Stochastic stability of solutions for a fourth-order stochastic differential equation with constant delay

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Abstract

In this paper, we present sufficient conditions to ensure the stochastic asymptotic stability of the zero solution for a specific type of fourth-order stochastic differential equation (SDE) with constant delay. By reducing the fourth-order SDE to a system of first-order SDEs, we utilize a fourth-order quadratic function to derive an appropriate Lyapunov functional. This functional is then employed to establish standard criteria for the nonlinear functions present in the SDE. The stability result obtained in this study is novel and extends the existing findings on stability in fourth-order differential equations. Additionally, we provide an illustrative example to demonstrate the significance and accuracy of our main result.

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1 Introduction

The growing utilization of differential equations in a wide range of fields, including the natural, engineering, and social sciences, has sparked significant research efforts in various branches of this subject. In this context, numerous techniques have been developed by authors to analyze the characteristics and behavior of solutions of differential equations. These techniques include Pontryagin's theory for quasi-polynomials, the Riccati technique, the averaging functions method, the contraction mapping principle, the coincidence and Leary–Schauder degree theory, and the Lyapunov direct method. Additionally, researchers have conducted background studies on systems of ordinary and functional differential equations see Ademola and Ogundiran [3], Burton [6, 7], Driver [9], Hale [11], Lakshmikantham et al. [16], Yoshizawa [26, 27]. Based on our examination of the pertinent literature, the researchers who have investigated stability, boundedness, square integrability, and the existence of unique periodic solutions for fourth-order ordinary and delay differential equations encompass, but are not restricted to: Ademola [1, 2], Adesina and Ogundare [4], Balamuralitharan [5], Cai and Meng [8], Kang and Si [12], Korkmaz and Tunç [15], Korkmaz [13, 14], Okoronkwo [17], Rahmane and Remili [19], Sadek [20],

Sinha [21], Tejumola and Tchegnani [22], Tunç [23–25]. These researchers utilized the direct method of Lyapunov, with the exception of [5] and [17], where the continuation theorem of coincidence degree theory with Banach space lemma techniques and Razumikhin type theorem were employed, respectively.

In 2021, the author of [2] provided conditions for the stability, boundedness, and existence of a unique periodic solution to a fourth-order differential equation.

$$x^{(4)} + \phi(\cdot)x''' + f\big(x'(t-r(t)), x''(t-r(t))\big) + g\big(x(t-r(t)), x'(t-r(t))\big) + h\big(x(t-r(t))\big) = p(\cdot),$$

where $\phi(\cdot) = \phi(t, x, x', x'', x''')$, $p(\cdot) = p(t, x, x', x'', x''', x(t-r(t)), x'(t-r(t)), x''(t-r(t)))$, the functions ϕ, f, g, h and p are continuous in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^4, \mathbb{R}^3, \mathbb{R}^2, \mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}^7$ respectively with $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$ and $0 \le r(t) \le \gamma$, $\gamma > 0$ being a constant, the derivatives $h'(x), g_x(x, y), g_y(x, y), f_x(x, y, z), f_y(x, y, z)$ and $f_z(x, y, z)$ exist and are continuous for all x, y, z. A significant tool employed in [2] is the renowned Lyapunov's functional, specifically designed for this purpose. Recently, Fatmi et al. [10] investigated and established standard criteria for the stability, boundedness, and square integrability of solutions for neutral type differential equations with variable delay. The specific form of the studied equation is given by:

$$\big(x'''(t) + \rho x'''(t-r(t))\big)' + a(t)\big(\varphi(x(t))x''(t)\big)' + b(t)\big(g(x(t))x'(t)\big)' + c(t)f(x(t))x'(t)$$
$$+ d(t)h\big(x(t-r(t))\big) = \psi\big(t, x(t), x'(t), x''(t), x'''(t)\big),$$

where r represents a bounded delay, with $0 \le r(t) \le r_1$, $-r_3 \le r'(t) \le r_2$, $0 < r_2 < 1$, and $r_3 > 0$. The functions a, b, c, d are continuously differentiable, while the functions f, g, h, and ϕ are also continuously differentiable.

Without a doubt, the two equations are significantly more comprehensive compared to prior research on fourth-order differential equations with variable delay. Furthermore, equation (1.1) stands out due to its inclusion of an extra term (stochastic term), making it more valuable than any previously documented fourth-order differential equations. Therefore, this research work possesses a novelty factor. The motivations behind this research can be found in [1, 2, 4, 5, 8, 10, 12–15], where ordinary, delay, and neutral differential differential equations were discussed.

The objective of this study is to analyze the stochastic asymptotic stability of the zero solution for a fourth-order SDE with a constant delay. The equation can be expressed as follows:

$$x^{(4)}(t) + a\dddot{x}(t) + h\big(\ddot{x}(t)\big) + \phi\big(\dot{x}(t-r)\big) + f\big(x(t-r)\big) + \sigma x(t)\dot{\omega}(t) = 0. \tag{1.1}$$

We transform (1.1) to a system of first-order differential equation as follows

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = u,$$

$$\dot{u} = -au - h(z) - \phi(y) - f(x) - \sigma x(t)\dot{\omega}(t) + \int_{t-r}^{t} \phi'\big(y(\rho)\big)z(\rho)\,d\rho \tag{1.2}$$

$$+ \int_{t-r}^{t} f'\big(x(\rho)\big)y(\rho)\,d\rho,$$

where a and σ are two positive constants, r is a positive constant delay whose value will be determined later, the functions h, ϕ and f are continuous in their respective arguments of $\mathbb{R}$ with $\phi(0) = f(0) = 0$, $\dot{\omega}(t)$ is m-standard Brownian motion defined on the probability space. In addition, it is also supposed the derivatives $\phi'(y)$ and $f'(x)$ are continuous for all x and y.

2 Main result

We will express and demonstrate the stability result of (1.1) in this section. The results of the stability test are shown below.

