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New multivalued F-contraction mappings involving α-admissibility with an application

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Abstract

In this article, we obtain some fixed-point results involving α-admissibility for multivalued F-contractions in the framework of partial $\mathfrak{b}$-metric spaces. Appropriate illustrations are provided to support the main results. Finally, an application is developed by demonstrating the existence of a solution to an integral equation.

Mathematics Subject Classification: 47H10; 54H25

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1 Introduction and preliminaries

In 1922, Banach [6] proposed the well-known Banach contraction principle (BCP), which employed a contraction mapping in the domain of complete metric spaces. Later, it was regarded as an effective approach for locating unique fixed points. According to the BCP, in a complete metric space $(\mathcal{M}, d^*)$, a mapping $f : \mathcal{M} \to \mathcal{M}$ satisfying the contraction condition on $\mathcal{M}$, i.e.,

$$d^*(f\zeta, f\beta) \le cd^*(\zeta, \beta),$$

for all $\zeta, \beta \in \mathcal{M}$, provided $c \in [0, 1)$, has a unique fixed point.

The BCP was generalized using varieties of mappings on several extensions of metric spaces. In 1969, Nadler [7] generalized the BCP for multivalued mappings. In order to optimize a variety of approximation theory problems, it is much more advantageous to use proper fixed-point results for multivalued transformations. The notion of F-contractions was introduced by Wardowski [15]. Altun et al. [2] focused on the existence of the fixed point for multivalued F-contractions and proved certain fixed-point theorems on the setting of metric spaces. Many extensions and generalizations of BCP were produced and the existence and uniqueness of fixed-point were proved. Ali et al. [1] introduced the notion of α-F-admissible type mappings in the setting of uniform spaces. One can see many interesting results on α-F mappings in [3–5, 18].

In 2014, Shukla [12] gave a new direction for extending the metric space. He blended the principles of a partial metric space [9] and a $\mathfrak{b}$-metric space [10, 11] together and proposed a new notion of a partial $\mathfrak{b}$-metric space to present a fine interpretation of BCP in such a space. Kumar et al. [8] extended these results to partial metric spaces and proved fixed point results for multivalued F-contraction mappings. Kumar et al. [8] presented an article in April 2021, using multivalued F-mappings in partial metric spaces. A sound generalization of BCP under this new direction was given. One can see more work in the papers [16, 17, 19] and the references therein. Motivated by his work, an idea of extending the BCP in the globe of a partial $\mathfrak{b}$-metric space by integrating the notion of α-admissibility introduced by Samet et al. [13] under multivalued F-contractions, is presented.

Take $\mathbb{R}^+ = [0,\infty)$ and denote by $\mathbb{N}$ the set of positive integers. Throughout the article, the compact subset of the underlying space $\mathcal{M}$ will be denoted by $K(\mathcal{M})$. Let us now look at some essential concepts and consequences that will set a foundation for our main result.

Definition 1.1 [12] Let $\mathcal{M} \neq \phi$ and $\mathfrak{b} \geq 1$ be any real number. A map $p_\mathfrak{b} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfying the following properties on $\mathcal{M}$ is called a partial $\mathfrak{b}$ metric on $\mathcal{M}$:

$\acute{p}_b(1)$: $p_\mathfrak{b}(m_1,m_2) = p_\mathfrak{b}(m_1,m_1) = p_\mathfrak{b}(m_2,m_2)$ if and only if $m_1 = m_2$;

$\acute{p}_b(2)$: $p_\mathfrak{b}(m_1,m_2) \geq p_\mathfrak{b}(m_1,m_1)$;

$\acute{p}_b(3)$: $p_\mathfrak{b}(m_1,m_2) = p_\mathfrak{b}(m_2,m_1)$;

$\acute{p}_b(4)$: $p_\mathfrak{b}(m_1,m_2) \leq \mathfrak{b}\{p_\mathfrak{b}(m_1,m_3) + p_\mathfrak{b}(m_3,m_2)\} - p_\mathfrak{b}(m_3,m_3)$,for all $m_1,m_2,m_3 \in \mathcal{M}$.

The pair $(\mathcal{M},p_\mathfrak{b})$ is said to be a partial $\mathfrak{b}$-metric space (P$\mathfrak{b}$MS).

Example 1.2 Let $\mathcal{M} = \mathbb{R}^+$. We define $p_\mathfrak{b} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by

$$p_\mathfrak{b}(m_1,m_2) = |m_1 - m_2|^q + \left[\max\{m_1,m_2\}\right]^q, \quad \text{for all } m_1,m_2 \in \mathcal{M}.$$

Let $q > 1$ be any constant, then $(\mathcal{M},p_\mathfrak{b})$ is a P$\mathfrak{b}$MS with $\mathfrak{b} = 2^{q-1}$.

Definition 1.3 Let $(\mathcal{M},p_\mathfrak{b})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$. Let $\{m_\xi\}$ be a sequence in $\mathcal{M}$ and $m_0 \in \mathcal{M}$ be any arbitrary element.

