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Midpoint-type inequalities via twice-differentiable functions on tempered fractional integrals

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Abstract

In this paper, we obtain an equality involving tempered fractional integrals for twice-differentiable functions. By using this equality, we establish several left Hermite–Hadamard-type inequalities for the case of tempered fractional integrals. Moreover, we derive our results by using special cases of obtained theorems.

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1 Introduction

C. Hermite and J. Hadamard introduced Hermite–Hadamard-type inequalities for convex functions. Let us consider that $\mathfrak{F}: I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\sigma, \delta \in I$ with $\sigma < \delta$. Then, the following double inequality holds:

$$\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) \leq \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} \mathfrak{F}(x)\,dx \leq \frac{\mathfrak{F}(\sigma)+\mathfrak{F}(\delta)}{2}. \tag{1}$$

If $\mathfrak{F}$ is concave, then both inequalities in (1) are valid in the reverse direction. Many papers have been considered in order to obtain midpoint- and trapezoid-type inequalities, which give bounds for the left- and right-hand side of the inequality (1), respectively. For example, Dragomir and Agarwal first proved trapezoid-type inequalities for the case of convex functions in [12] and Kirmacı first obtained midpoint-type inequalities for convex functions in [18]. Iqbal et al. [16] investigated some fractional midpoint-type inequalities for convex functions. Sarikaya et al. generalized (1) for fractional integrals. The authors also investigated some corresponding trapezoid-type inequalities in [35]. Moreover, in [11] Dragomir proved Hermite–Hadamard-type inequalities for the case of coordinated convex functions. In addition to this, trapezoid- and midpoint-type inequalities for coordinated convex functions were established in [34] and [19], respectively. Furthermore, in [37], several fractional midpoint-type inequalities were established for coordinated convex functions. In addition, they proved Hermite–Hadamard inequalities and several trapezoidand midpoint-type inequalities for the case of generalized fractional integrals. We refer to [8, 10, 27] for further information about these kinds of inequalities.

Some Hermite–Hadamard and Simpson-type inequalities were established for functions whose absolute values of derivatives are convex in [31]. Barani et al. [6] proved Hermite–Hadamard-type inequalities for the case of twice-differentiable convex functions. In [28], J. Park considered new estimates in generalizations of Hadamard, Ostrowski, and Simpson-type inequalities for functions whose second derivatives in absolute value at certain powers are convex and quasiconvex functions. Moreover, some new generalized fractional integral inequalities of midpoint- and trapezoid-type for twice-differentiable convex functions are obtained in [24]. Furthermore, in [7], Budak et al. established some midpoint- and trapezoid-type inequalities for functions whose second derivatives in absolute value are convex. See [15, 32, 33] for results related to these types of inequalities involving twice-differentiable functions.

Numerous authors have considered fractional integral inequalities and applications by using Riemann–Liouville fractional integrals. For example, a variant of Hermite–Hadamard inequalities in Riemann–Liouville fractional integral forms was investigated in [30]. Moreover, in [14], some left Hermite–Hadamard-type inequalities were established for the case of Riemann–Liouville fractional integrals. See [13, 17, 23] and the references therein for further information and properties of Riemann–Liouville fractional integrals. While a considerable number of mathematicians has studied Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals, some authors have also considered Hermite–Hadamard inequalities for the case of other types of fractional integrals such as k-fractional integrals, Hadamard fractional integrals, tempered fractional integrals, conformable fractional integrals, etc. For example, we refer the reader to [1–5, 26] and the references cited therein.

Tempered fractional calculus is a branch of mathematics that extends the concept of fractional calculus. In [9], the definitions of fractional integration with exponential kernels and weak singular were firstly reported in Buschman's earlier work. For the other different definitions of the tempered fractional integration, see the books [22, 29, 36] and references therein. In [25], several Hermite–Hadamard-type inequalities were established associated with tempered fractional integrals for the case of convex functions which cover the previously published result for Riemann integrals and Riemann–Liouville fractional integrals.

The primary goal of this article is to present and prove left Hermite–Hadamard-type inequalities for tempered fractional integrals. The entire research structure takes four sections, including the introduction. In Sect. 2, we provide the basic definitions and facts from the fractional calculus theory. In Sect. 3, we establish an equality involving tempered fractional integrals for the case of twice-differentiable functions. By utilizing this equality, we prove midpoint-type inequalities for functions whose second derivatives are convex. We also present some remarks. Some conclusions and further directions of research are discussed in Sect. 4.

2 Preliminaries

We will now introduce the necessary mathematical preliminaries from fractional calculus theory, which will be utilized in the rest of this paper.