Theorem 2.1 *In addition to the fundamental presumptions that apply to the functions h, ϕ and f, assume that there are positive constants b, c, d, ℓ, m, c_1, M, and δ such that satisfy the following requirements*:

(i) $\frac{h(z)}{z} \geq b$, *for all* $z \neq 0$;

(ii) $\phi(0) = 0$, $c \leq \frac{\phi(y)}{y} \leq c_1$, *for all* $y \neq 0$ *and* $|\phi'(y)| \leq M$, *for all* y;

(iii) $f(0) = 0$, $\ell \leq \frac{f(x)}{x} \leq m$, *for all* $x \neq 0$ *and* $f'(x) \leq |f'(x)| \leq d$, *for all* x;

(iv) $c < ab$, $\delta \leq abc - c^2 - a^2 d$;

(v) $a\ell > 2(1 + d_1)\sigma^2$;

(vi) $\frac{\delta}{2ac\delta_0} \leq \varepsilon \leq \min\{\frac{2ab}{c}, \frac{2c}{a}\}$.

Then the zero solution of (1.1) *is stochastically asymptotically stable, provided that*

$$r < \min\left\{ \frac{a\ell - 2(1 + d_1)\sigma^2}{4a(M + b)}, \frac{(a + c)b}{4M_1}, \frac{\delta}{8acM_2}, \frac{a + c}{4(M + b)(1 + d_1)} \right\},$$

where

$$
\begin{aligned}
M_1 &:= (M + b)(b + d_2) + b(a + b + c + d_1 + d_2 + 2), \\
M_2 &:= (M + b)(1 + c) + M(a + b + c + d_1 + d_2 + 2), \\
d_1 &:= \varepsilon + \frac{1}{a}, \qquad d_2 := \varepsilon + \frac{d}{c}, \delta_0 := ab + bcd^{-1} > 0.
\end{aligned}
\tag{2.1}
$$

Remark 2.1 Upon comparing with the existing literature, we have observed various similarities and differences:

(i) If $h(\ddot{x}(t)) = b\ddot{x}(t)$, $\phi(\dot{x}(t - r)) = c\dot{x}$, $f(x(t - r)) = dx$ and $\sigma x(t)\dot{\omega}(t) = 0$, equation (1.1) reduces to a fourth-order linear ordinary differential equation

$$x^{(4)}(t) + a\dddot{x}(t) + b\ddot{x}(t) + c\dot{x}(t) + dx(t) = 0, \tag{2.2}$$

and hypotheses (i)-(vi) of Theorem 2.1 Routh–Hurtwitz conditions $a > 0$, $b > 0$, $c > 0$, $ab > c$, $abc - c^2 > a^2 d$ and $d > 0$ for asymptotic stability of the trivial solution of (2.2);

(ii) Hypotheses (i), (ii), (iv), (vi) of Theorem 2.1 coincide with hypotheses (ii), (iv), (v), and (3.6) Theorem 3.1 in [2], respectively;

(iii) If the double integrals

$$2\int_{-r}^{t}\int_{t+s}^{t} \mu_3 u^2(\theta))\, d\theta\, ds = 0,$$

in equation (3.3) of [2], then the functional defined by (2.3) coincide with the one in [2];

(iv) Substantial parts of our hypotheses in Theorem 2.1 coincide with the hypotheses in Theorem 5 of [1], as seen in items (ii) and (iii) above;

(v) In the recent publication by [10], only a few of our hypotheses align, primarily because of the intricate nature of the main tools employed in both studies; and

(vi) The inclusion of the stochastic term in equation (1.1) enhances numerous exceptional research works already present in the literature.

Proof of Theorem 2.1 Let $X_t = (x_t, y_t, z_t, u_t)$ be any solution of the system (1.2). We define a Lyapunov functional $W(X_t) = W(x_t, y_t, z_t, u_t)$ as follows to demonstrate the Theorem 2.1

$$
\begin{aligned}
2W(X_t) = {} & 2d_2 \int_0^x f(\zeta)\, d\zeta + 2 \int_0^y \phi(\eta)\, d\eta + (1 + d_1)u^2 + \left(a - d_2 + bd_1 + c^2\right)z^2 \\
& + \left(bd_2 - d_1 d + b^2\right)y^2 + a^2 x^2 + 2f(x)y + 2d_1 f(x)z + 2(d_2 + b)yu \\
& + 2(ad_2 + cd_1 + bc)yz + 2(1 + c)zu + 2abxy + 2acxz + 2axu \\
& + 2 \int_{-r}^t \int_{t+s}^t \left(\lambda y^2(\vartheta) + \mu z^2(\vartheta)\right) d\vartheta\, ds,
\end{aligned}
\tag{2.3}
$$

where λ and μ are two positive constants, which will be established later in the proof.

Since the above Lyapunov functional $2W(X_t)$ clearly meets the condition $W(0,0,0,0) = 0$, then we may rewrite (2.3) as follows

$$
\begin{aligned}
2W(X_t) = {} & 2d_2 \int_0^x f(\zeta)\, d\zeta - \frac{1}{c}f^2(x) + 2 \int_0^y \phi(\eta)\, d\eta - cy^2 + \left(d_1 - \frac{1}{a}\right)u^2 \\
& + \left(bd_1 - d_2 - cd_1^2\right)z^2 + \left(bd_2 - d_1 d - ad_2^2\right)y^2 + \frac{1}{a}(ad_2 y + az + u)^2 \\
& + \frac{1}{c}\left(f(x) + cy + cd_1 z\right)^2 + (ax + by + cz + u)^2 \\
& + 2 \int_{-r}^t \int_{t+s}^t \left(\lambda y^2(\vartheta) + \mu z^2(\vartheta)\right) d\vartheta\, ds.
\end{aligned}
$$

