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Radial solutions of p-Laplace equations with nonlinear gradient terms on exterior domains

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Abstract

This paper studies the existence of radial solutions of the boundary value problem of *p*-Laplace equation with gradient term

$$\begin{cases} -\Delta_p u = K(|x|)f(|x|, u, |\nabla u|), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, $N \geq 3$, $1 < p \leq 2$, $K : [r_0, \infty) \to \mathbb{R}^+$, and $f : [r_0, \infty) \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ are continuous. Under certain inequality conditions of f, the existence results of radial solutions are obtained.

Mathematics Subject Classification: 35J25; 35J60; 47H10; 47N20

Keywords: p-Laplace equation; Radial solution; Exterior domain; Gradient term

1 Introduction

The boundary value problems of p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ have important application background. These problems have been raised in many different fields of applied mathematics and mechanics, such as diffusion problems, nonlinear elasticity, non-Newtonian fluids, etc. For the Laplace operator case ($p = 2$), these problems have been extensively and deeply studied, and a large number of research results have been achieved. But for the p-Laplace operator cases ($p \neq 2$), the problems are still being explored, and research results are very limited. In this paper, we consider the existence of radial solution for the boundary value problem(BVP) of p-Laplace equation with gradient term

$$\begin{cases} -\Delta_p u = K(|x|)f(|x|, u, |\nabla u|), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{1.1}$$

in the exterior domain $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, where $N \in \mathbb{N}$ and ≥ 3, $r_0 > 0$, and $p > 1$ are positive constants, $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on $\partial\Omega$, $K : [r_0, \infty) \to \mathbb{R}^+$ and

$f : [r_0, \infty) \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, $\mathbb{R}^+ = [0, \infty)$. Set $J = [0, \infty)$, $q = \frac{p}{p-1}$. For the convenience, we make the following assumptions:

(A1) $K : J \to \mathbb{R}^+$ is continuous and $r^{q(N-1)}K(r)$ is bounded on J;

(A2) $f : J \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is continuous, and for $\forall M > 0, f(r, u, \eta)$ is uniformly continuous on $J \times [-M, M] \times [0, M]$; for every $(u, \eta) \in \mathbb{R} \times \mathbb{R}^+, f(\cdot, u, \eta)$ is bounded on J.

For the special case BVP(1.1) of $p = 2$ and the nonlinearity f without gradient terms, namely for the boundary value problem

$$\begin{cases} -\Delta u = K(|x|)f(u), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.2}$$

the existence of radial solutions has been considered by many authors, see [1–7]. The authors of references[1–7] obtained some existence results by using various nonlinear analysis methods, such as upper and lower solutions method, priori estimates technique, fixed point index theory, etc. In [7], Li and Zhang built an eigenvalue criterion for the existence of positive radial solutions of BVP(1.2), see [7, Theorem 1.1]. The eigenvalue criterion is related to the principle eigenvalue λ_1 of the corresponding linear eigenvalue problem, and it is an effective method to obtain positive solutions. Recently, Li and Wei [8] partially extended the result of [7] to the p-Laplace boundary value problem

$$\begin{cases} -\Delta_p u = K(|x|)f(u), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{1.3}$$

in the case of $1 < p < N$, see [8, Theorem 1.1]. BVP(1.3) has a variational structure, and the existence of its solution can be obtained by using critical point theory. For the case of bounded domains, see references [9–12].

This paper aims to study the existence of radial solutions for the general BVP(1.1) with gradient term. For the case of $p = 2$, the existence of radial solutions has been studied by some authors, see [13–17]. These authors discussed the existence of radial solutions by using upper and lower solutions method and fixed point index theory in cones. For the case of $p \neq 2$, since the p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is nonlinear, BVP(1.1) is difficult to discuss and the approach of $p = 2$ is not applicable. In this paper, we consider the case of $1 < p \leq 2$ and obtain an existence result of radial solutions. We introduce two positive constants:

$$H_0 = \frac{\sup_{r\in J} r^{q(N-1)}K(r)}{(q-1)^p(N-p)^p {r_0}^{q(N-p)}}, \qquad H_1 = \frac{(q-1)^p(N-p)^p}{{r_0}^p} H_0. \tag{1.4}$$

The main result of our paper is as follows.

