1}{2}}$ in expression (3.30), we arrive at

$$\left\|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right\| \leq \omega(\tilde{f},\delta_n)\left\|\mathcal{G}_{n,A}(\mu_0)-\mu_0\right\|+2\omega(\tilde{f},\delta_n)+\mathcal{N}\left\|\mathcal{G}_{n,A}(\mu_0;x)-\mu_0\right\| \tag{3.31}$$

or, equivalently

$$\left\|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right\| \leq \kappa\left\{\omega(\tilde{f},\delta_n)\left\|\mathcal{G}_{n,A}(\mu_0)-\mu_0\right\|+\omega(\tilde{f},\delta_n)+\left\|\mathcal{G}_{n,A}(\mu_0;x)-\mu_0\right\|\right\} \tag{3.32}$$

here $\kappa=\max\{2,\mathcal{N}\}$. Let $\epsilon>0$, then we can define the sets as follows:

$$\upsilon=\left\{k:\left\|\mathcal{G}_{k,A}(\tilde{f})-\tilde{f}\right\|\geq\epsilon\right\}, \tag{3.33}$$

$$\upsilon_1=\left\{k:\omega(\tilde{f},\delta_k)\left\|\mathcal{G}_{k,A}(\tilde{f})-\tilde{f}\right\|\geq\frac{\epsilon}{3\kappa}\right\}, \tag{3.34}$$

$$\upsilon_2=\left\{k:\omega(\tilde{f},\delta_k)\geq\frac{\epsilon}{3\kappa}\right\}, \tag{3.35}$$

$$\upsilon_3=\left\{k:\left\|\mathcal{G}_{n,A}(\mu_0;x)-\mu_0\right\|\geq\frac{\epsilon}{3\kappa}\right\}. \tag{3.36}$$

Then the inclusion $\upsilon\subset\bigcup_{j=1}^{3}\upsilon_j$ holds and yields

$$\begin{aligned}\frac{1}{h_mP_m}\sum_{n=0}^{m}p_n\sum_{k\in\upsilon}\beta_{nk}(i) &\leq \frac{1}{h_mP_m}\sum_{n=0}^{m}p_n\sum_{k\in\upsilon_1}\beta_{nk}(i)+\frac{1}{d_mP_m}\sum_{n=0}^{m}p_n\sum_{k\in\upsilon_2}\beta_{nk}(i) \\ &\quad+\frac{1}{c_mP_m}\sum_{n=0}^{m}p_n\sum_{k\in\upsilon_3}\beta_{nk}(i), \quad i\in\mathbb{N}.\end{aligned} \tag{3.37}$$

Through hypothesis (1) and (2), we have

$$\left\|\mathcal{G}_{n,A}(\tilde{f};x)-\tilde{f}(x)\right\| = [stat_{\mathfrak{B}}, p_n] - o(h_m), \tag{3.38}$$

where $h_m = \max\{c_m, d_m\}$. The proof is now concluded. □

4 Numerical examples

In this section, we establish the positive linear operators that include specific members of the Appell family and general polynomial families.

Example 4.1 In quantum physics, the Laguerre polynomials are extensively used to research the isotropic harmonic oscillator in three dimensions. Insights concerning the quantum mechanical behavior of these systems can be gleaned from their appearance in the solution of the Schrödinger equation for a single electron in such systems. Hermite polynomials can be used for function approximation and interpolation. Appropriate approximations of functions can be found by writing them down as a series of Hermite polynomials. This approximation method shines when the function in question oscillates or decays quickly.

Hermite-modified Laguerre polynomials ${}_Hf_n^{(\alpha,\lambda)}(x,y)$ are introduced by Raza et al. [29] as:

$$\sum_{n=0}^{\infty} {}_Hf_n^{(\alpha,\lambda)}(x,y)t^n = \frac{1}{(1-t)^{\alpha}} e^{x\lambda t + y(\lambda t)^2}, \tag{4.1}$$

which yields the following explicit representation of ${}_Hf_n^{(\alpha,\lambda)}(x,y)$:

$${}_Hf_n^{(\alpha,\lambda)}(x,y) = \sum_{r=0}^{n} \frac{(\alpha)_{n-r}\lambda^r H_r(x,y)}{r!(n-r)!}. \tag{4.2}$$