The equation above can be rearranged to take the following form

$$
\begin{aligned}
2W(X_t) = {} & \sum_{i=1}^5 W_i + \frac{1}{a}(ad_2 y + az + u)^2 + \frac{1}{c}\left(f(x) + cy + cd_1 z\right)^2 \\
& + (ax + by + cz + u)^2 + 2 \int_{-r}^t \int_{t+s}^t \left(\lambda y^2(\vartheta) + \mu z^2(\vartheta)\right) d\vartheta\, ds,
\end{aligned}
\tag{2.4}
$$

where

$$
W_1 := 2d_2 \int_0^x f(\zeta)\, d\zeta - \frac{1}{c}f^2(x), \qquad W_2 := 2 \int_0^y \phi(\eta)\, d\eta - cy^2,
$$

$$
W_3 := \left(bd_2 - d_1 d - ad_2^2\right)y^2, \qquad W_4 := \left(bd_1 - d_2 - cd_1^2\right)z^2, \qquad W_5 := \left(d_1 - \frac{1}{a}\right)u^2.
$$

Using the criterion (*iii*), (2.1), and the fact that $f'(x) \leq d$, we have

$$W_1 = 2\left(\varepsilon + \frac{d}{c}\right)\int_0^x f(\zeta)\,d\zeta - \frac{1}{c}f^2(x) = 2\int_0^x\left[\left(\varepsilon + \frac{d}{c}\right) - \frac{1}{c}f'(\zeta)\right]f(\zeta)\,d\zeta$$

$$\geq 2\varepsilon\int_0^x f(\zeta)\,d\zeta \geq \varepsilon\ell x^2.$$

from condition (*ii*) and $c \leq \frac{\phi(y)}{y}$, we have

$$W_2 = 2\int_0^y\left(\frac{\phi(\eta)}{\eta} - c\right)\eta\,d\eta \geq 0, \quad \text{for all } y.$$

From (2.1) and assumption (*iv*) of Theorem 2.1, we get

$$b - ad_2 - cd_1 = b - a\left(\varepsilon + \frac{d}{c}\right) - c\left(\varepsilon + \frac{1}{a}\right) = \frac{1}{ac}(abc - a^2d - c^2) - \varepsilon(a + c)$$

$$\geq \frac{\delta}{ac} - \varepsilon(a + c).$$

Keep in mind that, $abc > abc - c^2 > a^2d$ indicates that $a < bcd^{-1}$ and $c < ab$, so

$$b - ad_2 - cd_1 \geq \frac{\delta}{ac} - \varepsilon\left(bcd^{-1} + ab\right) = \frac{\delta}{ac} - \varepsilon\delta_0, \tag{2.5}$$

where $\delta_0 := ab + bcd^{-1} > 0$. We can determine the coefficient of y^2 in W_3 as a result of the estimates (2.5) and (2.1) mentioned before

$$bd_2 - d_1d - ad_2^2 = d_2(b - ad_2 - cd_1) + d_1(cd_2 - d) \geq d_2\left(\frac{\delta}{ac} - \varepsilon\delta_0\right) + c\varepsilon d_1.$$

Since $c\varepsilon d_1 > 0$, it tends that

$$bd_2 - d_1d - ad_2^2 \geq d_2\left(\frac{\delta}{ac} - \varepsilon\delta_0\right).$$

Create the constant $\varepsilon = \frac{\delta}{2ac\delta_0}$ now, so that

$$bd_2 - d_1d - ad_2^2 \geq \left(\frac{\delta}{2ac\delta_0} + \frac{d}{c}\right)\left(\frac{\delta}{2ac}\right) = \frac{\delta}{4a^2c^2\delta_0}(\delta + 2ad\delta_0).$$

Next, we obtain

$$W_3 \geq \frac{\delta}{4a^2c^2\delta_0}(\delta + 2ad\delta_0)y^2.$$

When (2.5) and (2.1) are combined, then the coefficient of z^2 in W_4 is obtained

$$bd_1 - d_2 - cd_1^2 = d_1(b - ad_2 - cd_1) + (ad_1 - 1)d_2$$

$$\geq d_1\left(\frac{\delta}{ac} - \varepsilon\delta_0\right), \quad \text{such that } a\varepsilon d_2 > 0$$

$$= \left(\varepsilon + \frac{1}{a}\right)\left(\frac{\delta}{ac} - \varepsilon\delta_0\right)$$

$$= \frac{\delta}{4a^2c^2\delta_0}(\delta + 2c\delta_0), \quad \text{since } \varepsilon = \frac{\delta}{2ac\delta_0}.$$

Therefore, we get

$$W_4 \geq \frac{\delta}{4a^2c^2\delta_0}(\delta + 2c\delta_0)z^2.$$

By using (2.1) and the fact that $\varepsilon = \frac{\delta}{2ac\delta_0}$, we additionally discover

$$W_5 = \left(d_1 - \frac{1}{a}\right)u^2 = \varepsilon u^2 = \frac{\delta}{2ac\delta_0}u^2.$$

Moreover, since $a > 0$ and $c > 0$, it follows that

$$\frac{1}{a}(ad_2y + az + u)^2 \geq 0, \qquad \frac{1}{c}\big(f(x) + cy + cd_1z\big)^2 \geq 0, \qquad (ax + by + cz + u)^2 \geq 0,$$

for all x, y, z, and u.

Finally, the double integral

$$2\int_{-r}^{t}\int_{t+s}^{t}\big(\lambda y^2(\vartheta) + \mu z^2(\vartheta)\big)\,d\vartheta\,ds,$$

is non-negative.

Given the discussion above, a positive constant β_1 exists that ensures

$$W(X_t) \geq \beta_1\big(x^2 + y^2 + z^2 + u^2\big), \quad \text{for all } t \geq 0, x, y, z, u, \tag{2.6}$$

where

$$\beta_1 := \frac{1}{2}\min\left\{\frac{\delta\ell}{2ac\delta_0}, \frac{\delta(\delta + 2ad\delta_0)}{4a^2c^2\delta_0}, \frac{\delta(\delta + 2c\delta_0)}{4a^2c^2\delta_0}, \frac{\delta}{2ac\delta_0}\right\}.$$

The Lyapunov functional $W(X_t)$ defined by inequality (2.3) is positively semi-definite, as demonstrated by inequality (2.6).