Theorem 1.1 *Let* $1 < p \leq 2$ *and assumptions* (A1) *and* (A2) *hold. If the nonlinear function* f *satisfies the following conditions*:

(F1) *There exist constants* $\alpha, \beta \geq 0$ *and* $C > 0$ *with* $H_0\alpha + H_1\beta < 1$ *such that*

$$f(r,\xi,\eta)\xi \leq \alpha|\xi|^p + \beta\eta^p + C, (r,\xi,\eta) \in J \times \mathbb{R} \times \mathbb{R}^+;$$

(F2) *For every given* $M > 0$, *there is a continuous monotone increasing function* $G_M : \mathbb{R}^+ \to (0,\infty)$ *satisfying*

$$\int_0^\infty \frac{\rho d\rho}{G_M(\rho)} = \infty \tag{1.5}$$

such that

$$\left|f(r,\xi,\eta)\right| \leq G_M\left(\eta^{p-1}\right) \quad \text{for all } (r,\xi,\eta) \in J \times [-M,M] \times \mathbb{R}^+, \tag{1.6}$$

then BVP(1.1) *has at least one radial solution.*

In Theorem 1.1, condition (F1) is a growth condition of $f(r,\xi,\eta)$ on ξ and η, and it allows $f(r,\xi,\eta)$ to have downward superlinear growth on ξ and η, and upward $(p-1)$-power growth. Condition (F2) is a Nagumo-type growth condition, and it restricts $f(r,\xi,\eta)$ to have at most $2(p-1)$-power growth on η.

The proof of Theorem 1.1 is presented in Sect. 3. Some preliminaries to discuss BVP(1.1) are given in Sect. 2. At the end of Sect. 3, an example to illustrate the applicability of Theorem 1.1 is presented.

2 Preliminaries

Let $u = u(|x|)$ be a radially symmetric solution of BVP (1.1) and $r = |x|$. By direct computation, we have

$$-\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -\left(\left|u'(r)\right|^{p-2}u'(r)\right)' - \frac{N-1}{r}\left|u'(r)\right|^{p-2}u'(r).$$

Hence u is a solution of the ordinary differential equation BVP in $[r_0,\infty)$

$$\begin{cases} -(|u'(r)|^{p-2}u'(r))' - \frac{N-1}{r}|u'(r)|^{p-2}u'(r) = K(r)f(r,u(r),|u'(r)|), & r \in J, \\ u'(r_0) = 0, \qquad u(\infty) = 0, \end{cases} \tag{2.1}$$

where $u(\infty) = \lim_{r\to\infty} u(r)$. Conversely, if $u(r)$ is a solution of BVP(2.1), then $u(|x|)$ is a radial solution of BVP(1.1). Hence, to discuss the radial solutions of BVP(1.1) just consider BVP (2.1).

For BVP(2.1), we make the variable transformation by

$$t = \left(\frac{r_0}{r}\right)^{(q-1)(N-p)} \qquad \text{i.e. } r = r_0 t^{-1/(q-1)(N-p)}, \tag{2.2}$$

and set

$$v(t) = u\left(r(t)\right), \quad t \in [0,1],$$

then BVP(2.1) is changed into the BVP in $(0,1]$

$$\begin{cases} -(|v'(t)|^{p-2}v'(t))' = a(t)f(r(t), v(t), b(t)|v'(t)|), & t \in (0,1], \\ v(0) = 0, \qquad v'(1) = 0, \end{cases} \tag{2.3}$$

where

$$a(t) = \frac{r^{q(N-1)}(t)K(r(t))}{(q-1)^p(N-p)^p {r_0}^{q(N-p)}}, \quad t \in (0,1], \tag{2.4}$$

$$b(t) = \frac{(q-1)(N-p)}{r_0} t^{\frac{N-1}{N-p}}, \quad t \in (0,1]. \tag{2.5}$$

