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A note on Lototsky–Bernstein bases

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Abstract

In this note, we study some approximation properties on a class of special Lototsky–Bernstein bases. We focus on approximation of $|x|$ on $[-1, 1]$ by an approximation process generated from fixed points on Lototsky–Bernstein bases. Our first result shows that the approximation procedure $p_n(x)$ to $|x|$ preserves good shapes on $[-1, 1]$. Moreover, some convergence results and inequalities are derived. Our second main result states that the rate convergence of the approximation is $O(n^{-2})$.

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1 Introduction

Let $\{p_i(x), 0 \le i \le n\}$ denote a sequence of real-valued functions on $[0, 1]$. Lototsky–Bernstein basis functions $b_{n,k}(x)$ for $0 \le x \le 1$ are defined as follows

$$b_{n,k}(x) = \sum_{\substack{K\cup L=\{1,2,\ldots,n\}\\ |L|=n-k,|K|=k}} \prod_{m\in L}\bigl(1 - p_m(x)\bigr) \prod_{l\in K} p_l(x). \tag{1.1}$$

The Lototsky–Bernstein operators L_n are defined for each function $f \in C[0, 1]$ by (see [11])

$$L_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x). \tag{1.2}$$

Throughout this paper, we always assume that $p_i(x) \in C[0, 1]$ $(1 \le i \le n)$, $0 < p_i(x) < 1$ for $x \in (0, 1)$, and that $p_i(0) = 0$, $p_i(1) = 1$. When $p_i(x) = x$, the operators L_n become the classical n-th order Bernstein operators (see [5]).

The class of Lototsky–Bernstein operators is of particular interest since the Lototsky–Bernstein basis functions $b_{n,k}(x)$ $(0 \le k \le n)$ are generated by special $p_i(x)$ $(1 \le i \le n)$. In this case, given a strictly increasing function $p_1(x)$ such that $p_1(0) = 0$, $p_1(1) = 1$, all the $p_j(x)$ $(j \ge 2)$ are determined recursively by (see [10])

$$p_{n+1}(x) = \frac{\sum_{k=0}^{n}[p_1(k/n) - p_1(k/(n + 1))]b_{n,k}(x)}{\sum_{k=0}^{n}[p_1((k + 1)/(n + 1)) - p_1(k/(n + 1))]b_{n,k}(x)}. \tag{1.3}$$

Set

$$Q_k := \frac{p_1(k/n) - p_1(k/(n+1))}{p_1((k+1)/(n+1)) - p_1(k/(n+1))} \in (0,1), \quad 0 \le k \le n.$$

These special Lototsky–Bernstein operators possess the following properties ([10, 11]):

(a) $L_n(f;x)$ preserve 1 and $p_1(x)$.

(b) $\lim_{n\to\infty} L_n(f;x) = f(x)$ uniformly on $[0,1]$ for any $f \in C[0,1]$.

(c) $p_n(x)$ $(n \ge 2)$ are increasing on $[0,1]$ with $\lim_{n\to\infty} p_n(x) = x$ uniformly in $[0,1]$ if $Q_{k+1} \ge Q_k$, for $0 \le k \le n-1$.

From (a) and (b) we know that there exist Lototsky–Bernstein operators $L_n(f;x)$ that can fix any increasing function $p_1(x)$ on $[0,1]$ with $p_1(0) = 0$, $p_1(1) = 1$ and still approximate all continuous functions uniformly on $[0,1]$. In (c), we give monotonicity and convergence of $p_n(x)$ with condition $Q_{k+1} \ge Q_k$, for $0 \le k \le n-1$. A natural question we want to ask is how fast of the limit $\lim_{n\to\infty} p_n(x) = x$ in (c) shall be. Since the form of the $p_n(x)$ from (1.3) is quite complicated for general $p_1(x)$, it therefore not easy to get the rate of the limit $\lim_{n\to\infty} p_n(x) = x$.