Definition 1 Let $\mathfrak{F} \in L_1[\sigma,\delta]$. The Riemann–Liouville integrals $J^{\alpha}_{\sigma+}\mathfrak{F}$ and $J^{\alpha}_{\delta-}\mathfrak{F}$ of order $\alpha > 0$ with $\sigma \geq 0$ are defined by

$$J^{\alpha}_{\sigma+}\mathfrak{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{x} (x-\mu)^{\alpha-1}\mathfrak{F}(\mu)\,dt, \quad x > \sigma, \tag{2}$$

and

$$J^{\alpha}_{\delta-}\mathfrak{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\delta} (\mu-x)^{\alpha-1}\mathfrak{F}(\mu)\,dt, \quad x < \delta, \tag{3}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}\,du.$$

Remark 1 In the case of $\alpha = 1$, the fractional integral becomes the classical integral.

The following are the fundamental definitions and new notations of tempered fractional operators that we will be using.

Definition 2 The incomplete gamma function and λ-incomplete gamma function are defined by

$$\curlyvee(\alpha,x) := \int_0^x \mu^{\alpha-1}e^{-\mu}\,dt$$

and

$$\curlyvee_{\lambda}(\alpha,x) := \int_0^x \mu^{\alpha-1}e^{-\lambda t}\,dt,$$

respectively. Here, $0 < \alpha < \infty$ and $\lambda \geq 0$.

Remark 2 (See [25]) For the real numbers $\alpha > 0$, $x, \lambda \geq 0$, and $\sigma < \delta$, we have

(1) $\curlyvee_{\lambda(\frac{\delta-\sigma}{2})}(\alpha,1) = \int_0^1 \mu^{\alpha-1}e^{-\lambda(\frac{\delta-\sigma}{2})\mu}\,dt = (\frac{2}{\delta-\sigma})^{\alpha} \curlyvee_{\lambda}(\alpha,\delta-\sigma)$,
(2) $\int_0^1 \curlyvee_{\lambda(\delta-\sigma)}(\alpha,x)\,dx = \frac{\curlyvee_{\lambda}(\alpha,\delta-\sigma)}{(\delta-\sigma)^{\alpha}} - \frac{\curlyvee_{\lambda}(\alpha+1,\delta-\sigma)}{(\delta-\sigma)^{\alpha+1}}$.

Definition 3 (See [20, 21]) The fractional tempered integral operators $\mathcal{J}^{(\alpha,\lambda)}_{\sigma+}\mathfrak{F}$ and $\mathcal{J}^{(\sigma,\lambda)}_{\delta-}\mathfrak{F}$ of order $\alpha > 0$ and $\lambda \geq 0$ are given by

$$\mathcal{J}^{(\alpha,\lambda)}_{\sigma+}\mathfrak{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{x} (x-\mu)^{\alpha-1}e^{-\lambda(x-\mu)}\mathfrak{F}(\mu)\,dt, \quad x \in [\sigma,\delta], \tag{4}$$

and

$$\mathcal{J}^{(\alpha,\lambda)}_{\delta-}\mathfrak{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\delta} (\mu-x)^{\alpha-1}e^{-\lambda(\mu-x)}\mathfrak{F}(\mu)\,dt, \quad x \in [\sigma,\delta], \tag{5}$$

respectively, for $\mathfrak{F} \in L_1[\sigma,\delta]$.

If we choose $\lambda = 0$, then the fractional integrals in (4) and (5) equal to the Riemann–Liouville fractional integral in (2) and (3), respectively.

3 Main results

In this section, we give several tempered fractional midpoint-type inequalities for the case of twice-differentiable functions. Let us first prove an identity in order to build midpoint-type inequalities.

Lemma 1 *If $\mathfrak{F} : [\sigma, \delta] \to \mathbb{R}$ is absolutely continuous on (σ, δ) and $\mathfrak{F}'' \in L_1([\sigma, \delta])$, then we have*

$$\begin{aligned} &\frac{\Gamma(\alpha)}{2(\delta-\sigma)^{\alpha}\, \Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) \\ &\quad= \frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\sum_{k=1}^{4} I_k. \end{aligned} \tag{6}$$

Here,

$$\begin{cases} I_1 = \int_0^{\frac{1}{2}} \varphi_\alpha(\lambda,\mu)\mathfrak{F}''(tb+(1-\mu)\sigma)\,dt, & I_3 = \int_{\frac{1}{2}}^{1} \psi_\alpha(\lambda,\mu)\mathfrak{F}''(tb+(1-\mu)\sigma)\,dt, \\ I_2 = \int_0^{\frac{1}{2}} \varphi_\alpha(\lambda,\mu)\mathfrak{F}''(ta+(1-\mu)\delta)\,dt, & I_4 = \int_{\frac{1}{2}}^{1} \psi_\alpha(\lambda,\mu)\mathfrak{F}''(ta+(1-\mu)\delta)\,dt, \end{cases}$$

with

$$\begin{cases} \varphi_\alpha(\lambda,\mu) = \mu\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\mu) - \Upsilon_{\lambda(\delta-\sigma)}(\alpha+1,\mu), \\ \psi_\alpha(\lambda,\mu) = \mu\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\mu) - \Upsilon_{\lambda(\delta-\sigma)}(\alpha+1,\mu) - \mu\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1) + \Upsilon_{\lambda(\delta-\sigma)}(\alpha+1,1). \end{cases}$$