(1) The sequence $\{m_\xi\}$ is called a convergent sequence with limit m_0 if

$$\lim_{\xi\to\infty} p_\mathfrak{b}(m_\xi,m_0) = p_\mathfrak{b}(m_0,m_0).$$

As an example, consider $\mathcal{M} = [0,1]$ and let $m_\xi = \{\frac{1}{\xi} : \xi \in \mathbb{N}\}$. Define a map $p_\mathfrak{b} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ by $p_\mathfrak{b}(m_1,m_2) = |m_1 - m_2|^5 + \nu$, where $\nu > 0$. It is easy to see that $(\mathcal{M},p_\mathfrak{b})$ is a P$\mathfrak{b}$MS with $\mathfrak{b} = 2^4$. Now,

$$\lim_{\xi\to\infty} p_\mathfrak{b}(m_\xi,0) = \lim_{\xi\to\infty} p_\mathfrak{b}\left(\frac{1}{\xi},0\right) = \lim_{\xi\to\infty}\left[\left|\frac{1}{\xi} - 0\right| + \nu\right] = p_\mathfrak{b}(0,0).$$

That is, $\{m_\xi\}$ is a convergent sequence in $(\mathcal{M},p_\mathfrak{b})$.

(2) A sequence $\{m_k\}$ in $\mathcal{M}$ becomes a Cauchy sequence if

$$\lim_{k,l\to\infty} p_\mathfrak{b}(m_k,m_l)$$

exists and is finite.

(3) $(\mathcal{M},p_\mathfrak{b})$ is called a complete P$\mathfrak{b}$MS if every Cauchy sequence converges in $\mathcal{M}$.

Some useful ideas concerning Hausdorff distance under the structure of P$\mathfrak{b}$MSs have been suggested by Felhi [14] and recently revised by Anwar et al. [3].

Definition 1.4 Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$, and $CB_{p_{\mathfrak{b}}}(\mathcal{M})$ be the collection of all nonempty bounded and closed subsets of $\mathcal{M}$. For $\mathcal{P}, \mathcal{Q} \in CB_{p_{\mathfrak{b}}}(\mathcal{M})$, the partial Hausdorff $\mathfrak{b}$-metric on $CB_{p_{\mathfrak{b}}}(\mathcal{M})$ induced by $p_{\mathfrak{b}}$ is given as follows:

$$\mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{Q}) = \max\{\delta_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{Q}), \delta_{p_{\mathfrak{b}}}(\mathcal{Q}, \mathcal{P})\},$$

where $\delta_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{Q}) = \sup\{p_{\mathfrak{b}}(p, \mathcal{Q}) : p \in \mathcal{P}\}$ and $\delta_{p_{\mathfrak{b}}}(\mathcal{Q}, \mathcal{P}) = \sup\{p_{\mathfrak{b}}(q, \mathcal{P}) : q \in \mathcal{Q}\}$.

Lemma 1.5 *Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$. Consider two nonempty subsets $\mathcal{P}, \mathcal{P}^* \in CB_{p_{\mathfrak{b}}}(\mathcal{M})$, and $k^* > 1$. For some $p \in \mathcal{P}$, there exists $q \in \mathcal{P}^*$ so that*

$$p_{\mathfrak{b}}(p, q) \leq k^* \mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{P}^*).$$

Lemma 1.6 *Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$, then for two nonempty subsets $\mathcal{P}, \mathcal{P}^* \in CB_{p_{\mathfrak{b}}}(\mathcal{M})$, and for each $p \in \mathcal{P}$, we have*

$$p_{\mathfrak{b}}(p, \mathcal{P}^*) \leq \mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{P}^*).$$

A new concept was given by Wardowski [15] in 2012 by introducing Δ_f-family.

Definition 1.7 A mapping $\mathcal{F}$ from $(0, \infty)$ to $\mathbb{R}$ is a member of Δ_f-family if $\mathcal{F}$ satisfies these properties:

(F_1): $\mathcal{F}$ is strictly increasing, i.e.,

$$m_1 < m_2 \quad \Longrightarrow \quad \mathcal{F}(m_1) < \mathcal{F}(m_2), \quad \text{for all } m_1, m_2 \in \mathbb{R}.$$

(F_2): For every positive term sequence $\{m_\xi : \xi \in \mathbb{N}\}$,

$$\lim_{n \to \infty} m_\xi = 0 \quad \Longleftrightarrow \quad \lim_{n \to \infty} \mathcal{F}(m_\xi) = -\infty.$$

(F_3): If we have $\gamma \in (0, 1)$, then $\lim_{\xi \to 0^+} \xi^\gamma \mathcal{F}(\xi) = 0$.

Example 1.8 Let $\mathcal{F} : (0, \infty) \to \mathbb{R}$ be defined as $\mathcal{F}(m) = \ln(m)$. $\mathcal{F}$ is a member of Δ_f-family.

Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$. This paper initiates the concept of new multivalued contraction mappings involving the Δ_f-family and a given function $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ in the context of a P$\mathfrak{b}$MS. We develop some fixed point results for such contractions. Furthermore, we illustrate our main result with concrete examples. An application is also presented for a deeper understanding of the obtained result.

2 Main results

We start with the following definition.

Definition 2.1 Consider a set $\mathcal{M} \neq \phi$ and let $S : \mathcal{M} \to 2^{\mathcal{M}}$ be a multivalued mapping. Given a function $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$. S is called a multivalued α-admissible mapping if for $m, n \in \mathcal{M}$, we have

$$\alpha(m, n) \geq 1 \quad \Longrightarrow \quad \alpha(m_0, n_0) \geq 1,$$

where $m_0 \in S(m)$ and $n_0 \in S(n)$.