If we take $A(t) = \frac{1}{(1-t)^{\alpha}}$, $\lambda = 1$ and $\phi(h,t) = e^{\lambda h t}$ in equation (1.7), we get the following positive linear operators involving Hermite-modified Laguerre polynomials ${}_Hf_n^{(\alpha,\lambda)}(x,y)$ as:

$$\mathcal{G}_{n,HL}(\tilde{f},x) = \frac{e^{-\frac{n}{2\beta_n}x-\frac{h}{4}}}{2^{\alpha}} \sum_{k=0}^{\infty} \frac{{}_Hf_k^{(\alpha,\lambda)}(\frac{n}{\beta_n}x;h)}{2^k k!} \tilde{f}\left(\frac{k}{n}\beta_n\right), \quad x \in [0,\infty). \tag{4.3}$$

Here, we replace t by $\frac{1}{2}$ instead of 1 for the existence of particular $A(t)$.

For $n = 80, 90, 100$; $\beta_n = n^{\frac{1}{5}}$; $h = 10$ and $\alpha = 1$, Fig. 1 depicts the convergence of the operator (4.3) to the function

$$\tilde{f}(x) = \sin x^2, \tag{4.4}$$

and for $n = 80, 90, 100$; $\beta_n = n^{\frac{1}{15}}$; $h = 0.0001$ and $\alpha = 1$, Fig. 2 depicts the convergence of the operator (4.3) to the function

$$\tilde{g}(x) = \sinh x^2. \tag{4.5}$$

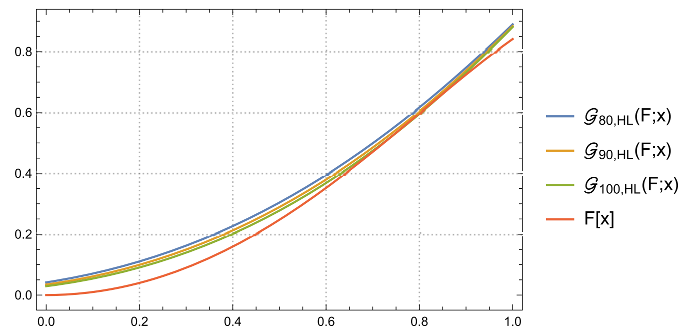

Figure 1 The convergence of operators $\mathcal{G}_{n,HL}(\tilde{f},x)$ to $\tilde{f}(x) = \sin x^2$

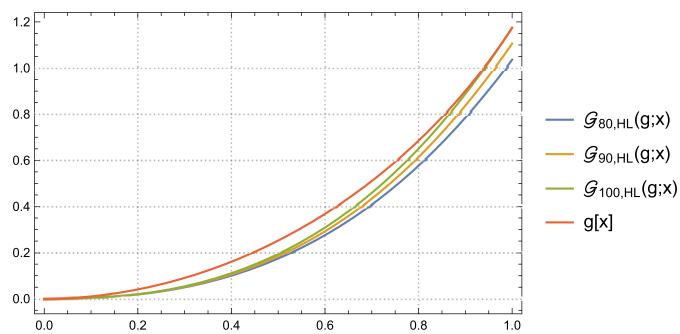

Figure 2 The convergence of operators $\mathcal{G}_{n,HL}(\tilde{g},x)$ to $\tilde{g}(x) = \sinh x^2$

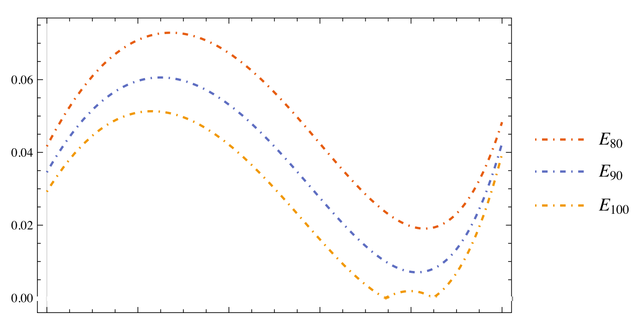

Figure 3 Graphical depiction of absolute error of operators $\mathcal{G}_{n,HL}(\tilde{f},x)$ to $\tilde{f}(x) = \sin x^2$

Further, we estimate the absolute error $E_n = |\mathcal{G}_{n,HL}(\tilde{f},x) - \tilde{f}(x)|$ and $\mathcal{E}_n = |\mathcal{G}_{n,HL}(\tilde{g},x) - \tilde{g}(x)|$ for different values of n and give the corresponding graph for the error depicting the convergence in Figs. 3 and 4. It can be clearly seen from Figs. 1–4 that for larger values of n, operator (4.3) converges to $\tilde{f}(x)$ and $\tilde{g}(x)$.