In this note, we shall consider a special $p_1(x) = x^2$ to solve above question to some extent. In this case, the recursive formula of $p_n(x)$ is as follows(see [11] Remark 7.8)

$$p_{n+1}(x) =: \frac{(2n+1)x^2}{1 + 2\sum_{k=1}^{n} p_k(x)}. \tag{1.4}$$

It is not difficult to inductively prove that all the $p_n(x)$ are even functions on $[-1,1]$, which entails that $\lim_{n\to\infty} p_n(x) = |x|$ uniformly on $[-1,1]$.

The function $|x|$ has been the focus of much research in approximation theory over the years. Its fundamental role in polynomial approximation is well illustrated by Lebesgue's proof of the Weierstrass approximation theorem, which is based solely on the fact that the single function $|x|$ can be approximated. For progress on approximation of $|x|$ by polynomials and rational functions, we refer to see [1–4, 6–9].

The primary goal of this paper is to study convergence properties of $p_n(x)$. The paper is organized as follows. In Sect. 2, we present some basic properties of $p_n(x)$. The inequalities $p_n(x) \le p_{n+1}(x)$ $(0 \le x \le 1/2,\ n \ge 2)$ and $p_n(x) \ge p_{n+1}(x)$ $(1/2 \le x \le 1,\ n \ge 2)$ are derived. Also we deduce two inequalities $\sum_{k=1}^{n}(x - p_k(x)) \le \frac{1-x}{2}$ $(0 \le x \le 1/2)$ and $\sum_{k=1}^{n}(x - p_k(x)) \ge \frac{1-x}{2}$ $(1/2 < x \le 1)$, and one limit $\sum_{k=1}^{\infty}(x - p_k(x)) = \frac{1-x}{2}$ $(0 < x \le 1)$. In Sect. 3, we prove the estimate $||x| - p_n(x)| \le \max\{e^{2\delta+1/\delta-4/3}, 4\} \cdot \frac{||x|-p_2(x)|}{n^2}$ on $[-1,\delta] \cup [\delta,1]$ for some fixed $0 < \delta < 1/2$.

2 Some preservation properties of $p_n(x)$

This section is devoted to some basic properties on $p_n(x)$. In [11] we prove that $p_n(\frac{1}{2}) = \frac{1}{2}$ for $n \ge 2$. Thus $p_n(x)$ $(n \ge 2)$ interpolates $|x|$ at the following set of 5 points: $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. Since $p_n(x)$ as well as $|x|$ are even functions, the study of the approximation may be restricted to the interval $[0,1]$.

Theorem 2.1 *Let $p_n(x)$ be defined in (1.4) for $x \in [0,1]$. Then for $n \ge 2$, we have $p_n(x) \le p_{n+1}(x) \le x$ when $0 \le x \le 1/2$, and $p_n(x) \ge p_{n+1}(x) \ge x$ when $1/2 < x \le 1$.*

Proof We need only to prove the desired result for $0 < x \le 1$. It is obvious to deduce from (1.4) that

$$\frac{p_{n+1}(x)}{p_n(x)} = \frac{(2n+1)x^2}{2p_n^2(x) + (2n-1)x^2}, \tag{2.1}$$

and

$$p_{n+1}(x) - x = \frac{(p_n(x) - x)[(2n-1)x^2 - 2xp_n(x)]}{2p_n^2(x) + (2n-1)x^2}, \tag{2.2}$$

with $p_2(x) = \frac{3x^2}{2x^2+1}$. Note that the following two inequalities $p_2(x) \le x$ for $0 < x \le 1/2$ and $p_2(x) \ge x$ for $1/2 < x \le 1$ hold. Setting $a_n(x) := (2n-1)x^2 - 2xp_n(x) \ge 0$, we now divide the proof into two cases.

Case I, $0 < x \le 1/2$. It is easy to see that $a_2(x) = 3x^2 - 2xp_2(x) \ge 3x^2 - 2x^2 > 0$. Thus from (2.2) we know that the sign of $p_3(x) - x$ is the same as the sign of $p_2(x) - x$, which means that $p_3(x) \le x$. Along this idea of deduction, by noting $a_n(x) \ge 0$, we can prove inductively from (2.2) that $p_n(x) \le x$. Thus from (2.1) and $p_n(x) \le x$, we have $p_{n+1}(x)/p_n(x) \ge 1$, i.e., $p_n(x) \le p_{n+1}(x)$.