Definition 2.2 Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$ and define a map $S : \mathcal{M} \to K(\mathcal{M})$. Then S is said to be a MV$\mathcal{F}$-contraction mapping if there are $\mathcal{F} \in \Delta_f$ – family and $\tau > 0$ such that

$$\mathcal{H}_{p\mathfrak{b}}(Sm_1, Sm_2) > 0 \quad \Longrightarrow \quad \tau + \mathcal{F}\big(\mathfrak{b}\mathcal{H}_{p\mathfrak{b}}(Sm_1, Sm_2)\big) \leq \mathcal{F}\big(\mathbb{M}(m_1, m_2)\big), \tag{2.1}$$

where

$$\mathbb{M}(m_1, m_2) = \max\Big\{p_{\mathfrak{b}}(m_1, m_2), p_{\mathfrak{b}}(m_1, Sm_1), p_{\mathfrak{b}}(m_2, Sm_2), \frac{p_{\mathfrak{b}}(m_1, Sm_2) + p_{\mathfrak{b}}(m_2, Sm_1)}{2\mathfrak{b}}\Big\}.$$

Definition 2.3 Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$. Given a function $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$. The mapping $S : \mathcal{M} \to K(\mathcal{M})$ is said to be a MV$\alpha\mathcal{F}$-contraction if there are $\mathcal{F} \in \Delta_f$ – family and $\tau > 0$ such that

$$\mathcal{H}_{p\mathfrak{b}}(Sm_1, Sm_2) > 0 \\ \Longrightarrow \quad \tau + \mathcal{F}\big(\alpha(m_1, m_2)\big(\mathfrak{b}\mathcal{H}_{p\mathfrak{b}}(Sm_1, Sm_2)\big)\big) \leq \mathcal{F}\big(\mathbb{M}(m_1, m_2)\big), \tag{2.2}$$

where

$$\mathbb{M}(m_1, m_2) = \max\Big\{p_{\mathfrak{b}}(m_1, m_2), p_{\mathfrak{b}}(m_1, Sm_1), p_{\mathfrak{b}}(m_2, Sm_2), \frac{p_{\mathfrak{b}}(m_1, Sm_2) + p_{\mathfrak{b}}(m_2, Sm_1)}{2\mathfrak{b}}\Big\}.$$

Lemma 2.4 *Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a complete P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$ and $S : \mathcal{M} \to K(\mathcal{M})$ be a MV$\mathcal{F}$-contraction mapping, then*

$$\lim_{\xi \to \infty} \mathfrak{b}^{\xi} v_{\xi} = 0,$$

where $v_{\xi} = p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})$ and $\xi = 0, 1, 2, \ldots$.

Proof We take an arbitrary $m_0 \in \mathcal{M}$. As Sm_0 is compact, it is nonempty, so we can choose $m_1 \in Sm_0$. If $m_1 \in Sm_1$, this means that m_1 is a fixed point of S trivially. Suppose $m_1 \notin Sm_1$. As Sm_1 is closed, so we have $p_{\mathfrak{b}}(m_1, Sm_1) > 0$. Also, we know that

$$p_{\mathfrak{b}}(m_1, Sm_1) \leq \mathcal{H}_{p_{\mathfrak{b}}}(Sm_0, Sm_1). \tag{2.3}$$

As Sm_1 is compact, so there exists $m_2 \in Sm_1$ such that

$$p_{\mathfrak{b}}(m_1, m_2) = p_{\mathfrak{b}}(m_1, Sm_1).$$

Thus,

$$p_{\mathfrak{b}}(m_1, m_2) \leq \mathcal{H}_{p_{\mathfrak{b}}}(Sm_0, Sm_1).$$

Similarly for $m_3 \in Sm_2$, we get

$$p_{\mathfrak{b}}(m_2, m_3) \leq \mathcal{H}_{p_{\mathfrak{b}}}(Sm_1, Sm_2),$$

which ultimately gives

$$p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}) \leq \mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1}).$$

This leads to

$$\mathfrak{b}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big) \leq \mathfrak{b}\big(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1})\big).$$

The condition (F_1) implies that

$$\mathcal{F}\big(\mathfrak{b}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \leq \mathcal{F}\big(\mathfrak{b}\big(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1})\big)\big). \tag{2.4}$$

By (2.1), we have

$$\mathcal{F}\big(\mathfrak{b}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \leq \mathcal{F}\big(\mathbb{M}(m_{\xi+1}, m_{\xi})\big) - \tau, \tag{2.5}$$

where

$$\begin{aligned}
\mathbb{M}(m_{\xi}, m_{\xi+1}) &= \max\bigg\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi}, Sm_{\xi}), p_{\mathfrak{b}}(m_{\xi+1}, Sm_{\xi+1}), \\
&\qquad \frac{p_{\mathfrak{b}}(m_{\xi}, Sm_{\xi+1}) + p_{\mathfrak{b}}(m_{\xi+1}, Sm_{\xi})}{2\mathfrak{b}}\bigg\} \\
&= \max\bigg\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}), \\
&\qquad \frac{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})}{2\mathfrak{b}}\bigg\} \\
&\leq \max\bigg\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}), \\
&\qquad \mathfrak{b}\bigg[\frac{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})}{2\mathfrak{b}}\bigg]\bigg\}. \\
&= \max\big\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big\}.
\end{aligned}$$

Assume that

$$\max\big\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big\} = p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}).$$

The inequality (2.5) yields

$$\tau + \mathcal{F}\big(\mathfrak{b}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \leq \mathcal{F}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big),$$

which is a contradiction. Therefore,

$$\max\big\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big\} = p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}).$$

It implies that

$$\mathcal{F}\big(\mathfrak{b}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \leq \mathcal{F}\big(p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1})\big).$$