BVP (2.3) is a boundary value problem of quasilinear ordinary differential equation with nonlinear derivative term and singularity at $t = 0$. A solution v of BVP(2.3) means that $v \in C^1[0,1]$ such that $|v'|^{p-2}v' \in C^1(0,1]$, and it satisfies equation (2.3). Hence, the solution of BVP(2.3) belongs to the subset of $C^1(I)$

$$\mathfrak{D} := \left\{ v \in C^1(I) | v(0) = 0, v'(1) = 0, \left|v'\right|^{p-2} v' \in C^1(0,1] \right\}. \tag{2.6}$$

If $v \in \mathfrak{D}$ is a solution of BVP(2.3), then we easily verify that $u(r) = v(t(r))$ is a solution of BVP (2.1) and $u(|x|)$ is a classical radial solution of BVP(1.1). Hence we discuss BVP(2.3) to obtain radial solutions of BVP (1.1). We will use the Leray–Schauder fixed point theorem on the completely continuous mapping to obtain the existence of BVP (2.3).

Let $I = [0,1]$. We use $C(I)$ to denote the Banach space of all continuous function $v(t)$ on I with the maximal module norm $\|v\|_C = \max_{t\in I} |v(t)|$, $C^1(I)$ denotes the Banach space of all continuous differentiable function on I with the norm $\|v\|_{C^1} = \max\{\|v\|_C, \|v'\|_C\}$. Let $C_B(0,1]$ be the Banach space of all bounded continuous function $w(t)$ on $(0,1]$ with the norm $\|w\|_\infty = \sup_{t\in(0,1]} |w(t)|$.

Given $h \in C_B(0,1]$, we consider the simple boundary value problem corresponding to BVP(2.3)

$$\begin{cases} -(|v'(t)|^{p-2}v'(t))' = a(t)h(t), & t \in (0,1], \\ v(0) = 0, \qquad v'(1) = 0. \end{cases} \tag{2.7}$$

Define a function Φ by

$$\Phi(v) = |v|^{p-2}v = |v|^{p-1} \operatorname{sgn} v, \quad v \in \mathbb{R}. \tag{2.8}$$

Clearly, $w = \Phi(v)$ is a strictly monotone increasing continuous function on $\mathbb{R}$ and its inverse function is given by

$$\Psi(w) := \Phi^{-1}(w) = |w|^{q-1} \operatorname{sgn} w, \quad w \in \mathbb{R}. \tag{2.9}$$

$v = \Psi(w)$ is also a strictly monotone increasing continuous function on $\mathbb{R}$.

Lemma 2.1 *For any given $h \in C_B(0,1]$, BVP (2.7) has a unique solution $v := Sh \in \mathfrak{D}$. Moreover, the solution operator $S : C_B(0,1] \to C^1(I)$ is compact continuous and satisfies*

$$S(\nu h) = \nu^{q-1} Sh, \quad h \in C_B(0,1], \nu \geq 0. \tag{2.10}$$

Proof By (2.4) and assumption (A1), the function $a(t)$ is nonnegative, bounded, and continuous on $(0,1]$, and

$$\|a\|_\infty = \sup_{r\in J} \frac{r^{q(N-1)} K(r)}{(q-1)^p (N-p)^p {r_0}^{q(N-p)}} = H_0 \tag{2.11}$$

for every $s \in (0,1]$,

$$\int_s^1 a(t)\,dt = \frac{1}{[(q-1)(N-p)]^{p-1} {r_0}^{q(N-p)}} \int_{r_0}^{r(s)} r^{N-1} K(r)\,dr > 0. \tag{2.12}$$