Case II, $1/2 < x \le 1$. Similarly, we have $a_2(x) = 3x^2 - 2xp_2(x) = \frac{3x^2(2x^2-2x+1)}{2x^2+1} > 0$. It follows as well that the sign of $p_3(x) - x$ is the same as the sign of $p_2(x) - x$, i. e., $p_3(x) \ge x$. From (2.1) we now have $p_3(x)/p_2(x) \le 1$, i.e., $p_3(x) \le p_2(x)$. The inequality $a_3(x) = 5x^2 - 2xp_3(x) \ge 5x^2 - 2xp_2(x) \ge a_2(x) > 0$ follows. Along this idea, we can deduce inductively that $p_n(x) \ge x$ and $p_n(x) \ge p_{n+1}(x)$. □

Theorem 2.2 *Let $p_n(x)$ be defined in (1.4) for $x \in [0,1]$. Then*

(1) $\sum_{k=1}^n (x - p_k(x)) \le \frac{1-x}{2}$, for $0 \le x \le \frac{1}{2}$ and $\sum_{k=1}^n (x - p_k(x)) \ge \frac{1-x}{2}$, for $\frac{1}{2} < x \le 1$;

(2) $\sum_{k=1}^\infty (x - p_k(x)) = \frac{1-x}{2}$, for $0 < x \le 1$ and $\sum_{k=1}^\infty (x - p_k(x)) = 0$, for $x = 0$.

Proof (1) It is trivial for $x = 0$. From (1.4) we have for $0 < x \le 1$

$$2\sum_{k=1}^n \bigl(x - p_k(x)\bigr) - 1 = 2nx - \frac{(2n+1)x^2}{p_{n+1}(x)}, \tag{2.3}$$

which, combined (2.3) with Theorem 2.1, implies that the first two inequalities follow.

(2) In the case of $0 < x \le 1/2$, we know from (1) that $\sum_{k=1}^n (x - p_k(x)) \le \frac{1-x}{2}$. Since $x - p_k(x) \ge 0$, the series $\sum_{k=1}^\infty (x - p_k(x))$ converges. Now by noting $p_n(x) \le p_{n+1}(x)$ from Theorem 2.1, it follows that $b_n(x) := x - p_n(x)$ is decreasing in n. Then

$$b_{[n/2]}(x) + b_{[n/2]+1}(x) + \cdots + b_n(x) \ge nb_n(x)/2, \tag{2.4}$$

which entails that $nb_n(x) = n(x - p_n(x)) \to 0$ as $n \to \infty$. In another case $1/2 \le x \le 1$, we know from (1) that $\sum_{k=2}^n (p_k(x) - x) \le (1-x)(x-1/2)$. By noting $p_{n+1}(x) \ge p_n(x)$ from Theorem 2.1, $p_n(x) - x$ is decreasing in n. It follows similarly that $n(p_n(x) - x) \to 0$ as $n \to \infty$. Thus, from (2.3), we have for $0 < x \le 1$

$$2\sum_{k=1}^n \bigl(x - p_k(x)\bigr) - 1 = \frac{2nx(p_{n+1}(x) - x) - x^2}{p_{n+1}(x)} \to -x, \quad n \to \infty. \tag{2.5}$$

The case of $x = 0$ is also trivial. □

Corollary 2.2.1 *Let* $p_n(x)$ *be defined in* (1.4) *for* $x \in [0,1]$. *Then*

(1) $x - p_n(x) = o(\frac{1}{n})$;

(2) $p_n(x) - p_{n+1}(x) = o(\frac{1}{n^2})$.

Proof The first result follows from (2.4) and the second one can be verified by the fact that

$$p_{n+1}(x) - p_n(x) = \frac{2p_n(x)(x + p_n(x))(x - p_n(x))}{2p_n^2(x) + (2n-1)x^2}. \tag{2.6}$$

□

3 Approximation estimate of $p_n(x)$

In this section, we shall give an estimate of the approximation of $p_n(x)$ to $|x|$ on $[-1,1]$.