For convenience, we are setting $\nu_{\xi} = p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})$, where $\xi = 0, 1, \ldots$. Clearly, $\nu_{\xi} > 0$ for all $\xi \in \mathbb{N}$. Now, substituting this into the above equation, we have

$$\tau + \mathcal{F}\big(\mathfrak{b}(\nu_{\xi})\big) \leq \mathcal{F}(\nu_{\xi-1}).$$

Iteratively,

$$\tau + \mathcal{F}\big(\mathfrak{b}^{\xi}(\nu_{\xi})\big) \leq \mathcal{F}\big(\mathfrak{b}^{\xi-1}(\nu_{\xi-1})\big).$$

We will get

$$\mathcal{F}(\mathfrak{b}^{\xi}(\nu_{\xi}) \leq \mathcal{F}\big(\mathfrak{b}^{\xi-1}(\nu_{\xi-1})\big) - \tau \leq \mathcal{F}\big(\mathfrak{b}^{\xi-2}(\nu_{\xi-2})\big) - 2\tau \leq \cdots \leq \mathcal{F}(\nu_0) - \xi\tau. \tag{2.6}$$

Hence,

$$\lim_{\xi\to\infty} \mathcal{F}\mathfrak{b}^{\xi}(\nu_{\xi}) = -\infty,$$

we have

$$\lim_{\xi\to\infty} \mathfrak{b}^{\xi}\nu_{\xi} = 0, \quad \text{by } (F_2).$$ □

Theorem 2.5 *Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a complete P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$, such that $p_{\mathfrak{b}}$ is a continuous mapping and $S : \mathcal{M} \to K(\mathcal{M})$ is a multivalued $\alpha\mathcal{F}$-contraction mapping. Suppose that*

(1) *S is continuous*;
(2) *S is an α-admissible mapping*;
(3) *there exist $m_0 \in \mathcal{M}$ and $m_1 \in Sm_0$ such that $\alpha(m_0, m_1) \geq 1$.*

Then S has a fixed point.

Proof For $m_0 \in \mathcal{M}$, we have by assumption $\alpha(m_0, m_1) \geq 1$ for some $m_1 \in Sm_0$. Similarly, for $m_2 \in Sm_1$, we have $\alpha(m_1, m_2) \geq 1$ and for any sequence $m_{\xi+1} \in Sm_{\xi}$, we get

$$\alpha(m_{\xi}, m_{\xi+1}) \geq 1 \quad \text{for all } \xi \in \mathbb{N} \cup \{0\}. \tag{2.7}$$

Now, by the contraction condition (2.2), we have

$$\tau + \mathcal{F}\big(\alpha(m_{\xi}, m_{\xi+1})\mathfrak{b}\big(\mathcal{H}_{p\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \leq \mathcal{F}\big(\mathbb{M}(m_{\xi+1}, m_{\xi})\big).$$

The inequality (2.7) implies that

$$\tau + \mathcal{F}\big(\mathfrak{b}\big(\mathcal{H}_{p\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \leq \mathcal{F}\big(\mathbb{M}(m_{\xi+1}, m_{\xi})\big),$$

where $\mathfrak{b} \geq 1$. We have

$$\mathcal{F}\big(\mathfrak{b}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \leq \mathcal{F}\big(\mathbb{M}(m_{\xi+1}, m_{\xi})\big) - \tau. \tag{2.8}$$

By lemma 2.4, one writes

$$\lim_{\xi\to\infty} \mathfrak{b}^{\xi} \nu_{\xi} = 0.$$

By (F_3), for any $\gamma \in (0,1)$

$$\lim_{\xi\to\infty} \left(\mathfrak{b}^{\xi} \nu_{\xi}\right)^{\gamma} \mathcal{F}\mathfrak{b}^{\xi}(\nu_{\xi}) = 0, \quad \forall \xi \in \mathbb{N}.$$

Using (2.6), one writes

$$\left(\mathfrak{b}^{\xi} \nu_{\xi}\right)^{\gamma} \left(\mathcal{F}\mathfrak{b}^{\xi}(\nu_{\xi}) - \mathcal{F}(\nu_0)\right) \leq -\left(\mathfrak{b}^{\xi} \nu_{\xi}\right)^{\gamma} \xi\tau \leq 0. \tag{2.9}$$

Now, as $\tau > 0$, we have

$$\lim_{\xi\to\infty} \left(\mathfrak{b}^{\xi} \nu_{\xi}\right)^{\gamma} \xi = 0.$$

So, there exists $\xi_1 \in \mathbb{N}$, such that

$$\left(\mathfrak{b}^{\xi} \nu_{\xi}\right)^{\gamma} \xi \leq 1, \quad \forall \xi \geq \xi_1.$$

It implies that

$$\mathfrak{b}^{\xi} \nu_{\xi} \leq \frac{1}{\xi^{\frac{1}{\gamma}}}. \tag{2.10}$$