Due to the fact that for all $x > 0$ and $y > 0$, $\frac{f(x)}{x} \leq m$ and $\frac{\phi(y)}{y} \leq c_1$, it follows that

$$2W(X_t) \leq 2d_2\int_0^x m\zeta\,d\zeta + 2\int_0^y c_1\eta\,d\eta + (1 + d_1)u^2 + \big(a - d_2 + bd_1 + c^2\big)z^2$$

$$+ \big(bd_2 - d_1d + b^2\big)y^2 + a^2x^2 + 2mxy + 2d_1mxz + 2(d_2 + b)yu$$

$$+ 2(ad_2 + cd_1 + bc)yz + 2(1 + c)zu + 2abxy + 2acxz + 2axu$$

$$+ 2\lambda\int_{t-r}^{t}(\vartheta - t + r)y^2(\vartheta)\,d\vartheta + 2\mu\int_{t-r}^{t}(\vartheta - t + r)z^2(\vartheta)\,d\vartheta.$$

Using the estimate $2|u_1 v_1| \le u_1^2 + v_1^2$, we may deduce that

$$
\begin{aligned}
2W(X_t) \le & \big\{ a(a + b + c + 1) + m(d_1 + d_2 + 1) \big\} \|x\|^2 \\
& + \big\{ d_1(c + d) + d_2(a + b + 1) + b(a + b + c + 1) + m + c_1 + \lambda r^2 \big\} \|y\|^2 \\
& + \big\{ d_1(b + c + m) + d_2(a + 1) + c(a + b + c + 1) + a + 1 + \mu r^2 \big\} \|z\|^2 \\
& + (a + b + c + d_1 + d_2 + 2) \|u\|^2.
\end{aligned}
$$

It follows that

$$
W(X_t) \le \beta_2 \big(x^2 + y^2 + z^2 + u^2 \big), \quad \beta_2 > 0, \tag{2.7}
$$

where

$$
\begin{aligned}
\beta_2 := \frac{1}{2} \max \big\{ & a(a + b + c + 1) + m(d_1 + d_2 + 1), a + b + c + d_1 + d_2 + 2, \\
& d_1(c + d) + d_2(a + b + 1) + b(a + b + c + 1) + m + c_1 + \lambda r^2, \\
& d_1(b + c + m) + d_2(a + 1) + c(a + b + c + 1) + a + 1 + \mu r^2 \big\}.
\end{aligned}
$$

Inequality (2.7) reveals that the functional $W(X_t)$ of (1.1) is decrescent.

The first partial derivative of the Lyapunov functional $W(X_t)$ with respect to the independent variable t, given that (X_t) is any solution of system (1.2), is

$$
\begin{aligned}
2\frac{dW(X_t)}{dt} = & 2d_2 f(x)y + 2\phi(y)z + 2\big(a - d_2 + bd_1 + c^2\big)zu + 2\big(bd_2 - d_1 d + b^2\big)yz \\
& + 2a^2 xy + 2f'(x)y^2 + 2f(x)z + 2d_1 f'(x)yz + 2d_1 f(x)u \\
& + 2(d_2 + b)zu + 2(ad_2 + cd_1 + bc)\big(yu + z^2\big) + 2(1 + c)u^2 \\
& + 2ab\big(xz + y^2\big) + 2ac(xu + yz) + 2ayu + 2\lambda ry^2 + 2\mu rz^2 \\
& - 2\lambda \int_{t-r}^{t} y^2(\vartheta)\, d\vartheta - 2\mu \int_{t-r}^{t} z^2(\vartheta))\, d\vartheta \\
& + 2\big\{ ax + (d_2 + b)y + (1 + c)z + (1 + d_1)u \big\} \times \Big(-au - h(z) - \phi(y) \\
& - f(x) + \int_{t-r}^{t} \phi'\big(y(\rho)\big)z(\rho)\, d\rho + \int_{t-r}^{t} f'\big(x(\rho)\big)y(\rho)\, d\rho \Big).
\end{aligned}
$$

It frequently

$$
\begin{aligned}
\frac{dW(X_t)}{dt} = & -axf(x) - (b + d_2)y\phi(y) + \big(ab + f'(x)\big)y^2 - (1 + c)zh(z) \\
& - a(1 + d_1)u^2 + (1 + c)u^2 + (ad_2 + cd_1 + bc)z^2 - bf(x)y - a\phi(y)x \\
& + a^2 xy - cf(x)z - ah(z)x + abxz - f(x)u - a^2 xu + acxu - c\phi(y)z \\
& - (b + d_2)h(z)y + \big(d_1 f'(x) + ac + bd_2 - d_1 d + b^2\big)yz - (1 + d_1)\phi(y)u \\
& - abyu + (a + cd_1 + bc)yu - (1 + d_1)h(z)u - aczu + \big(b + bd_1 + c^2\big)zu
\end{aligned}
$$

$$+ \lambda r y^2 + \mu r z^2 - \lambda \int_{t-r}^{t} y^2(\vartheta) \, d\vartheta - \mu \int_{t-r}^{t} z^2(\vartheta)) \, d\vartheta$$

$$+ \{ ax + (d_2 + b)y + (1 + c)z + (1 + d_1)u \}$$

$$\times \left(\int_{t-r}^{t} \phi'(y(\rho)) z(\rho) \, d\rho + \int_{t-r}^{t} f'(x(\rho)) y(\rho) \, d\rho \right).$$