Theorem 3.1 *For* $x \in [-1,-\delta] \cup [\delta,1]$ *with some fixed* $0 < \delta < \frac{1}{2}$, *let* $p_n(x)$ $(n \geq 2)$ *be defined in* (1.4). *Then*

$$\big||x| - p_n(x)\big| \leq \max\big\{e^{2\delta+1/\delta-4/3}, 4\big\} \cdot \frac{||x| - p_2(x)|}{n^2}. \tag{3.1}$$

Proof We need only to prove for $x \in (0,1]$. It follows from (2.2) that

$$p_{n+1}(x) - x = \big(p_n(x) - x\big)\frac{2n - 1 - 2\frac{p_n(x)}{x}}{2n - 1 + 2(\frac{p_n(x)}{x})^2}. \tag{3.2}$$

In the following, we divide the proof into two cases.

Case I, $\frac{1}{2} \leq x \leq 1$. We have from Theorem 2.1 that $x \leq p_n(x) \leq p_2(x) = \frac{3x^2}{2x^2+1}$ $(n \geq 2)$. Thus $1 \leq \frac{p_n(x)}{x} \leq \frac{3}{2}$. Now the equality (3.2) can deduce that for $n \geq 2$

$$p_{n+1}(x) - x \leq \big(p_n(x) - x\big)\frac{2n - 1 - 2}{2n - 1 + 2} = \big(p_n(x) - x\big)\frac{2n-3}{2n+1}. \tag{3.3}$$

It follows from iterated inequality (3.3) that

$$\begin{aligned} p_n(x) - x &\leq \big(p_2(x) - x\big) \cdot \frac{1}{5} \cdot \frac{3}{7} \cdot \frac{5}{9} \cdots \frac{2n-9}{2n-5} \cdot \frac{2n-7}{2n-3} \cdot \frac{2n-5}{2n-1} \\ &\leq \frac{3(p_2(x) - x)}{(2n-3)(2n-1)} \\ &= \frac{3(p_2(x) - x)}{(2 - \frac{3}{n})(2 - \frac{1}{n})n^2} \leq \frac{3(p_2(x) - x)}{(2 - \frac{3}{2})(2 - \frac{1}{2})n^2} = \frac{4(p_2(x) - x)}{n^2}. \end{aligned} \tag{3.4}$$

Case II, $\delta \leq x < \frac{1}{2}$. It follows from (2.1) that

$$\begin{aligned} &\frac{p_{n+1}(x) - x}{p_n(x) - x} \\ &\quad = 1 + \frac{-4}{2n+1} + \frac{(x - p_n(x))((4x + 2p_n(x)) - 8(p_n(x) + x)/(2n+1))}{(2n-1)x^2 + 2p_n^2(x)} \\ &\quad = 1 + \frac{-2}{n} + \frac{2}{n(2n+1)} + \frac{(x - p_n(x)) \cdot ((4 + 2\frac{p_n(x)}{x}) - 8(\frac{p_n(x)}{x} + 1)/(2n+1))}{x \cdot ((2n-1) + 2(\frac{p_n(x)}{x})^2)}. \end{aligned} \tag{3.5}$$

By using Theorem 2.1, $p_2(x) = \frac{3x^2}{2x^2+1} \le p_n(x) \le x$ $(n \ge 2)$. We know that the sign of the last term of (3.5) is positive for $n \ge 2$. By taking logarithm to both sides of (3.5) and using the inequality $\log(1+x) \le x$ $(x > -1)$, it follows that for $n \ge 2$

$$\begin{aligned}
&\log\bigl(x - p_{n+1}(x)\bigr) - \log\bigl(x - p_n(x)\bigr) \\
&\quad\le \frac{-2}{n} + \frac{2}{n(2n+1)} + \frac{(x - p_n(x)) \cdot ((4 + 2\frac{p_n(x)}{x}) - 8(\frac{p_n(x)}{x} + 1)/(2n+1))}{x \cdot ((2n-1) + 2(\frac{p_n(x)}{x})^2)} \\
&\quad\le \frac{-2}{n} + \frac{2}{n(2n+1)} + \frac{(x - p_n(x)) \cdot ((4 + 2\frac{p_n(x)}{x}))}{x \cdot ((2n-1) + 2(\frac{p_n(x)}{x})^2)} \\
&\quad\le \frac{-2}{n} + \frac{4}{(2n-1)(2n+1)} + \frac{x - p_n(x)}{x} \cdot \frac{6}{2n-1} \\
&\quad\le \frac{-2}{n} + 2\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) + \frac{2(x - p_n(x)}{x}.
\end{aligned} \tag{3.6}$$