Now, we will prove that $\{m_{\xi}\}$ is a Cauchy sequence in $\mathcal{M}$. For this, let $\xi, l \in \mathbb{N}$ provided that $\xi > l \geq \xi_1$. Using the triangular inequality of a P$\mathfrak{b}$MS, we have

$$\begin{aligned}
p_{\mathfrak{b}}(m_{\xi}, m_{\eta}) &\leq \mathfrak{b}\left\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + p_{\mathfrak{b}}(m_{\xi+1}, m_{\eta})\right\} - p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+1}) \\
&\leq \mathfrak{b}\left\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + p_{\mathfrak{b}}(m_{\xi+1}, m_{\eta})\right\} \\
&\leq \mathfrak{b}p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + \mathfrak{b}^2\left\{p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}) + p_{\mathfrak{b}}(m_{\xi+2}, m_{\eta})\right\} \\
&\quad - p_{\mathfrak{b}}(m_{\xi+2}, m_{\xi+2}) \\
&\leq \mathfrak{b}p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + \mathfrak{b}^2\left\{p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}) + p_{\mathfrak{b}}(m_{\xi+2}, m_{\eta})\right\} \\
&\quad \vdots \\
&= \mathfrak{b}p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + \mathfrak{b}^2\{p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}) + \cdots + \mathfrak{b}^{l-\xi} p_{\mathfrak{b}}(m_{\eta-1}, m_{\eta}) \\
&= \sum_{\beta=\xi}^{\eta-1} \mathfrak{b}^{\beta-\xi+1} p_{\mathfrak{b}}(m_{\beta}, m_{\beta+1}) \\
&\leq \sum_{\beta=\xi}^{\infty} \mathfrak{b}^{\beta} p_{\mathfrak{b}}(m_{\beta+1}, m_{\beta+2}) \\
&= \sum_{\beta=\xi}^{\infty} \mathfrak{b}^{\beta} \nu_{\beta}
\end{aligned}$$

$$\leq \sum_{\beta=\xi}^{\infty} \frac{1}{\beta^{\frac{1}{\gamma}}}.$$

The convergence of the series $\sum_{\beta=1}^{\infty} \frac{1}{\beta^{\frac{1}{\gamma}}}$ implies that $\lim_{\xi\to\infty} p_{\mathfrak{b}}(m_\xi, m_\eta) = 0$, which shows $\{m_\xi\}$ is a Cauchy sequence in $\mathcal{M}$. Since $\mathcal{M}$ is complete, there exists $m^* \in \mathcal{M}$ such that

$$\lim_{\xi\to\infty} p_{\mathfrak{b}}(m_\xi, m^*) = p_{\mathfrak{b}}(m^*, m^*) = 0. \tag{2.11}$$

We claim that m^* is a fixed point of S, that is,

$$p_{\mathfrak{b}}(m^*, Sm^*) = p_{\mathfrak{b}}(m^*, m^*).$$

Suppose $p_{\mathfrak{b}}(m^*, Sm^*) > 0$. So, there exists $k_0 \in \mathbb{N}$ such that $p_{\mathfrak{b}}(m_\xi, Sm^*) > 0$ for all $\xi > k_0$. We have

$$p_{\mathfrak{b}}(m_\xi, Sm^*) \leq \mathcal{H}_{p\mathfrak{b}}(Sm_{\xi+1}, Sm^*).$$

By using our contraction condition and taking limit $\xi \to \infty$, we have

$$\begin{aligned}\tau + \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)) &\leq \tau + \mathcal{F}(\alpha(m^*, m^*)\mathcal{H}_{p\mathfrak{b}}(Sm^*, Sm^*)) \\ &\leq \mathcal{F}(\mathbb{M}(m^*, m^*)) \\ &\leq \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)),\end{aligned}$$

where,

$$\begin{aligned}\mathbb{M}(m^*, m^*) &= \max\Big\{p_{\mathfrak{b}}(m^*, m^*), p_{\mathfrak{b}}(m^*, Sm^*), p_{\mathfrak{b}}(m^*, Sm^*), \\ &\qquad \frac{p_{\mathfrak{b}}(m^*, Sm^*) + p_{\mathfrak{b}}(Sm^*, m^*)}{2\mathfrak{b}}\Big\} \\ &\leq p_{\mathfrak{b}}(m^*, Sm^*).\end{aligned}$$

It yields that

$$\tau + \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)) \leq \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)).$$

Since $\tau > 0$, the above relation yields a contradiction, therefore $p_{\mathfrak{b}}(m^*, Sm^*) = 0$. Also,

$$p_{\mathfrak{b}}(m^*, m^*) = 0.$$

This gives $m^* \in \bar{S}m^* = Sm^*$. Proving that m^* is a fixed point of S. □

Example 2.6 Let $\mathcal{M} = \{0, 1, 2, 3, \ldots\}$ and $p_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be defined as

$$p_{\mathfrak{b}}(\zeta, \nu) = |\zeta - \nu|^q + \big[\max\{\zeta, \nu\}\big]^q \quad \text{for all } \zeta, \nu \in \mathcal{M}.$$

It is easy to check that $(\mathcal{M}, p_{\mathfrak{b}})$ is a complete P$\mathfrak{b}$MS with $\mathfrak{b} = 2^{q-1}$, where $q > 1$. We also define a multivalued map $S : \mathcal{M} \to 2^{\mathcal{M}}$ by

$$S\zeta = \begin{cases} \{0,1\}, & \text{if } \zeta = 0,1, \\ \{\zeta - 1, \zeta\} & \text{otherwise.} \end{cases}$$

Consider $\alpha : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ as

$$\alpha(\zeta,\nu) = \begin{cases} 2, & \text{if } \zeta,\nu \in \{0,1\}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Let $\zeta_0 = 0$, $\zeta_1 = 1$, then $S\zeta_0 = \{0,1\}$ and $\zeta_1 = \{0,1\}$. Giving $\alpha(\zeta_0,\zeta_1) = \alpha(0,1) = 2 > 1$, for some $\zeta_2 = 0 \in S\zeta_1$, we get $\alpha(\zeta_1,\zeta_2) = \alpha(1,0) = 2 > 1$. That is, S is an α-admissible map.