The above equation can be rearranged as follows:

$$\frac{dW(X_t)}{dt} = -\frac{1}{4} \sum_{i=6}^{12} W_i + W_{13} + \lambda r y^2 + \mu r z^2$$

$$- \lambda \int_{t-r}^{t} y^2(\vartheta) \, d\vartheta - \mu \int_{t-r}^{t} z^2(\vartheta)) \, d\vartheta, \tag{2.8}$$

where

$$W_6 := a \frac{f(x)}{x} x^2 + \left\{ (b + d_2) \frac{\phi(y)}{y} - \left(ab + f'(x) \right) \right\} y^2 + \left\{ (1 + c) \frac{h(z)}{z} - (ad_2 + cd_1 + bc) \right\} z^2$$

$$+ \{ a(1 + d_1) - (1 + c) \} u^2,$$

$$W_7 := a \frac{f(x)}{x} x^2 + 4 \left\{ b \frac{f(x)}{x} + a \frac{\phi(y)}{y} - a^2 \right\} xy + \left\{ (b + d_2) \frac{\phi(y)}{y} - \left(ab + f'(x) \right) \right\} y^2,$$

$$W_8 := a \frac{f(x)}{x} x^2 + 4 \left\{ c \frac{f(x)}{x} + b \frac{h(z)}{z} - ab \right\} xz + \left\{ (1 + c) \frac{h(z)}{z} - (ad_2 + cd_1 + bc) \right\} z^2,$$

$$W_9 := a \frac{f(x)}{x} x^2 + 4 \left\{ \left(\frac{f(x)}{x} + a^2 \right) - ac \right\} xu + \{ a(1 + d_1) - (1 + c) \} u^2,$$

$$W_{10} := \left\{ (b + d_2) \frac{\phi(y)}{y} - \left(ab + f'(x) \right) \right\} y^2 + \left\{ (1 + c) \frac{h(z)}{z} - (ad_2 + cd_1 + bc) \right\} z^2$$

$$+ 4 \left\{ c \frac{\phi(y)}{y} + (b + d_2) \frac{h(z)}{z} - \left(d_1 f'(x) + ac + bd_2 - d_1 d + b^2 \right) \right\} yz,$$

$$W_{11} := \left\{ (b + d_2) \frac{\phi(y)}{y} - \left(ab + f'(x) \right) \right\} y^2 + \{ a(1 + d_1) - (1 + c) \} u^2$$

$$+ 4 \left\{ (1 + d_1) \frac{\phi(y)}{y} + ab - (a + cd_1 + bc) \right\} yu,$$

$$W_{12} := \left\{ (1 + c) \frac{h(z)}{z} - (ad_2 + cd_1 + bc) \right\} z^2 + \{ a(1 + d_1) - (1 + c) \} u^2$$

$$+ 4 \left\{ (1 + d_1) \frac{h(z)}{z} + ac - \left(b + bd_1 + c^2 \right) \right\} zu,$$

$$W_{13} := \{ ax + (d_2 + b)y + (1 + c)z + (1 + d_1)u \} \left(\int_{t-r}^{t} \phi'(y(\rho)) z(\rho) \, d\rho \right.$$

$$\left. + \int_{t-r}^{t} f'(x(\rho)) y(\rho) \, d\rho \right).$$

From conditions $(i) - (iii)$, we have

$$W_6 \ge a\ell x^2 + \{ (b + d_2)c - ab - d \} y^2 + \{ (1 + c)b - (ad_2 + cd_1 + bc) \} z^2$$

$$+ \{ a(1 + d_1) - (1 + c) \} u^2.$$

In light of the fact that $d_2 = \varepsilon + \frac{d}{c}$ and $\varepsilon = \frac{2ab}{c}$, then the coefficient of y^2 becomes

$$(b + d_2)c - ab - d = \left(\frac{2ab}{c} + \frac{d}{c} + b \right)c - ab - d = (a + c)b.$$

Additionally, since $\varepsilon = \frac{\delta}{2ac\delta_0}$ and from (2.5), we find the coefficient of z^2 as

$$b - ad_2 - cd_1 \geq \frac{\delta}{ac} - \varepsilon\delta_0 = \frac{\delta}{2ac}.$$

Furthermore, since $d_1 = \varepsilon + \frac{1}{a}$ and $\varepsilon = \frac{2c}{a}$, we may express the coefficient of u^2 as follows

$$a(1 + d_1) - (1 + c) = \left(1 + \varepsilon + \frac{1}{a} \right)a - (1 + c) = \varepsilon a + a - c = a + c.$$

Using the above analysis, we can

$$W_6 \geq a\ell x^2 + b(a + c)y^2 + \frac{\delta}{2ac}z^2 + (a + c)u^2,$$

for all x, y, z, and u.

Following that, since $W_i(i = 7, \ldots, 12)$ are quadratic functions of two variables, we can normally create a 2×2-symmetric Hassian matrix A, which is positive definite, if and only if, $\alpha > 0$ and $\alpha\gamma > \beta^2$.

In $W_i(i = 7, \ldots, 12)$, the positive coefficients of x^2, y^2, z^2 and u^2 are then found for $\alpha > 0$. For instance, the discriminant $\beta^2 < \alpha\gamma$ produces the estimate below

$$4\left\{ b\frac{f(x)}{x} + a\frac{\phi(y)}{y} - a^2 \right\}^2 < \left\{ a\frac{f(x)}{x} \right\}\left\{ (b + d_2)\frac{\phi(y)}{y} - \left(ab + f'(x) \right) \right\}.$$

Applying the aforementioned inequality, we obtain

$$W_7 \geq \left\{ \sqrt{a\frac{f(x)}{x}}|x| - \sqrt{(b + d_2)\frac{\phi(y)}{y} - \left(ab + f'(x) \right)}|y| \right\}^2 \geq 0, \quad \text{for all } x \text{ and } y.$$