Proof With the help of integration by parts, we obtain

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left[\mu\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\mu) - \Upsilon_{\lambda(\delta-\sigma)}(\alpha+1,\mu)\right]\mathfrak{F}''\big(tb+(1-\mu)\sigma\big)\,dt \\ &= \left[\mu\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\mu) - \Upsilon_{\lambda(\delta-\sigma)}(\alpha+1,\mu)\right]\frac{\mathfrak{F}'(tb+(1-\mu)\sigma)}{\delta-\sigma}\Big|_0^{\frac{1}{2}} \\ &\quad - \frac{1}{\delta-\sigma}\int_0^{\frac{1}{2}} \Upsilon_{\lambda(\delta-\sigma)}(\alpha,\mu)\mathfrak{F}'\big(tb+(1-\mu)\sigma\big)\,dt \\ &= \frac{1}{(\delta-\sigma)}\left[\frac{1}{2}\Upsilon_{\lambda(\delta-\sigma)}\left(\alpha,\frac{1}{2}\right) - \Upsilon_{\lambda(\delta-\sigma)}\left(\alpha+1,\frac{1}{2}\right)\right]\mathfrak{F}'\left(\frac{\sigma+\delta}{2}\right) \\ &\quad - \frac{1}{\delta-\sigma}\left[\frac{\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\mu)\mathfrak{F}(tb+(1-\mu)\sigma)}{\delta-\sigma}\Big|_0^{\frac{1}{2}}\right. \\ &\quad \left. - \frac{1}{\delta-\sigma}\int_0^{\frac{1}{2}} \mu^{\alpha-1}e^{-\lambda(\delta-\sigma)\mu}\mathfrak{F}\big(tb+(1-\mu)\sigma\big)\,dt\right] \\ &= \frac{1}{(\delta-\sigma)}\left[\frac{1}{2}\Upsilon_{\lambda(\delta-\sigma)}\left(\alpha,\frac{1}{2}\right) - \Upsilon_{\lambda(\delta-\sigma)}\left(\alpha+1,\frac{1}{2}\right)\right]\mathfrak{F}'\left(\frac{\sigma+\delta}{2}\right) \\ &\quad - \frac{\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\frac{1}{2})}{(\delta-\sigma)^2}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right) + \frac{1}{(\delta-\sigma)^2}\int_0^{\frac{1}{2}} \mu^{\alpha-1}e^{-\lambda(\delta-\sigma)\mu}\mathfrak{F}\big(tb+(1-\mu)\sigma\big)\,dt. \end{aligned} \tag{7}$$

Then, similarly we have

$$I_2 = -\frac{1}{(\delta-\sigma)}\left[\frac{1}{2}\Upsilon_{\lambda(\delta-\sigma)}\left(\alpha,\frac{1}{2}\right)-\Upsilon_{\lambda(\delta-\sigma)}\left(\alpha+1,\frac{1}{2}\right)\right]\mathfrak{F}'\left(\frac{\sigma+\delta}{2}\right) \tag{8}$$
$$-\frac{\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\frac{1}{2})}{(\delta-\sigma)^2}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{1}{(\delta-\sigma)^2}\int_0^{\frac{1}{2}}\mu^{\alpha-1}e^{-\lambda(\delta-\sigma)\mu}\mathfrak{F}\big(ta+(1-\mu)\delta\big)\,dt,$$

$$I_3 = -\frac{1}{(\delta-\sigma)}\left[\frac{1}{2}\Upsilon_{\lambda(\delta-\sigma)}\left(\alpha,\frac{1}{2}\right)-\Upsilon_{\lambda(\delta-\sigma)}\left(\alpha+1,\frac{1}{2}\right)\right. \tag{9}$$
$$\left.-\frac{1}{2}\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)+\Upsilon_{\lambda(\delta-\sigma)}(\alpha+1,1)\right]\mathfrak{F}'\left(\frac{\sigma+\delta}{2}\right)$$
$$+\frac{\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\frac{1}{2})-\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}{(\delta-\sigma)^2}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$+\frac{1}{(\delta-\sigma)^2}\int_{\frac{1}{2}}^{1}\mu^{\alpha-1}e^{-\lambda(\delta-\sigma)\mu}\mathfrak{F}\big(tb+(1-\mu)\sigma\big)\,dt,$$