Then taking sum of both sides of (3.6), and combining the inequality $\sum_{k=1}^{n} \frac{1}{k} > \log n + \frac{1}{2}$ and (1) of Theorem 2.2, we get

$$\begin{aligned}
&\log\bigl(x - p_n(x)\bigr) - \log\bigl(x - p_2(x)\bigr) \\
&\quad\le -2\sum_{k=2}^{n-1} \frac{1}{k} + 2\sum_{k=2}^{n-1}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) + \frac{2}{x}\sum_{k=2}^{n-1}\bigl(x - p_k(x)\bigr) \\
&\quad\le -2(\log n - 1/2) + \frac{2}{3} + \frac{2}{x} \cdot \left(\frac{1-x}{2} - \bigl(x - x^2\bigr)\right) \\
&\quad\le -2\log n + \frac{5}{3} + \frac{(1-x)(1-2x)}{x}.
\end{aligned} \tag{3.7}$$

It follows from (3.7) that

$$x - p_n(x) \le e^{5/3+(1-x)(1-2x)/x} \cdot \frac{x - p_2(x)}{n^2}. \tag{3.8}$$

On the other hand, we have for $x \in [\delta, 1/2)$

$$2\sqrt{2} - 4/3 \le 5/3 + (1-x)(1-2x)/x = 2x + 1/x - 4/3 \le 2\delta + 1/\delta - 4/3. \tag{3.9}$$

Summing (3.4), (3.8), and (3.9)

$$\bigl||x| - p_n(x)\bigr| \le \max\bigl\{e^{2\delta+1/\delta-4/3}, 4\bigr\} \cdot \frac{||x| - p_2(x)|}{n^2}, \tag{3.10}$$

which implies that (3.1) follows. □

Remark 1 When $\delta \to 0^+$ in (3.1), the constant $\max\{e^{2\delta+1/\delta-4/3}, 4\}$ may tend to infinity. If the control constant $\max\{e^{2\delta+1/\delta-4/3}, 4\}$ can be improved to a constant which is independent of δ remains open.

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References

1. Brutman, L.: On rational interpolation to $|x|$ at the adjusted Chebyshev nodes. J. Approx. Theory **95**, 146–152 (1998)
2. Brutman, L., Passow, E.: On rational interpolation to $|x|$. Constr. Approx. **13**, 381–391 (1997)
3. Brutman, L., Passow, E.: Rational interpolation to $|x|$ at the Chebyshev nodes. Bull. Aust. Math. Soc. **56**, 81–86 (1997)
4. Castor, M.J., Gallardo, J.M., Marquina, A.: A class of incomplete Riemann solvers based on uniform rational approximations to the absolute value function. J. Sci. Comput. **60**, 363–389 (2014)
5. Davis, P.J.: Interpolation and Approximation. Dover, New York (1975)
6. Newman, D.J.: Rational approximation to $|x|$. Mich. Math. J. **11**, 11–14 (1964)
7. Newman, D.J., Reddy, A.R.: Rational approximation to $|x|/(1+x^{2m})$ on $(-\infty,+\infty)$. J. Approx. Theory **19**, 231–238 (1977)
8. Reddy, A.R.: On rational approximation to $|x|$. J. Approx. Theory **58**, 244–245 (1989)
9. Xie, T.F., Zhou, S.P.: The asymptotic property of approximation to $|x|$ by Newman's rational operators. Acta Math. Hung. **103**(4), 313–319 (2004)
10. Xu, X.W., Goldman, R.: On Lototsky-Bernstein operators and Lototsky-Bernstein bases. Comput. Aided Geom. Des. **68**, 48–59 (2019)
11. Xu, X.W., Zeng, X.M., Goldman, R.: Shape preserving properties of univariate Lototsky-Bernstein operators. J. Approx. Theory **224**, 13–42 (2017)

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