For any given $h \in C_B(0,1]$, we verify that

$$v(t) = \int_0^t \Psi\left(\int_s^1 a(\tau)h(\tau)\,d\tau\right) ds := Sh(t), \quad t \in I, \tag{2.13}$$

is a solution of BVP(2.7). By (2.12), the function defined by

$$H(s) = \int_s^1 a(\tau)h(\tau)\,d\tau, \quad s \in I,$$

is continuous on I. Hence, $\Psi(H(s))$ is continuous on I, and

$$v(t) = \int_0^t \Psi\big(H(s)\big)\,ds, \quad t \in I,$$

is continuously differentiable on H. This means that $v \in C^1(I)$ and $v'(t) = \Psi(H(t))$ for $t \in I$, so we have

$$v'(t) = \Psi\left(\int_t^1 a(\tau)h(\tau)\,d\tau\right), \quad t \in I. \tag{2.14}$$

Using Φ to act on both sides of this equation, we obtain that

$$\left|v'(t)\right|^{p-2} v'(t) = \Phi\big(v'(t)\big) = \int_t^1 a(\tau)h(\tau)\,d\tau, \quad t \in I.$$

This implies that $(|v'(t)|^{p-2} v'(t) \in C^1(0,1]$ and

$$\left(\left|v'(t)\right|^{p-2} v'(t)\right)' = -a(t)h(t), \quad t \in (0,1].$$

Hence, $v \in \mathfrak{D}$, and it is a solution of BVP(2.7).

Conversely, if $v \in \mathfrak{D}$ is a solution of BVP(2.7), we show that v can be expressed by (2.13). Integrating equation (2.13) on $(t,1]$, we have

$$\left|v'(t)\right|^{p-2} v'(t) = H(t), \quad t \in [0,1].$$

Using Φ to act on both sides of this equation, we obtain that

$$v'(t) = \Psi\big(H(t)\big), \quad t \in [0,1].$$

Integrating this equation on $[0,t]$, we have

$$v(t) = \int_0^t \Psi\big(H(s)\big)\,ds, \quad t \in [0,1].$$

That is, v is expressed by (2.13). Hence, BVP(2.7) has a unique solution $v = Sh$.

Finally, we prove that the operator $S : C_B(0,1] \to C^1(I)$ is compact continuous. By (2.13) and (2.14) and the continuity of Ψ, we easily see that $S : C_B(0,1] \to C^1(I)$ is continuous. For any bounded set $D \subset C_B(0,1]$, by (2.13) and (2.14) we can show that $S(D)$ and its derivative set $\{v' | v \in S(D)\}$ are bounded equicontinuous sets in $C(I)$. By the Ascoli–Arzéla theorem, $S(D)$ is a precompact subset of $C^1(I)$. Thus, $S : C_B(0,1] \to C^1(I)$ is compact continuous.

By expression (2.13) of the solution operator S, we can directly verify that S satisfies (2.10). □

Lemma 2.2 *Let* $1 < p \le 2$, $[a,b] \subset \mathbb{R}^+$, $w \in C^+[a,b]$. *Then*

$$\int_a^b \Phi\big(w(t)\big)\,dt \le (b-a)^{2-p}\Phi\left(\int_a^b w(t)\,dt\right). \tag{2.15}$$

Proof Since $\Phi''(v) < 0$ in $(0,+\infty)$, it follows that $\Phi(v)$ is an upper convex function on $\mathbb{R}^+$. Hence, $\Phi(v)$ satisfies Jensen's inequality on $\mathbb{R}^+$. That is, for any $v_1, v_2, \ldots, v_n \in \mathbb{R}^+$, and $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{R}^+$ with $\mu_1 + \mu_2 + \cdots + \mu_n = 1$, $\Phi(v)$ satisfies the inequality

$$\sum_{k=1}^n \mu_k \Phi(v_k) \le \Phi\left(\sum_{k=1}^n \mu_k v_k\right). \tag{2.16}$$

For any partition of $[a,b]$,

$$\Delta : a = t_0 < t_1 < \cdot < t_n = b,$$

setting $\Delta t_k = t_k - t_{k-1}$, $k = 1,2,\ldots,n$, by Jensen's inequality (2.16), we have

$$\frac{1}{b-a}\sum_{k=1}^n \Phi\big(w(t_k)\big)\Delta t_i \le \Phi\left(\frac{1}{b-a}\sum_{k=1}^n w(t_k)\Delta t_i\right).$$