and

$$I_4 = \frac{1}{(\delta-\sigma)}\left[\frac{1}{2}\Upsilon_{\lambda(\delta-\sigma)}\left(\alpha,\frac{1}{2}\right)-\Upsilon_{\lambda(\delta-\sigma)}\left(\alpha+1,\frac{1}{2}\right)\right. \tag{10}$$
$$\left.-\frac{1}{2}\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)+\Upsilon_{\lambda(\delta-\sigma)}(\alpha+1,1)\right]\mathfrak{F}'\left(\frac{\sigma+\delta}{2}\right)$$
$$+\frac{\Upsilon_{\lambda(\delta-\sigma)}(\alpha,\frac{1}{2})-\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}{(\delta-\sigma)^2}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$+\frac{1}{(\delta-\sigma)^2}\int_{\frac{1}{2}}^{1}\mu^{\alpha-1}e^{-\lambda(\delta-\sigma)\mu}\mathfrak{F}\big(ta+(1-\mu)\delta\big)\,dt.$$

Adding (7)–(10), we get

$$\sum_{k=1}^{4}I_k = \frac{1}{(\delta-\sigma)^2}\left[\int_0^1\mu^{\alpha-1}e^{-\lambda(\delta-\sigma)\mu}\mathfrak{F}\big(tb+(1-\mu)\sigma\big)\,dt\right. \tag{11}$$
$$\left.+\int_0^1\mu^{\alpha-1}e^{-\lambda(\delta-\sigma)\mu}\mathfrak{F}\big(ta+(1-\mu)\delta\big)\,dt\right]$$
$$-\frac{2\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}{(\delta-\sigma)^2}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$=\frac{\Gamma(\alpha)}{(\delta-\sigma)^{\alpha+2}}\left[\frac{1}{\Gamma(\alpha)}\int_\sigma^\delta(x-\sigma)^{\alpha-1}e^{-\lambda(\mu-x)}\mathfrak{F}(x)\,dx\right.$$
$$\left.+\frac{1}{\Gamma(\alpha)}\int_\sigma^\delta(\delta-x)^{\alpha-1}e^{-\lambda(x-\mu)}\mathfrak{F}(x)\,dx\right]$$
$$-\frac{2\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}{(\delta-\sigma)^2}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$=\frac{\Gamma(\alpha)}{(\delta-\sigma)^{\alpha+2}}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\frac{2\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}{(\delta-\sigma)^2}\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right).$$

If we multiply both sides of (11) by $\frac{(\delta-\sigma)^2}{2\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}$, then we have (6) simultaneously. This finishes the proof of Lemma 1. □

Theorem 1 *Assume that the assumptions of Lemma 1 hold. If $|\mathfrak{F}''|$ is convex on $[\sigma,\delta]$, then we have the following midpoint-type inequality*:

$$\begin{aligned}&\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^{\alpha}\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\&\quad\leq\frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\Omega_1(\lambda,\alpha)+\Omega_2(\lambda,\alpha)\right]\left[\left|\mathfrak{F}''(\sigma)\right|+\left|\mathfrak{F}''(\delta)\right|\right].\end{aligned}$$

Here,

$$\begin{cases}\Omega_1(\lambda,\alpha)=\int_0^{\frac{1}{2}}|\varphi_\alpha(\lambda,\mu)|\,dt,\\ \Omega_2(\lambda,\alpha)=\int_{\frac{1}{2}}^1|\psi_\alpha(\lambda,\mu)|\,dt.\end{cases}\tag{12}$$

Proof If we take modulus in equation (6) and apply the triangle inequality, then we have

$$\begin{aligned}&\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^{\alpha}\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\&\quad\leq\frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\int_0^{\frac{1}{2}}|\varphi_\alpha(\lambda,\mu)|\left|\mathfrak{F}''\big(tb+(1-\mu)\sigma\big)\right|dt\right.\\&\quad+\int_0^{\frac{1}{2}}|\varphi_\alpha(\lambda,\mu)|\left|\mathfrak{F}''\big(ta+(1-\mu)\delta\big)\right|dt\\&\quad\left.+\int_{\frac{1}{2}}^1|\psi_\alpha(\lambda,\mu)|\left|\mathfrak{F}''\big(tb+(1-\mu)\sigma\big)\right|dt+\int_{\frac{1}{2}}^1|\psi_\alpha(\lambda,\mu)|\left|\mathfrak{F}''\big(ta+(1-\mu)\delta\big)\right|dt\right].\end{aligned}\tag{13}$$