Define $\mathcal{F} : (0,\infty) \to \mathbb{R}$ as $\mathcal{F}(\zeta) = \ln(\zeta) + \zeta$. It can be observed easily that $\mathcal{F}$ is a member of Δ_f-family. Now, applying $\mathcal{F}$ on our contraction condition, one gets

$$\tau + \mathcal{F}\big(\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu)\big) \le \mathcal{F}\big(\mathbb{M}(\zeta,\nu)\big).$$

That is,

$$\begin{aligned} &\tau + \ln\big\{\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu)\big\} + \alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu) \\ &\quad \le \ln\big(\mathbb{M}(\zeta,\nu)\big) + \mathbb{M}(\zeta,\nu). \end{aligned}$$

Hence,

$$\tau + \alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu) - \mathbb{M}(\zeta,\nu) \le \ln\big(\mathbb{M}(\zeta,\nu)\big) - \ln\big\{\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu)\big\}.$$

Therefore,

$$e^{\tau + \alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu) - \mathbb{M}(\zeta,\nu)} \le \frac{\mathbb{M}(\zeta,\nu)}{\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu)}$$

That is,

$$\frac{\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu)}{\mathbb{M}(\zeta,\nu)} e^{\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu) - \mathbb{M}(\zeta,\nu)} \le e^{-\tau}. \tag{2.12}$$

Now,

$$\begin{aligned} \delta_{p_{\mathfrak{b}}}\big(\mathcal{P},\mathcal{P}^*\big) &= \delta_{p_{\mathfrak{b}}}(S\zeta,S\nu) \\ &= \max\big\{p_{\mathfrak{b}}(\zeta,S\nu), p_{\mathfrak{b}}(\zeta-1,S\nu)\big\} \\ &= \max\big\{\inf\big\{p_{\mathfrak{b}}(\zeta,\nu), p_{\mathfrak{b}}(\zeta,\nu-1)\big\}, \inf\big\{p_{\mathfrak{b}}(\zeta-1,\nu), p_{\mathfrak{b}}(\zeta-1,\nu-1)\big\}\big\} \\ &= \max\big\{|\zeta-\nu|^q + \zeta^q, |\zeta-\nu-2|^q + \zeta^q\big\} \\ &= |\zeta-\nu|^q + \zeta^q. \end{aligned}$$

Similarly, we can calculate

$$\delta_{p_{\mathfrak{b}}}\left(\mathcal{P}^{*},\mathcal{P}\right)=|\zeta-\nu|^{q}+\zeta^{q}.$$

Hence,

$$\begin{aligned}\mathcal{H}_{p_{\mathfrak{b}}}\left(\mathcal{P},\mathcal{P}^{*}\right)&=\max\left\{|\zeta-\nu|^{q}+\zeta^{q},|\zeta-\nu|^{q}+\zeta^{q}\right\}\\&=|\zeta-\nu|^{q}+\zeta^{q}.\end{aligned}\tag{2.13}$$

Also,

$$\mathbb{M}(\zeta,\nu)\geq p_{\mathfrak{b}}(\zeta,\nu)=|\zeta-\nu|^{q}+\zeta^{q}.\tag{2.14}$$

Setting these both in the contraction condition, we get

$$\begin{aligned}&\frac{\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu)}{\mathbb{M}(\zeta,\nu)}e^{(\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu))-\mathbb{M}(\zeta,\nu)}\\&=\frac{|\zeta-\nu|^{q}+\zeta^{q}}{2\mathbb{M}(\zeta,\nu)}e^{\frac{1}{2}(|\zeta-\nu|^{q}+\zeta^{q})-\mathbb{M}(\zeta,\nu)}\quad\text{using (2.25)}\\&\leq\frac{|\zeta-\nu|^{q}+\zeta^{q}}{2|\zeta-\nu|^{q}+\zeta^{q}}e^{\frac{1}{2}(|\zeta-\nu|^{q}+\zeta^{q})-|\zeta-\nu|^{q}+\zeta^{q}}\quad\text{using (2.26)}\\&=\frac{1}{2}e^{\frac{-1}{2}(|\zeta-\nu|^{q}+\zeta^{q})}\\&=\frac{1}{2}e^{-\tau}\\&<e^{-\tau}.\end{aligned}$$

This implies that (2.12) is satisfied with $\tau=\frac{1}{2}(|\zeta-\nu|^{q}+\zeta^{q})$, which is a positive number for $\zeta\neq\nu$. All conditions of Theorem 2.5 are true, and 0 and 1 are two fixed points of S.

Theorem 2.7 *Let* $(\mathcal{M},p_{\mathfrak{b}})$ *be a complete P*$\mathfrak{b}$*MS with* $\mathfrak{b}\geq 1$ *such that* $p_{\mathfrak{b}}$ *is a continuous mapping. Let* $S:\mathcal{M}\to CB_{p_{\mathfrak{b}}}(\mathcal{M})$ *be a MV*$\alpha\mathcal{F}$*-contraction mapping and* $B\subset(0,\infty)$ *with* $\inf B>0$. *Suppose that*

(1) *S is continuous;*
(2) *S is an α-admissible mapping;*
(3) *there exist $m_0\in\mathcal{M}$ and $m_1\in Sm_0$ such that $\alpha(m_0,m_1)\geq 1$;*
(4) *$\mathcal{F}(\inf B)=\inf\mathcal{F}(B)$, where $\mathcal{F}\in\Delta_f$ – family.*

Then S has a fixed point.