Letting $\|\Delta\| := \max_{1\le k\le n} \to 0$, by the definition of Riemann integral, we have

$$\frac{1}{b-a}\int_a^b \Phi\big(w(t)\big)\,dt \le \Phi\left(\frac{1}{b-a}\int_a^b w(t)\,dt\right).$$

Hence, (2.15) holds. □

Now we consider BVP(2.3). Let $f : J \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfy assumption (A1). Define a mapping $F : C^1(I) \to C_B(0,1]$ by

$$F(v)(t) := f\big(r(t), v(t), b(t)\big|v(t)\big|\big), \quad t \in I. \tag{2.17}$$

By assumption (A2), we easily verify that $F : C^1(I) \to C_B(0,1]$ is continuous and maps every bounded subset of $C^1(I)$ into a bounded subset of $C_B(0,1]$. Hence, by the compact continuity of the operator $S : C_B(0,1] \to C^1(I)$, the composite mapping

$$A = S \circ F : C^1(I) \to C^1(I) \tag{2.18}$$

is compact continuous. By the definitions of S, the solution of BVP(2.3) is equivalent to the fixed point of A. We will find the fixed point of A by using the following Leray–Schauder fixed point theorem of compact continuous mapping[18].

Lemma 2.3 *Let X be a Banach space, $A : X \to X$ be a compact continuous mapping. If the set of solutions of the equation family*

$$v = \mu A v, \quad 0 < \mu < 1,$$

is a bounded subset of X, then A has a fixed point.

3 Proof of the main result

Proof of Theorem 1.1 Let $A : C^1(I) \to C^1(I)$ be the mapping defined by (2.18). Then A is compact continuous and the solution of BVP(2.3) is equivalent to the fixed point of A. Hence, if $v \in C^1(I)$ is a fixed point of A, then $v(t)$ is a solution of BVP (2.3), and $u = v((r_0/|x|)^{(q-1)(N-p)})$ is a classical positive radial solution of BVP (1.1). We use Lemma 2.3 to show that A has a fixed point. For this, we consider the family of equations

$$v = \mu A v, \quad 0 < \mu < 1. \tag{3.1}$$

We need to prove that the set of solutions of (3.1) is bounded in $C^1(I)$.

Let $v_0 \in C^1(I)$ be a solution of (3.1) for $\mu_0 \in (0,1)$. By (2.10), $v_0 = \mu_0 A v_0 = \mu_0 S(F(v_0)) = S(\mu_0^{p-1} F(v_0))$. By the definition of S, v_0 is the unique solution of BVP(2.7) for $h = \mu_0^{p-1} F(v_0) \in C_B(0,1]$. Hence $v_0 \in \mathfrak{D}$ satisfies the differential equation

$$\begin{cases} -(|v_0'(t)|^{p-2} v_0'(t))' = \mu_0^{p-1} a(t) f(r(t), v_0(t), b(t)|v_0(t)|), & t \in (0,1], \\ v(0) = 0, \qquad v'(1) = 0. \end{cases} \tag{3.2}$$

By the boundary condition of v_0, we easily see that

$$\|v_0\|_p \leq \left\|v_0'\right\|_p. \tag{3.3}$$

Multiplying equation (3.2) by $v_0(t)$, by condition (F1) we have

$$\begin{aligned} -\left(\left|v_0'(t)\right|^{p-2} v_0'(t)\right)' v_0(t) &= \mu_0^{p-1} a(t) f\left(r(t), v_0(t), b(t)\left|v_0(t)\right|\right) v_0(t) \\ &\leq \mu_0^{p-1} a(t) \left(\alpha \left|v_0(t)\right|^p + \beta b^p(t) \left|v_0'(t)\right|^p + C\right) \\ &\leq \|a\|_\infty \left(\alpha \left|v_0(t)\right|^p + \beta b^p(1) \left|v_0'(t)\right|^p + C\right) \\ &= H_0 \alpha \left|v_0(t)\right|^p + H_1 \beta \left|v_0'(t)\right|^p + H_0 C, \quad t \in (0,1]. \end{aligned}$$