By using the convexity of $|\mathfrak{F}''|$, we have

$$\begin{aligned}&\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^{\alpha}\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right|\\&\quad\leq\frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\int_0^{\frac{1}{2}}|\varphi_\alpha(\lambda,\mu)|\left[\mu\left|\mathfrak{F}''(\delta)\right|+(1-\mu)\left|\mathfrak{F}''(\sigma)\right|\right]dt\right.\\&\quad+\int_0^{\frac{1}{2}}|\varphi_\alpha(\lambda,\mu)|\left[\mu\left|\mathfrak{F}''(\sigma)\right|+(1-\mu)\left|\mathfrak{F}''(\delta)\right|\right]dt\\&\quad+\int_{\frac{1}{2}}^1|\psi_\alpha(\lambda,\mu)|\left[\mu\left|\mathfrak{F}''(\delta)\right|+(1-\mu)\left|\mathfrak{F}''(\sigma)\right|\right]dt\\&\quad\left.+\int_{\frac{1}{2}}^1|\psi_\alpha(\lambda,\mu)|\left[\mu\left|\mathfrak{F}''(\sigma)\right|+(1-\mu)\left|\mathfrak{F}''(\delta)\right|\right]dt\right]\\&\quad=\frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\int_0^{\frac{1}{2}}|\varphi_\alpha(\lambda,\mu)|\,dt+\int_{\frac{1}{2}}^1|\psi_\alpha(\lambda,\mu)|\,dt\right]\left[\left|\mathfrak{F}''(\sigma)\right|+\left|\mathfrak{F}''(\delta)\right|\right].\end{aligned}$$

This ends the proof of Theorem 1. □

Remark 3 If we choose $\lambda = 0$ in Theorem 1, then the following midpoint-type inequality holds:

$$\left|\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathfrak{F}(\delta)+J_{\delta-}^{\alpha}\mathfrak{F}(\sigma)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\ \leq \frac{(\delta-\sigma)^2}{2(\alpha+1)}\left(\frac{1}{\alpha+2}+\frac{\alpha-3}{8}\right)\left[\left|\mathfrak{F}''(\sigma)\right|+\left|\mathfrak{F}''(\delta)\right|\right],$$

which is presented in [14, Theorem 2.2].

Remark 4 If we let $\alpha = 1$ and $\lambda = 0$ in Theorem 1, then we obtain the midpoint-type inequality

$$\left|\frac{1}{(\delta-\sigma)}\int_{\sigma}^{\delta}\mathfrak{F}(\mu)\,dt-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \leq \frac{(\delta-\sigma)^2}{48}\left(\left|\mathfrak{F}''(\sigma)\right|+\left|\mathfrak{F}''(\delta)\right|\right),$$

which is given in [33, Theorem 5].

Theorem 2 *Let us consider that the assumptions of Lemma 1 hold. If, moreover,* $|\mathfrak{F}''|^q$, $q > 1$ *is convex on* $[\sigma,\delta]$, *then*

$$\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^{\alpha}\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\ \leq \frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\left(\int_0^{\frac{1}{2}}\left|\varphi_{\alpha}(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^1\left|\psi_{\alpha}(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\right] \\ \times\left[\left(\frac{3|\mathfrak{F}''(\delta)|^q+|\mathfrak{F}''(\sigma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\mathfrak{F}''(\sigma)|^q+|\mathfrak{F}''(\delta)|^q}{8}\right)^{\frac{1}{q}}\right] \\ \leq \frac{(\delta-\sigma)^2}{2^{\frac{3}{q}-1}\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\left(\int_0^{\frac{1}{2}}\left|\varphi_{\alpha}(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^1\left|\psi_{\alpha}(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\right] \\ \times\left[\left|\mathfrak{F}''(\sigma)\right|+\left|\mathfrak{F}''(\delta)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof Let us first apply Hölder's inequality in (13). Then, we get

$$\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^{\alpha}\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\ \leq \frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\left(\int_0^{\frac{1}{2}}\left|\varphi_{\alpha}(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\mathfrak{F}''\big(tb+(1-\mu)\sigma\big)\right|^q dt\right)^{\frac{1}{q}}\right. \\ +\left(\int_0^{\frac{1}{2}}\left|\varphi_{\alpha}(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\mathfrak{F}''\big(ta+(1-\mu)\delta\big)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left|\psi_{\alpha}(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\mathfrak{F}''\big(tb+(1-\mu)\sigma\big)\right|^q dt\right)^{\frac{1}{q}} \\ \left.+\left(\int_{\frac{1}{2}}^1\left|\psi_{\alpha}(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\mathfrak{F}''\big(ta+(1-\mu)\delta\big)\right|^q dt\right)^{\frac{1}{q}}\right].$$