Given the reasoning above, we can designate the functions $W_i(i = 8, \ldots, 12)$ as W_7

$$W_8 \geq \left\{ \sqrt{a\frac{f(x)}{x}}|x| - \sqrt{(1 + c)\frac{h(z)}{z} - (ad_2 + cd_1 + bc)}|z| \right\}^2 \geq 0, \quad \text{for all } x \text{ and } z,$$

$$W_9 \geq \left\{ \sqrt{a\frac{f(x)}{x}}|x| - \sqrt{a(1 + d_1) - (1 + c)}|u| \right\}^2 \geq 0, \quad \text{for all } x \text{ and } u,$$

$$W_{10} \geq \left\{ \sqrt{(b + d_2)\frac{\phi(y)}{y} - \left(ab + f'(x) \right)}|y| - \sqrt{(1 + c)\frac{h(z)}{z} - (ad_2 + cd_1 + bc)}|z| \right\}^2 \geq 0,$$

$$W_{11} \geq \left\{ \sqrt{(b + d_2)\frac{\phi(y)}{y} - \left(ab + f'(x) \right)}|y| - \sqrt{a(1 + d_1) - (1 + c)}|u| \right\}^2 \geq 0,$$

for all y and u,

$$W_{12} \geq \left\{ \sqrt{(1+c)\frac{h(z)}{z} - (ad_2 + cd_1 + bc)|z|} - \sqrt{a(1+d_1) - (1+c)}|u| \right\}^2 \geq 0,$$

for all z and u.

Since $|f'(x)| \leq d$ and $\phi'(y)| \leq M$, and using the inequality $2|u_1 v_1| \leq u_1^2 + v_1^2$, then we get

$$W_{13} \leq \frac{1}{2}r(M+d)\left\{ax^2 + (d_2 + b)y^2 + (1+c)z^2 + (1+d_1)u^2\right\}$$
$$+ \frac{1}{2}(a + b + c + d_1 + d_2 + 2) \int_{t-r}^{t} \left(Mz^2(\rho) + dy^2(\rho)\right)\} d\rho.$$

When estimations $W_i (i = 6, \ldots, 13)$ are combined, the result (2.8) is

$$\frac{dW(X_t)}{dt} \leq -\frac{1}{4}\left\{a\ell x^2 + b(a+c)y^2 + \frac{\delta}{2ac}z^2 + (a+c)u^2\right\}$$
$$+ \frac{1}{2}r(M+d)\left\{ax^2 + (d_2 + b)y^2 + (1+c)z^2 + (1+d_1)u^2\right\}$$
$$+ \frac{1}{2}(a+b+c+d_1+d_2+2) \int_{t-r}^{t} \left(Mz^2(\rho) + dy^2(\rho)\right) d\rho \tag{2.9}$$
$$+ \lambda r y^2 + \mu r z^2 - \lambda \int_{t-r}^{t} y^2(\vartheta)\,d\vartheta - \mu \int_{t-r}^{t} z^2(\vartheta))\,d\vartheta.$$

The derivative of the Lyapunov functional $W(X_t)$ in equation (2.3) along with any solution $X_t = (x_t, y_t, z_t, u_t)$ of system (1.2) can be calculated using the Itô formula as

$$\mathcal{L}W(X_t) = \frac{dW(X_t)}{dt} + \frac{1}{2}(1+d_1)\sigma^2 x^2.$$

From (2.9), it follows that

$$\mathcal{L}W(X_t) \leq -\frac{1}{4}\left\{a\ell x^2 + b(a+c)y^2 + \frac{\delta}{2ac}z^2 + (a+c)u^2\right\} + \frac{1}{2}(1+d_1)\sigma^2 x^2$$
$$+ \frac{1}{2}r(M+d)\left\{ax^2 + (d_2 + b)y^2 + (1+c)z^2 + (1+d_1)u^2\right\}$$
$$+ \frac{1}{2}(a+b+c+d_1+d_2+2) \int_{t-r}^{t} \left(Mz^2(\rho) + dy^2(\rho)\right) d\rho$$
$$+ \lambda r y^2 + \mu r z^2 - \lambda \int_{t-r}^{t} y^2(\vartheta)\,d\vartheta - \mu \int_{t-r}^{t} z^2(\vartheta))\,d\vartheta.$$

Then we achieve

$$\mathcal{L}W(X_t) \leq -\frac{1}{4}\left\{a\ell - 2ra(M+d) - 2(1+d_1)\sigma^2\right\}x^2$$
$$- \frac{1}{4}\left\{b(a+c) - 2r(M+d)(d_2+b) - 4\lambda r\right\}y^2$$
$$- \frac{1}{4}\left\{\frac{\delta}{2ac} - 2r(M+d)(1+c) - 4\mu r\right\}z^2 \tag{2.10}$$
$$- \frac{1}{4}\left\{a + c - 2r(M+d)(1+d_1)\right\}u^2$$

$$+ \frac{1}{2}\{d(a + b + c + d_1 + d_2 + 2) - 2\lambda\} \int_{t-r}^{t} y^2(\vartheta)\, d\vartheta$$

$$+ \frac{1}{2}\{M(a + b + c + d_1 + d_2 + 2) - 2\mu\} \int_{t-r}^{t} z^2(\vartheta)\, d\vartheta.$$

If we let

$$\lambda = \frac{1}{2}d(a + b + c + d_1 + d_2 + 2), \quad \text{and} \quad \mu = \frac{1}{2}M(a + b + c + d_1 + d_2 + 2).$$

The disparity (2.10) therefore becomes

$$\begin{aligned}
\mathcal{L}W(X_t) \leq{} & -\frac{1}{4}\{a\ell - 2ra(M + d) - 2(1 + d_1)\sigma^2\}x^2 \\
& -\frac{1}{4}\{b(a + c) - 2r(M + d)(d_2 + b) - 2d(a + b + c + d_1 + d_2 + 2)r\}y^2 \\
& -\frac{1}{4}\left\{\frac{\delta}{2ac} - 2r(M + d)(1 + c) - 2M(a + b + c + d_1 + d_2 + 2)r\right\}z^2 \\
& -\frac{1}{4}\{a + c - 2r(M + d)(1 + d_1)\}u^2.
\end{aligned} \tag{2.11}$$