Integrating this inequality on $(0,1]$, then using integration by parts for the left-hand side and (3.3), we obtain that

$$\begin{aligned}\|v_0'\|_p^p &\le H_0\alpha\|v_0\|_p^p + H_1\beta\|v_0'\|_p^p + H_0C\\ &\le (H_0\alpha + H_1\beta)\|v_0'\|_p^p + H_0C.\end{aligned}$$

From this inequality it follows that

$$\|v_0'\|_p \le \left(\frac{H_0C}{1-(H_0\alpha+H_1\beta)}\right)^{1/p} := M. \tag{3.4}$$

Hence, for every $t \in I$, we have

$$\begin{aligned}|v_0(t)| &= |v_0(t) - v_0(0)|\\ &= \left|\int_0^t v_0'(s)\,ds\right|\\ &\le \int_0^t |v_0'(s)|\,ds \le \int_0^1 |v_0'(s)|\,ds \le \|v_0'\|_p \le M.\end{aligned}$$

This means that

$$\|v_0\|_C \le M. \tag{3.5}$$

For this $M > 0$, by assumption (F2), there is a monotone increasing function $G_M : \mathbb{R}^+ \to (0,\infty)$ satisfying (1.5) such that (1.6) holds. From (1.6) and (3.5) it follows that

$$\begin{aligned}\bigl|f\bigl(r(t), v_0(t), b(t)|v_0'(t)|\bigr)\bigr| &\le G_M\bigl(|b(t)v_0'(t)|^{p-1}\bigr)\\ &\le G_M\bigl(|b(1)v_0'(t)|^{p-1}\bigr), \quad t \in (0,1].\end{aligned}$$

By this and equation (3.2), we have

$$-\bigl(|v_0'(t)|^{p-2}v_0'(t)\bigr)' \le a(t)G_M\bigl(|b(1)v_0'(t)|^{p-1}\bigr), \quad t \in (0,1]. \tag{3.6}$$

By (1.5), there exists a constant $M_1 > 0$ such that

$$\int_0^{M_1} \frac{\rho\,d\rho}{G_M(\rho)} > \|a\|_\infty b^{2(p-1)}(1)\Phi(2M). \tag{3.7}$$

Choosing the positive constant

$$M_2 := \max\bigl\{M, M_1^{q-1}/b(1)\bigr\}, \tag{3.8}$$

we show that

$$\|v_0'\|_C \le M_2. \tag{3.9}$$

It may be set $\|v_0'\|_C > 0$. Since $v_0'(1) = 0$, by the maximum theorem of continuous functions, there exists $t_1 \in [0,1)$ such that

$$\|v_0'\|_C = \max_{t\in I}|v_0'(t)| = v_0'(t_1). \tag{3.10}$$

There are two cases: $v_0'(t_1) > 0$ or $v_0'(t_1) < 0$. We only consider the case $v_0'(t_1) > 0$, and the other case can be treated in the same way. Set

$$s_1 = \inf\{s \in (t_1,1] | v_0'(s) = 0\}.$$

Then $t_1 < s_1 \leq 1$, and on $[t_1,s_1]$, $v_0'(t)$ satisfies

$$v_0'(t) > 0, \quad t \in [t_1,s_1); \qquad v_0'(s_1) = 0. \tag{3.11}$$

Hence, by inequality (3.6), we have

$$\begin{aligned} -\frac{((b(1)v_0'(t))^{p-1})'(b(1)v_2'(t))^{p-1}}{G_M((b(1)v_0'(t))^{p-1})} &\leq a(t)b^{2(p-1)}(1)\big(v_2'(t)\big)^{p-1} \\ &\leq \|a\|_\infty b^{2(p-1)}(1)\Phi\big(v_0'(t)\big), \quad t \in [t_1,s_1]. \end{aligned}$$