By using the convexity of $|\mathfrak{F}''|^q$, we get

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^{\alpha}\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\
&\leq \frac{(\delta-\sigma)^2}{2\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\left(\int_0^{\frac{1}{2}}\left|\varphi_\alpha(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\mu\left|\mathfrak{F}''(\delta)\right|^q+(1-\mu)\left|\mathfrak{F}''(\sigma)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\int_0^{\frac{1}{2}}\left|\varphi_\alpha(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\mu\left|\mathfrak{F}''(\sigma)\right|^q+(1-\mu)\left|\mathfrak{F}''(\delta)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{1}\left|\psi_\alpha(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[\mu\left|\mathfrak{F}''(\delta)\right|^q+(1-\mu)\left|\mathfrak{F}''(\sigma)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}\left|\psi_\alpha(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[\mu\left|\mathfrak{F}''(\sigma)\right|^q+(1-\mu)\left|\mathfrak{F}''(\delta)\right|^q\right]dt\right)^{\frac{1}{q}}\right] \\
&= \frac{(\delta-\sigma)^2}{2\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\left(\int_0^{\frac{1}{2}}\left|\varphi_\alpha(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^{1}\left|\psi_\alpha(\lambda,\mu)\right|^p dt\right)^{\frac{1}{p}}\right] \\
&\quad\times\left[\left(\frac{3|\mathfrak{F}''(\delta)|^q+|\mathfrak{F}''(\sigma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\mathfrak{F}''(\sigma)|^q+|\mathfrak{F}''(\delta)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Let $\sigma_1=|\mathfrak{F}''(\sigma)|^q$, $\delta_1=3|\mathfrak{F}''(\delta)|^q$, $\sigma_2=3|\mathfrak{F}''(\sigma)|^q$, and $\delta_2=|\mathfrak{F}''(\delta)|^q$ for the proof of the second inequality. Using the facts that

$$\sum_{k=1}^{n}(\sigma_k+\delta_k)^s\leq\sum_{k=1}^{n}\sigma_k^s+\sum_{k=1}^{n}\delta_k^s,\quad 0\leq s<1,$$

and $1+3^{\frac{1}{q}}\leq 4$, the desired result of Theorem 2 can be obtained straightforwardly. □

Remark 5 Let us consider $\lambda=0$ in Theorem 2. Then, the following midpoint-type inequality holds:

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathfrak{F}(\delta)+J_{\delta-}^{\alpha}\mathfrak{F}(\sigma)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\
&\leq\frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[\left(\frac{1}{2^{(p(1+\alpha)+1)}(p(1+\alpha)+1)}\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^{1}\left|\mu^{\alpha+1}-(1+\alpha)\mu+\alpha\right|^p dt\right)^{\frac{1}{p}}\right] \\
&\quad\times\left[\left(\frac{3|\mathfrak{F}''(\delta)|^q+|\mathfrak{F}''(\sigma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\mathfrak{F}''(\sigma)|^q+|\mathfrak{F}''(\delta)|^q}{8}\right)^{\frac{1}{q}}\right] \\
&\leq\frac{(\delta-\sigma)^2}{2^{\frac{3}{q}-1}(\alpha+1)}\left[\left(\frac{1}{2^{(p(1+\alpha)+1)}(p(1+\alpha)+1)}\right)^{\frac{1}{p}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}\left|\mu^{\alpha+1}-(1+\alpha)\mu+\alpha\right|^p dt\right)^{\frac{1}{p}}\right]\left(\left|\mathfrak{F}''(\delta)\right|+\left|\mathfrak{F}''(\sigma)\right|\right),
\end{aligned}$$

which is presented in [14, Theorem 2.5].

Remark 6 If we assign $\alpha = 1$ and $\lambda = 0$ in Theorem 2, then we have

$$\begin{aligned}
&\left|\frac{1}{(\delta-\sigma)}\int_\sigma^\delta \mathfrak{F}(\mu)\,dt - \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\
&\quad\le \frac{(\delta-\sigma)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{F}''(\delta)|^q + |\mathfrak{F}''(\sigma)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|\mathfrak{F}''(\sigma)|^q + |\mathfrak{F}''(\delta)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\quad\le \frac{(\delta-\sigma)^2}{16}\left(\frac{4}{2p+1}\right)^{\frac{1}{p}}\left(\left|\mathfrak{F}''(\delta)\right| + \left|\mathfrak{F}''(\sigma)\right|\right),
\end{aligned}$$

which are given in [7, Corollary 4.8].