The inequality (2.11) claims that a positive constant β_3 exists, such that

$$\mathcal{L}W(X_t) \leq -\beta_3(x^2 + y^2 + z^2 + u^2), \tag{2.12}$$

for all x, y, z, and u, provided that the following inequality hold

$$r < \min\left\{\frac{a\ell - 2(1 + d_1)\sigma^2}{4a(M + b)}, \frac{(a + c)b}{4M_1}, \frac{\delta}{8acM_2}, \frac{a + c}{4(M + b)(1 + d_1)}\right\},$$

such that

$$M_1 = (M + b)(b + d_2) + b(a + b + c + d_1 + d_2 + 2),$$

$$M_2 = (M + b)(1 + c) + M(a + b + c + d_1 + d_2 + 2).$$

Since from (2.6), (2.7), and (2.12) the Lyapunov functional $W(X_t)$ in (2.3) satisfies all the following requirements

$$\beta_1(|x|) \leq W(X_t) \leq \beta_2(|x|),$$

$$\mathcal{L}W(X_t) \leq -\beta_3(|x|).$$

Then the zero solution of (1.1) is stochastically asymptotically stable.

This completes the proof of Theorem 2.1. □

3 Example

A fourth-order SDE with constat delay r is shown below.

$$\begin{aligned}
& x^{(4)}(t) + 2\ddot{x}(t) + \left(6 + \frac{1}{2 + |\ddot{x}(t)|}\right)\dddot{x}(t) + 2\dot{x}(t - r) + \sin(\dot{x}(t - r)) \\
& + \left(1 + \frac{1}{1 + x^2(t - r)}\right)x(t - r) + \frac{1}{2}x(t)\dot{\omega}(t) = 0.
\end{aligned} \tag{3.1}$$

Its analogous first-order differential equation system is stated as

$$\dot{x} = y,$$
$$\dot{y} = z,$$
$$\dot{z} = u,$$
$$\dot{u} = -2u - \left(6 + \frac{1}{2 + |z|}\right)z - (2y + \sin y) - \left(1 + \frac{1}{1 + x^2}\right)x - \frac{1}{2}x(t)\dot{\omega}(t)$$
$$+ \int_{t-r}^{t} \{8 + \cos y(\rho)\}z(\rho)\,d\rho + \int_{t-r}^{t} \left\{1 + \frac{1 - x^2(\rho)}{(1 + x^2(\rho))^2}\right\}y(\rho)\,d\rho,$$

(3.2)

When we contrast the two systems (1.2) and (3.2), we find that

1. The positive constants a and σ have values $a = 2$ and $\sigma = 1/2$.
2. The function

$$h(z) = 6z + \frac{z}{2 + |z|},$$

since $2 + |z| \geq 2$, for all z and $h(0) = 0$, it follows that

$$\frac{h(z)}{z} = 6 + \frac{1}{2 + |z|} \geq 6,$$

it tends to $b = 6$, for all $z \neq 0$. The function $\frac{h(z)}{z}$ is shown in Fig. 1.

3. The function

$$\phi(y) = 2y + \sin y, \quad \text{it tends to} \quad \frac{\phi(y)}{y} = 2 + \frac{\sin y}{y}, \quad \text{for all } y \neq 0.$$

Since $-1 \leq \frac{\sin y}{y} \leq 1$, then we get $1 \leq \frac{\phi(y)}{y} \leq 3$, and the derivative of the function $\phi(y)$ with respect y becomes