Proof We take an arbitrary $m_0\in\mathcal{M}$. As Sm, the set of all images of $m\in\mathcal{M}$, is nonempty for all values in $\mathcal{M}$, we can choose $m_1\in Sm_0$. If $m_1\in Sm_1$, this means that m_1 is a fixed point of S. So suppose $m_1\notin Sm_1$. As Sm_1 is closed, we have

$$p_{\mathfrak{b}}(m_1,Sm_1)>0.$$

Also, we know that

$$p_{\mathfrak{b}}(m_1,Sm_1)\leq\mathcal{H}_{p_{\mathfrak{b}}}(Sm_0,Sm_1).$$

We have

$$\mathcal{F}\big(p_{\mathfrak{b}}(m_1, Sm_1)\big) \le \mathcal{F}\big(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_0, Sm_1)\big), \text{by} F_2. \tag{2.15}$$

Using (*4*)

$$\mathcal{F}\big(p_{\mathfrak{b}}(m_1, Sm_1)\big) = \inf_{g \in Sm_1} \mathcal{F}\big(p_{\mathfrak{b}}(m_1, g)\big).$$

That is,

$$\inf_{g \in Sm_1} \mathcal{F}\big(p_{\mathfrak{b}}(m_1, g)\big) \le \mathcal{F}\big(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_0, Sm_1)\big). \tag{2.16}$$

As Sm_1 is compact, so we can find a $m_2 \in Sm_1$ such that

$$\inf_{g \in Sm_1} \mathcal{F}\big(p_{\mathfrak{b}}(m_1, g)\big) = \mathcal{F}\big(p_{\mathfrak{b}}(m_1, m_2)\big).$$

From (2.15),

$$\mathcal{F}\big(p_{\mathfrak{b}}(m_1, m_2)\big) \le \mathcal{F}\big(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_0, Sm_1)\big). \tag{2.17}$$

Similarly, for $m_3 \in Sm_2$, we get

$$\mathcal{F}\big(p_{\mathfrak{b}}(m_2, m_3)\big) \le \mathcal{F}\big(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_1, Sm_2)\big),$$

which ultimately gives

$$\mathcal{F}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big) \le \mathcal{F}\big(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1})\big).$$

As $\mathfrak{b} \ge 1$, so we can write

$$\mathcal{F}\big(\mathfrak{b}\big(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \le \mathcal{F}\big(\mathfrak{b}\big(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1})\big)\big). \tag{2.18}$$

For $m_0 \in \mathcal{M}$ by assumption, $\alpha(m_0, m_1) \ge 1$ for some $m_1 \in Sm_0$. Similarly, for some $m_2 \in Sm_1$, we have $\alpha(m_1, m_2) \ge 1$ and for any sequence $m_{\xi+1} \in Sm_{\xi}$, we may write

$$\alpha(m_{\xi}, m_{\xi+1}) \ge 1 \quad \text{for all } \xi \in \mathbb{N} \cup \{0\}. \tag{2.19}$$

Using (2.2), we have

$$\tau + \mathcal{F}\big(\alpha(m_{\xi}, m_{\xi+1})\big(\mathcal{H}_{p\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \le \mathcal{F}\big(\mathbb{M}(m_{\xi+1}, m_{\xi})\big),$$

The inequality (2.19) implies that

$$\tau + \mathcal{F}\big(\mathfrak{b}\big(\mathcal{H}_{p\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\big)\big) \le \mathcal{F}\big(\mathbb{M}(m_{\xi+1}, m_{\xi})\big).$$

Using (2.18), we have

$$\mathcal{F}\big(\mathfrak{b}\big(p_{\mathfrak{b}}(m_{\xi+1},m_{\xi+2})\big)\big) \le \mathcal{F}\big(\mathbb{M}(m_{\xi+1},m_{\xi})\big) - \tau. \tag{2.20}$$

Now, using Lemma 2.4, one writes

$$\lim_{\xi\to\infty} \mathfrak{b}^{\xi} \nu_{\xi} = 0,$$

Now, by (F_3), for any $\gamma \in (0,1)$ and for all $\xi \in \mathbb{N}$,

$$\lim_{\xi\to\infty} \big(\mathfrak{b}^{\xi}\nu_{\xi}\big)^{\gamma} \mathcal{F}\mathfrak{b}^{\xi}(\nu_{\xi}) = 0.$$

It implies that

$$\big(\mathfrak{b}^{\xi}\nu_{\xi}\big)^{\gamma}\big(\mathcal{F}\mathfrak{b}^{\xi}(\nu_{\xi}) - \mathcal{F}(\nu_0)\big) \le -\big(\mathfrak{b}^{\xi}\nu_{\xi}\big)^{\gamma}\xi\tau \le 0. \tag{2.21}$$

As $\tau > 0$, we have

$$\lim_{\xi\to\infty} \big(\mathfrak{b}^{\xi}\nu_{\xi}\big)^{\gamma}\xi = 0.$$

So there exists $\xi_1 \in \mathbb{N}$ such that $(\mathfrak{b}^{\xi}\nu_{\xi})^{\gamma}\xi \le 1$ for all $\xi \ge \xi_1$. Then

$$\mathfrak{b}^{\xi}\nu_{\xi} \le \frac{1}{\xi^{\frac{1}{\gamma}}}. \tag{2.22}$$

Next, we prove that $\{m_{\xi}\}$ is a Cauchy sequence in $\mathcal{M}$. For this, following the same steps as done in Theorem 2.5, one can easily have

$$\lim_{\xi\to\infty} p_{\mathfrak{b}}\big(m_{\xi},m^*\big) = p_{\mathfrak{b}}\big(m^*,m^*\big) = 0. \tag{2.23}$$