Theorem 3 *Suppose that the assumptions of Lemma 1 hold. If* $|\mathfrak{F}''|^q$, $q \ge 1$ *is convex on* $[\sigma,\delta]$, *then*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^\alpha\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma) + \mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right] - \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\
&\quad\le \frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)} \\
&\qquad\times\left[\left(\Omega_1(\lambda,\alpha)\right)^{1-\frac{1}{q}}\left[\left(\left|\mathfrak{F}''(\delta)\right|^q\Omega_3(\lambda,\alpha) + \left|\mathfrak{F}''(\sigma)\right|^q\left(\Omega_1(\lambda,\alpha)-\Omega_3(\lambda,\alpha)\right)\right)^{\frac{1}{q}}\right.\right. \\
&\qquad\left.+\left(\left|\mathfrak{F}''(\sigma)\right|^q\Omega_3(\lambda,\alpha) + \left|\mathfrak{F}''(\delta)\right|^q\left(\Omega_1(\lambda,\alpha)-\Omega_3(\lambda,\alpha)\right)\right)^{\frac{1}{q}}\right] + \left(\Omega_2(\lambda,\alpha)\right)^{1-\frac{1}{q}} \\
&\qquad\times\left[\left(\left|\mathfrak{F}''(\delta)\right|^q\Omega_4(\lambda,\alpha) + \left|\mathfrak{F}''(\sigma)\right|^q\left(\Omega_2(\lambda,\alpha)-\Omega_4(\lambda,\alpha)\right)\right)^{\frac{1}{q}}\right. \\
&\qquad\left.\left.+\left(\left|\mathfrak{F}''(\sigma)\right|^q\Omega_4(\lambda,\alpha) + \left|\mathfrak{F}''(\delta)\right|^q\left(\Omega_2(\lambda,\alpha)-\Omega_4(\lambda,\alpha)\right)\right)^{\frac{1}{q}}\right]\right].
\end{aligned}$$

Here, $\Omega_1(\lambda,\alpha)$ *and* $\Omega_2(\lambda,\alpha)$ *are defined in* (12) *and*

$$\begin{cases}
\Omega_3(\lambda,\alpha) = \int_0^{\frac{1}{2}} \mu|\varphi_\alpha(\lambda,\mu)|\,dt, \\
\Omega_4(\lambda,\alpha) = \int_{\frac{1}{2}}^1 \mu|\psi_\alpha(\lambda,\mu)|\,dt.
\end{cases}$$

Proof By applying the power-mean inequality in (13), we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^\alpha\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma) + \mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right] - \mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\
&\quad\le \frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\left(\int_0^{\frac{1}{2}}\left|\varphi_\alpha(\lambda,\mu)\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\varphi_\alpha(\lambda,\mu)\right|\left|\mathfrak{F}''\left(tb+(1-\mu)\sigma\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad+\left(\int_0^{\frac{1}{2}}\left|\varphi_\alpha(\lambda,\mu)\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\varphi_\alpha(\lambda,\mu)\right|\left|\mathfrak{F}''\left(ta+(1-\mu)\delta\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_{\frac{1}{2}}^1\left|\psi_\alpha(\lambda,\mu)\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\psi_\alpha(\lambda,\mu)\right|\left|\mathfrak{F}''\left(tb+(1-\mu)\sigma\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad\left.+\left(\int_{\frac{1}{2}}^1\left|\psi_\alpha(\lambda,\mu)\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\psi_\alpha(\lambda,\mu)\right|\left|\mathfrak{F}''\left(ta+(1-\mu)\delta\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

It is known that $|\mathfrak{F}''|^q$ is convex. Then, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha)}{2(\delta-\sigma)^{\alpha}\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathfrak{F}(\sigma)+\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathfrak{F}(\delta)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\
&\quad\leq \frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)}\left[\left(\int_0^{\frac{1}{2}}\left|\varphi_{\alpha}(\lambda,\mu)\right|dt\right)^{1-\frac{1}{q}}\right. \\
&\qquad\times\left(\int_0^{\frac{1}{2}}\left|\varphi_{\alpha}(\lambda,\mu)\right|\left[\mu\left|\mathfrak{F}''(\delta)\right|^q+(1-\mu)\left|\mathfrak{F}''(\sigma)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_0^{\frac{1}{2}}\left|\varphi_{\alpha}(\lambda,\mu)\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\varphi_{\alpha}(\lambda,\mu)\right|\left[\mu\left|\mathfrak{F}''(\sigma)\right|^q+(1-\mu)\left|\mathfrak{F}''(\delta)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_{\frac{1}{2}}^{1}\left|\psi_{\alpha}(\lambda,\mu)\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\left|\psi_{\alpha}(\lambda,\mu)\right|\left[\mu\left|\mathfrak{F}''(\delta)\right|^q+(1-\mu)\left|\mathfrak{F}''(\sigma)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&\qquad\left.+\left(\int_{\frac{1}{2}}^{1}\left|\psi_{\alpha}(\lambda,\mu)\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\left|\psi_{\alpha}(\lambda,\mu)\right|\left[\mu\left|\mathfrak{F}''(\sigma)\right|^q+(1-\mu)\left|\mathfrak{F}''(\delta)\right|^q\right]dt\right)^{\frac{1}{q}}\right] \\
&\quad=\frac{(\delta-\sigma)^2}{2\,\Upsilon_{\lambda(\delta-\sigma)}(\alpha,1)} \\
&\qquad\times\left[\left(\Omega_1(\lambda,\alpha)\right)^{1-\frac{1}{q}}\left[\left(\left|\mathfrak{F}''(\delta)\right|^q\Omega_3(\lambda,\alpha)+\left|\mathfrak{F}''(\sigma)\right|^q\left(\Omega_1(\lambda,\alpha)-\Omega_3(\lambda,\alpha)\right)\right)^{\frac{1}{q}}\right.\right. \\
&\qquad\left.+\left(\left|\mathfrak{F}''(\sigma)\right|^q\Omega_3(\lambda,\alpha)+\left|\mathfrak{F}''(\delta)\right|^q\left(\Omega_1(\lambda,\alpha)-\Omega_3(\lambda,\alpha)\right)\right)^{\frac{1}{q}}\right]+\left(\Omega_2(\lambda,\alpha)\right)^{1-\frac{1}{q}} \\
&\qquad\times\left[\left(\left|\mathfrak{F}''(\delta)\right|^q\Omega_4(\lambda,\alpha)+\left|\mathfrak{F}''(\sigma)\right|^q\left(\Omega_2(\lambda,\alpha)-\Omega_4(\lambda,\alpha)\right)\right)^{\frac{1}{q}}\right. \\
&\qquad\left.\left.+\left(\left|\mathfrak{F}''(\sigma)\right|^q\Omega_4(\lambda,\alpha)+\left|\mathfrak{F}''(\delta)\right|^q\left(\Omega_2(\lambda,\alpha)-\Omega_4(\lambda,\alpha)\right)\right)^{\frac{1}{q}}\right]\right]. \qquad \square
\end{aligned}$$