$$\left|\phi'(y)\right| = |2 + \cos y| \leq 3, \quad \text{for all } y \neq 0.$$

Figure 1 The function $\frac{h(z)}{z}$ and its bounds for $z \in [-20, 20]$

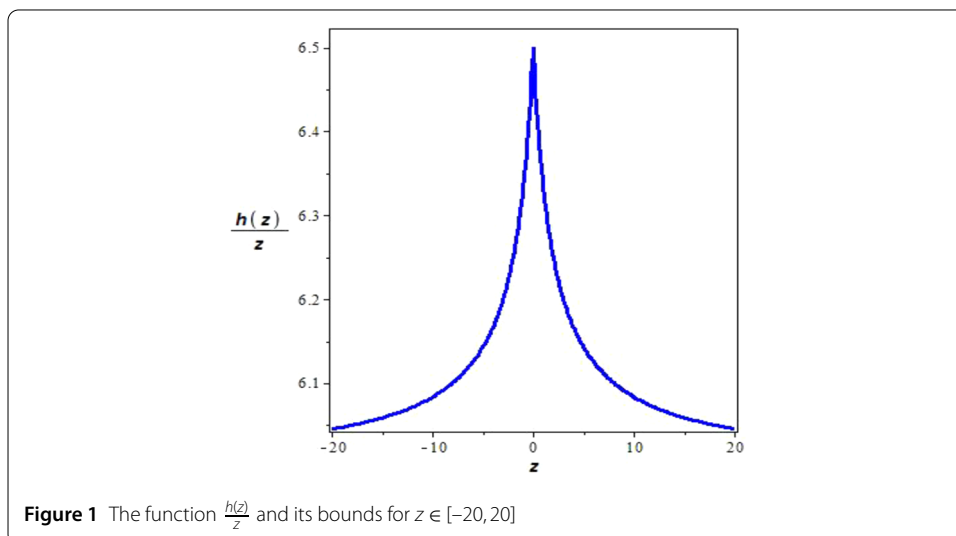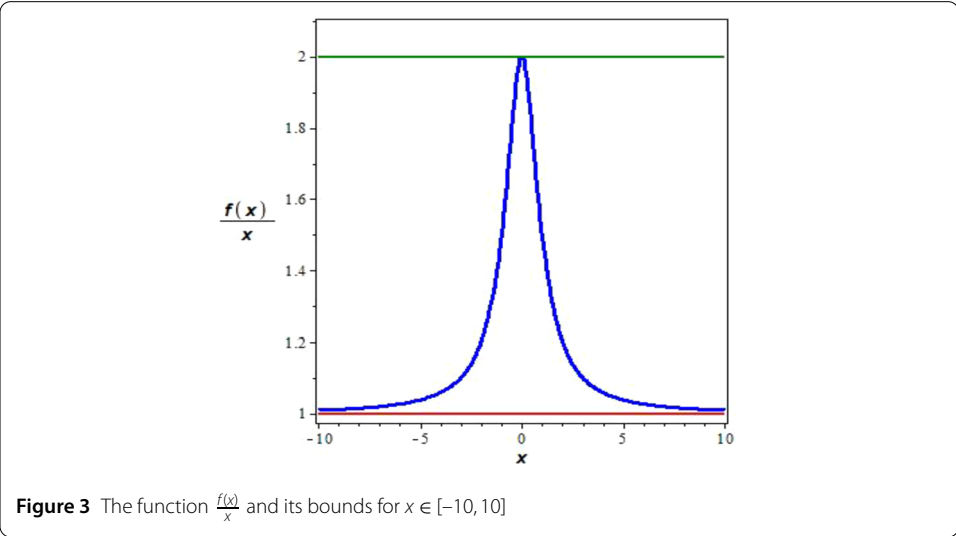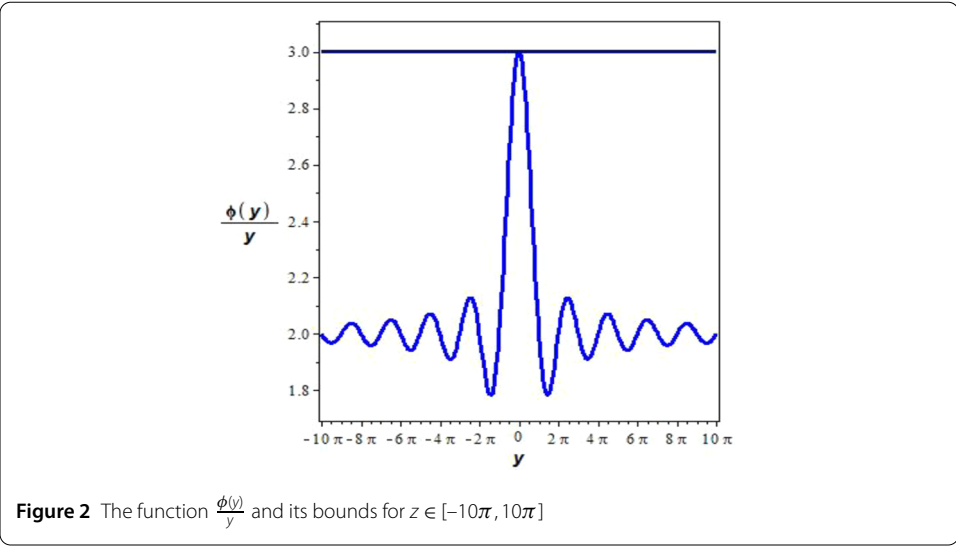**Figure 2** The function $\frac{\phi(y)}{y}$ and its bounds for $z \in [-10\pi, 10\pi]$

Figure 3 The function $\frac{f(x)}{x}$ and its bounds for $x \in [-10, 10]$

Then we state that $c = 1$, $c_1 = 3$, and $M = 3$. Path of the function $\frac{\phi(y)}{y}$ is depicted in Fig. 2.

4. The function

$$f(x) = x + \frac{x}{1 + x^2}, \quad \text{fulfills} \quad f(0) = 0 \quad \text{and} \quad \frac{f(x)}{x} = 1 + \frac{1}{1 + x^2}, \quad \text{for all } x \neq 0.$$

As a result, $1 \le \frac{f(x)}{x} \le 2$, the derivative of the function f with respect to x is

$$f'(x) = 1 + \frac{1 - x^2}{(1 + x^2)^2} \le 1 + \frac{1}{1 + x^2} \le 2, \quad \text{and} \quad \left| f'(x) \right| \le 2.$$

It gives that $\ell = 1$, $m = 2$, and $d = 2$. Figure 3 shows the function $\frac{f(x)}{x}$ and its bounds for $x \in [-10, 10]$.

5. Conditions $(iv)–(vi)$ of Theorem 2.1 have inequalities that are as follows:

$$ab - c = 11 > 0,$$

$$abc - c^2 - a^2 d = 3 \geq \delta.$$

Choose $\delta = 2.8$, then we can get the subsequent constants

$$\delta_0 = ab + bcd^{-1} = 15 > 0,$$

$$0.047 \leq \varepsilon \leq \min\{24, 1\}, \quad \text{it tends to} \quad 0.047 \leq \varepsilon \leq 1, \quad \text{here } \varepsilon = 0.05 \text{ is chosen.}$$

Also,

$$d_1 = \varepsilon + \frac{1}{a} = 0.55, \quad \text{and} \quad d_2 = \varepsilon + \frac{d}{c} = 2.05.$$

Therefore, the condition $2 = a\ell > 2(1 + d_1)\sigma^2 \cong 0.775$, is satisfied.

6. The value of the constant r is such that

$$r < 0.00497 = \min\{0.038, 0.0395, 0.00497, 0.121\}.$$

As a result, all the elements above satisfy all of the requirements of Theorem 2.1.

Therefore, the zero solution of (3.1) is stochastically asymptotically stable.

4 Conclusions

This paper discusses a specific fourth-order nonlinear SDE with a constant delay. The paper obtains standard criteria for the stochastic asymptotic stability of the equation. The effectiveness of the functional used in the investigation greatly aids in proving the main result. The paper's stochastic stability result enhances, incorporates, and supplements the existing stability results in the literature. Additionally, a relevant example is provided to verify the hypotheses.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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