Integrating both sides of this inequality on $[t_1,s_1]$ and making the variable transformation $\rho = (b(1)v_0{}'(t))^{p-1}$ for the left-hand side, using (2.8) and (3.11) for the right-hand side, we have

$$\int_0^{(b(1)v_0'(t_1))^{p-1}} \frac{\rho\, d\rho}{G_M(\rho)} \leq \|a\|_\infty b^{2(p-1)}(1)\int_{t_1}^{s_1}\Phi\big(v_0'(t)\big)\,dt. \tag{3.12}$$

By (3.11), $v_0' \in C^+[t_1,s_1]$. Hence $v_0(t)$ is increasing on $[t_1,s_1]$ and $0 \leq v_0(s_1) - v_0(t_1) \leq 2M$. By Lemma 2.2, we have

$$\begin{aligned} \int_{t_1}^{s_1}\Phi\big(v_0'(t)\big)\,dt &\leq (s_1-t_1)^{2-p}\Phi\left(\int_{t_1}^{s_1} v_0'(t)\,dt\right) \\ &= (s_1-t_1)^{2-p}\Phi\big(v_0(s_1)-v_0(t_1)\big) \leq \Phi(2M). \end{aligned}$$

Hence from (3.12) it follows that

$$\int_0^{(b(1)v_0'(t_1))^{p-1}} \frac{\rho\, d\rho}{G_M(\rho)} \leq \|a\|_\infty b^{2(p-1)}(1)\Phi(2M). \tag{3.13}$$

Combining this inequality and (3.7), we obtain that

$$\big(b(1)v_0'(t_1)\big)^{p-1} \leq M_1. \tag{3.14}$$

From this inequality it follows that

$$\|v_0'\|_C = v_0'(t_1) \leq M_1^{q-1}/b(1) \leq M_2.$$

Hence, (3.9) holds. By (3.9) and (3.3), we have

$$\|v_0\|_{C^1} = \max\left\{\|v_0\|_C, \left\|v_0'\right\|_C\right\} \leq M_2. \tag{3.15}$$

Hence, the set of solutions of equation family (3.1) is bounded in $C^1(I)$. By Lemma 2.3, A has a fixed point in $C^1(I)$, which is a solution of BVP(2.3).

The proof of Theorem 1.1 is complete. □

Example 3.1 Consider the boundary value problem of p-Laplace operator on the exterior of unit ball $\Omega = \{x \in \mathbb{R}^N : |x| > 1\}$

$$\begin{cases} -\Delta_p u = K(|x|)(c_0 + c_1|u|^{\alpha-2}u - c_2|\nabla u|^\beta u), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{3.16}$$

where $N \geq 3$, $1 < p \leq 2$, $K : [1, +\infty) \to \mathbb{R}^+$ is continuous and satisfies assumption (A1), c_0, c_1, c_2, α, β are positive constants. Corresponding to BVP(1.1), the nonlinearity is

$$f(r,\xi,\eta) = c_0 + c_1|\xi|^{\alpha-2}\xi - c_2\eta^\beta\xi, \quad r \geq 1, \xi \in \mathbb{R}, \eta \in \mathbb{R}^+. \tag{3.17}$$

From this it follows that

$$f(r,\xi,\eta)\xi \leq c_0|\xi| + c_1|\xi|^\alpha, \quad r \geq 1, \xi \in \mathbb{R}, \eta \in \mathbb{R}^+. \tag{3.18}$$

By this and Young's inequality, it is easy to prove that, when $1 < \alpha < p$, $f(r,\xi,\eta)$ satisfies condition (F1). By (3.17), when $0 < \beta \leq 2(p-1)$, $f(r,\xi,\eta)$ satisfies condition (F2). Hence, by Theorem 1.1, when $1 < \alpha < p$, $f(r,\xi,\eta)$ and $0 < \beta \leq 2(p-1)$, BVP(3.16) has at least one radial solution.

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Data availability
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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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