In Table 1, we compute the error of approximation of $\tilde{g}(x) = \sinh x^2$ at different points of interval for the different choices of sequence b_n.

Example 4.2 Truncated exponential polynomials exhibit a versatile range of behaviors and can approximate a wide range of functions. The capacity of truncated exponential functions to approximate functions with exponential growth and decay is their primary benefit.

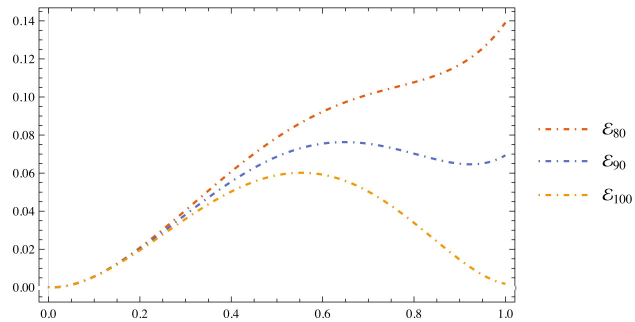

Figure 4 Graphical depiction of absolute error of operators $\mathcal{G}_{n,HL}(\tilde{g},x)$ to $\tilde{g}(x) = \sinh x^2$

Table 1 Error of approximation process for $\tilde{g}(x) = \sinh x^2$

	$\mathcal{E}_n$	Error bound at				
		$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
$b_n = n^{\frac{1}{5}}$	80	0.0226843	0.081178	0.159339	0.259574	0.423336
	90	0.0225464	0.0786976	0.150769	0.239849	0.386168
	100	0.0223686	0.0762322	0.142426	0.220774	0.350322
$b_n = n^{\frac{1}{10}}$	80	0.0214987	0.0670784	0.112372	0.152757	0.223047
	90	0.0210021	0.0626152	0.098046	0.120594	0.163077
	100	0.0204862	0.0582361	0.0841233	0.0894469	0.105094
$b_n = n^{\frac{1}{15}}$	80	0.00568912	0.0404805	0.0787168	0.101278	0.116962
	90	0.00566921	0.0382039	0.0686854	0.0754437	0.0649501
	100	0.00563257	0.0359504	0.058903	0.0503647	0.0145585

The Gould-Hopper polynomials are used in the context of the DVR technique to generalize the wavefunctions of quantum systems across a discrete basis. Eigenvalues, eigenfunctions, and other characteristics of quantum systems can be roughly approximated using this enlargement.

Two-variable truncated exponential-Gould-Hopper polynomials ${}_eH_n^{(d+1,r)}(x,y)$ are given as [5]:

$$\sum_{n=0}^{\infty} {}_eH_n^{(d+1,r)}(x,y)\frac{t^n}{n!} = \frac{e^{xt+yt^{d+1}}}{1-t^r} \tag{4.6}$$

yielding the following explicit representation of ${}_eH_n^{(d+1,r)}(x,y)$:

$${}_eH_n^{(d+1,r)}(x,y) = n!\sum_{k=0}^{[\frac{n}{r}]}\sum_{l=0}^{[\frac{n-rk}{d+1}]}\frac{x^{n-rk-(d+1)l}y^l}{l!(n-rk-(d+1)l)!}. \tag{4.7}$$

If we take $A(t) = \frac{1}{(1-t^r)}$, and $\phi(h,t) = e^{yt^{d+1}}$ in equation (1.7), we get the following positive linear operators involving 2-variable truncated exponential-Gould-Hopper polynomials ${}_eH_n^{(d+1,r)}(x,y)$ as:

$$\mathcal{G}_{n,eH}(\tilde{f},x) = \left(1-\left(\frac{1}{2}\right)^r\right)e^{-\frac{n}{2\beta_n}x-\frac{h}{2^{d+1}}}\sum_{k=0}^{\infty}\frac{{}_eH_k^{(d+1,r)}(\frac{n}{\beta_n}x;h)}{2^k k!}\tilde{f}\left(\frac{k}{n}\beta_n\right), \quad x\in[0,\infty). \tag{4.8}$$

Here, we replace t by $\frac{1}{2}$ instead of 1 for the existence of particular $A(t)$.