Remark 7 Consider $\lambda = 0$ in Theorem 3. Then, the following midpoint-type inequality holds:

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathfrak{F}(\delta)+J_{\delta-}^{\alpha}\mathfrak{F}(\sigma)\right]-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \\
&\quad\leq\frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[\left(\frac{1}{2^{\alpha+2}(\alpha+2)}\right)\left[\left(\frac{(\alpha+2)|\mathfrak{F}''(\delta)|^q+(\alpha+4)|\mathfrak{F}''(\sigma)|^q}{2(\alpha+3)}\right)^{\frac{1}{q}}\right.\right. \\
&\qquad\left.+\left(\frac{(\alpha+2)|\mathfrak{F}''(\sigma)|^q+(\alpha+4)|\mathfrak{F}''(\delta)|^q}{2(\alpha+3)}\right)^{\frac{1}{q}}\right]+\left(\phi_1(\alpha)\right)^{1-\frac{1}{q}} \\
&\qquad\times\left[\left[\left(\phi_2(\alpha)\right)\left|\mathfrak{F}''(\delta)\right|^q+\left(\phi_1(\alpha)-\phi_2(\alpha)\right)\left|\mathfrak{F}''(\sigma)\right|^q\right]^{\frac{1}{q}}\right. \\
&\qquad\left.\left.+\left[\left(\phi_2(\alpha)\right)\left|\mathfrak{F}''(\sigma)\right|^q+\left(\phi_1(\alpha)-\phi_2(\alpha)\right)\left|\mathfrak{F}''(\delta)\right|^q\right]^{\frac{1}{q}}\right]\right],
\end{aligned}$$

where

$$\begin{cases}
\phi_1(\alpha)=\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}+\frac{\alpha-3}{8}, \\
\phi_2(\alpha)=\frac{2^{\alpha+3}-1}{2^{\alpha+3}(\alpha+3)}+\frac{2\alpha-7}{24}.
\end{cases}$$

This result coincides with [14, Theorem 2.7].

Remark 8 Let us consider $\alpha = 1$ and $\lambda = 0$ in Theorem 3. Then, we have the midpoint-type inequality

$$\left|\frac{1}{(\delta-\sigma)}\int_{\sigma}^{\delta}\mathfrak{F}(\mu)\,dt-\mathfrak{F}\left(\frac{\sigma+\delta}{2}\right)\right| \leq \frac{(\delta-\sigma)^2}{48}\left[\left(\frac{3|\mathfrak{F}''(\delta)|^q+5|\mathfrak{F}''(\sigma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\mathfrak{F}''(\sigma)|^q+5|\mathfrak{F}''(\delta)|^q}{8}\right)^{\frac{1}{q}}\right],$$

which is given in [31, Proposition 5].

4 Conclusion

In this paper, we proved an equality involving tempered fractional integrals for the case of twice-differentiable functions. By using this equality, we established midpoint-type inequalities for tempered fractional integrals. Moreover, our results generalize known results from the literature.

In a future research, exploring the ideas and results related to midpoint-type inequalities using tempered fractional integrals could pave the way for new directions in this field of mathematics. Moreover, one can try to generalize our results by utilizing a different version of convex function classes or another type fractional integral operators. Finally, this suggests that using tempered fractional integrals with quantum calculus may lead to similar inequalities for convex functions.

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Declarations

Competing interests
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Author contributions
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