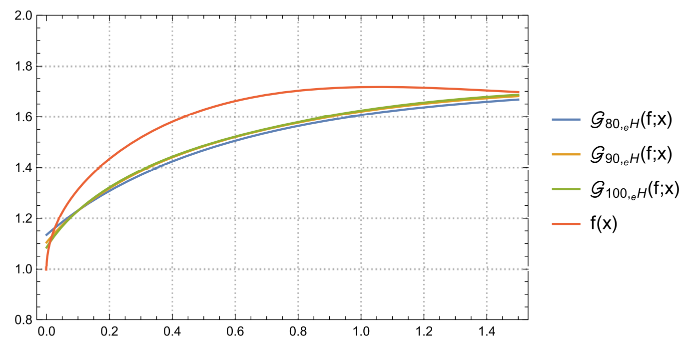

Figure 5 The convergence of operators $\mathcal{G}_{n,_eH}(\tilde{f},x)$ to $\tilde{f}(x)=\sqrt{x}+e^{-\frac{x^2}{3}}$

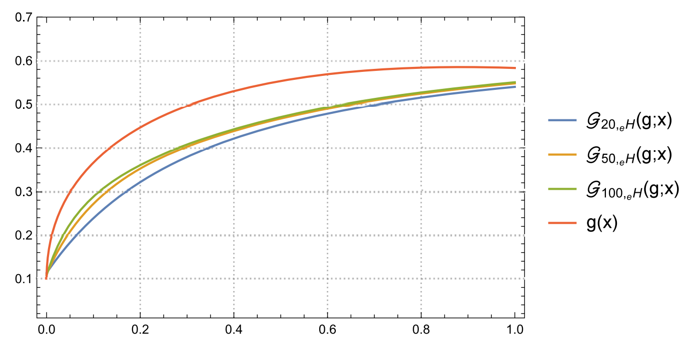

Figure 6 The convergence of operators $\mathcal{G}_{n,_eH}(\tilde{g},x)$ to $\tilde{g}(x)=-\frac{x}{2}+\frac{1}{10+x^2+x^3}+\sqrt{x}$

For $n=20,50,100$; $\beta_n=n^{\frac{1}{3}}$; $h=0.0001$; $r=2$ and $d=3$, Fig. 5 depicts the convergence of operator (4.8) to the function

$$\tilde{f}(x)=\sqrt{x}+e^{-\frac{x^2}{3}}, \tag{4.9}$$

and for $n=20,50,100$; $\beta_n=n^{\frac{1}{8}}$; $h=0.001$; $r=5$ and $d=3$, Fig. 6 depicts the convergence of operator (4.8) to the function

$$\tilde{g}(x)=-\frac{x}{2}+\frac{1}{10+x^2+x^3}+\sqrt{x}. \tag{4.10}$$

Further, we estimate the absolute error $\mathcal{E}_n=|\mathcal{G}_{n,_eH}(\tilde{f},x)-\tilde{f}(x)|$ and $E_n=|\mathcal{G}_{n,_eH}(\tilde{g},x)-\tilde{g}(x)|$ for different values of n and give the corresponding graph for the error depicting the convergence in Figs. 7 and 8. It can be clearly seen from Figs. 5–8 that for larger values of n, operator (4.8) converges to $\tilde{f}(x)$ and $\tilde{g}(x)$.

In Table 2, we compute the error of approximation of $\tilde{g}(x)=-\frac{x}{2}+\frac{1}{10+x^2+x^3}+\sqrt{x}$ at different points of interval for the different choices of sequence b_n.

5 Concluding remarks

With the help of the general-Appell polynomials, we present the Chlodowsky generalization of Szász operators. The convergence properties of the sequence of operators in (1.7) are established. The numerical evaluation is done in Wolfram Mathematica.

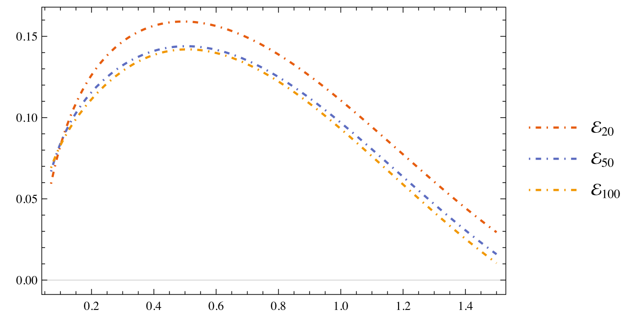

Figure 7 Graphical depiction of absolute error of operators $\mathcal{G}_{n,eH}(\tilde{f},x)$ to $\tilde{f}(x) = \sqrt{x} + e^{-\frac{x^2}{3}}$

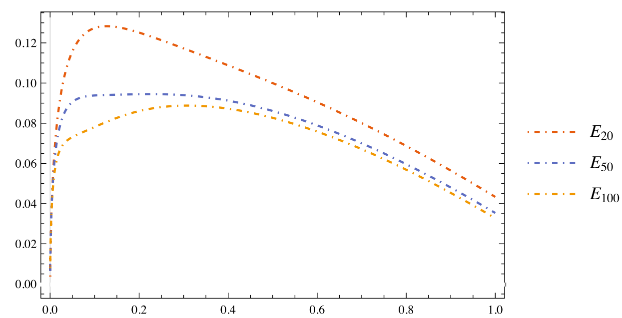

Figure 8 Graphical depiction of absolute error of operators $\mathcal{G}_{n,eH}(\tilde{g},x)$ to $\tilde{g}(x) = -\frac{x}{2} + \frac{1}{10+x^2+x^3} + \sqrt{x}$

Table 2 Error of approximation process for $\tilde{g}(x) = -\frac{x}{2} + \frac{1}{10+x^2+x^3} + \sqrt{x}$

	E_n	Error bound at				
	$n =$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
$b_n = n^{\frac{1}{4}}$	20	0.147529	0.12774	0.104113	0.0789145	0.0515653
	50	0.110057	0.0990441	0.084131	0.0639238	0.0392311
	100	0.0937739	0.0909359	0.0787186	0.0594359	0.0351593
$b_n = n^{\frac{1}{6}}$	20	0.132132	0.114179	0.0941172	0.0714188	0.0455463
	50	0.0985856	0.0931308	0.0802711	0.0607755	0.0363954
	100	0.0877984	0.0881726	0.0765825	0.0575355	0.0333826
$b_n = n^{\frac{1}{8}}$	20	0.125123	0.108813	0.0904582	0.0687138	0.0433252
	50	0.0944476	0.0912384	0.0789394	0.0596288	0.0353382
	100	0.0860583	0.0872578	0.0758307	0.0568539	0.032739

The absolute error of operators (1.7) can be computed by comparing the approximations obtained using the operators with the actual values of the functions $\tilde{g}(x) = \sinh x^2$ and $\tilde{g}(x) = -\frac{x}{2} + \frac{1}{10+x^2+x^3} + \sqrt{x}$ at different points within the interval $[0,1]$. Tables 1 and 2 present the computed absolute errors for different choices of the sequence b_n. As we correctly observed, for smaller values of b_n, the absolute error of the approximation tends to decrease. This suggests that using smaller b_n values yields more accurate results when approximating the functions $\tilde{g}(x) = \sinh x^2$ and $\tilde{g}(x) = -\frac{x}{2} + \frac{1}{10+x^2+x^3} + \sqrt{x}$ with the given operators (1.7).

In further studies, researchers may look at generalizing and modifying these operators for better approximation.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
N.R. and M.K. wrote the main manuscript text and M.M. reviewed and finalized the manuscript. All authors reviewed the manuscript.

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Received: 7 August 2023 Accepted: 6 February 2024 Published online: 20 February 2024

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