We claim that m^* is a fixed point of S. Suppose that $p_{\mathfrak{b}}(m^*,Sm^*) > 0$, this means there exists $k_0 \in \mathbb{N}$ such that we have $p_{\mathfrak{b}}(m_{\xi},Sm^*) > 0$ for all $\xi > k_0$. One writes

$$p_{\mathfrak{b}}\big(m_{\xi},Sm^*\big) \le \mathcal{H}_{p\mathfrak{b}}\big(Sm_{\xi+1},Sm^*\big).$$

Using (2.2) and taking limit $\xi \to \infty$, we have

$$\begin{aligned}\tau + \mathcal{F}\big(p_{\mathfrak{b}}\big(m^*,Sm^*\big)\big) &\le \tau + \mathcal{F}\big(\alpha\big(m^*,m^*\big)\mathcal{H}_{p\mathfrak{b}}\big(Sm^*,Sm^*\big)\big) \\ &\le \mathcal{F}\big(\mathbb{M}\big(m^*,m^*\big)\big) \\ &\le \mathcal{F}\big(p_{\mathfrak{b}}\big(m^*,Sm^*\big)\big),\end{aligned}$$

where

$$\mathbb{M}\big(m^*,m^*\big) = \max\Big\{p_{\mathfrak{b}}\big(m^*,m^*\big),p_{\mathfrak{b}}\big(m^*,Sm^*\big),p_{\mathfrak{b}}\big(m^*,Sm^*\big),$$

$$\left.\frac{p_{\mathfrak{b}}(m^*,Sm^*)+p_{\mathfrak{b}}(Sm^*,m^*)}{2\mathfrak{b}}\right\}$$
$$\leq p_{\mathfrak{b}}\big(m^*,Sm^*\big).$$

It implies that

$$\tau+\mathcal{F}\big(p_{\mathfrak{b}}\big(m^*,Sm^*\big)\big)\leq\mathcal{F}\big(p_{\mathfrak{b}}\big(m^*,Sm^*\big)\big).$$

Since $\tau>0$, the above relation yields a contradiction. Thus,

$$p_{\mathfrak{b}}\big(m^*,Sm^*\big)=0.$$

Also, $p_{\mathfrak{b}}(m^*,m^*)=0$. This gives $m^*\in\bar{S}m^*=Sm^*$. Hence, m^* is a fixed point of S. □

Example 2.8 Let $\mathcal{M}=\{m_\zeta=1-(\frac{1}{2})^\zeta:\zeta\in\mathbb{N}\}$ and $p_{\mathfrak{b}}:\mathcal{M}\times\mathcal{M}\to[0,\infty)$ be defined by

$$p_{\mathfrak{b}}(\zeta,\nu)=|\zeta-\nu|^2+\big[\max\{\zeta,\nu\}\big]^2\text{for all}\zeta,\nu\in\mathcal{M}.$$

One can easily verify that $(\mathcal{M},p_{\mathfrak{b}})$ is a complete P$\mathfrak{b}$MS with $\mathfrak{b}=2$. We also define a multivalued map $S:\mathcal{M}\to2^{\mathcal{M}}$ by

$$Sm=\begin{cases}\{m_1\}, & m=m_1,\\ \{m_\zeta,m_{\zeta+1}\}, & m=m_\zeta,\zeta=2,3,\ldots.\end{cases}$$

Consider $\alpha(m_\zeta,m_\nu)=1$ and $\mathbb{M}(m_\zeta,m_\nu)=p_{\mathfrak{b}}(m_\zeta,m_\nu)$. Take $\mathcal{F}:(0,\text{infty})\to\mathbb{R}$ as $\mathcal{F}(\zeta)=\ln(\zeta)+\zeta$. Hence, the contraction condition will take the following form:

$$\frac{\mathcal{H}_{p_{\mathfrak{b}}}(Sm_\zeta,Sm_\nu)}{\mathbb{M}(m_\zeta,m_\nu)}e^{\mathcal{H}_{p_{\mathfrak{b}}}(Sm_\zeta,Sm_\nu)-\mathbb{M}(m_\zeta,m_\nu)}\leq e^{-\tau}.\tag{2.24}$$

Now, we verify this condition for the following two possible cases:

Case I

If $\mathcal{H}_{p_{\mathfrak{b}}}(Sm_\zeta,Sm_1)>0$ and $\nu=1$, we have

$$\begin{aligned}\delta_{p_{\mathfrak{b}}}(Sm_\zeta,Sm_1)&=\max\big\{p_{\mathfrak{b}}(m_\zeta,Sm_1),p_{\mathfrak{b}}(m_{\zeta+1},Sm_1)\big\}\\&=\max\big\{|m_\zeta-m_1|^2+(m_\zeta)^2,|m_{\zeta+1}-m_1|^2+(m_{\zeta+1})^2\big\}\\&=|m_{\zeta+1}-m_1|^2+(m_{\zeta+1})^2.\end{aligned}$$

In the same manner,

$$\delta_{p_{\mathfrak{b}}}(Sm_1,Sm_\zeta)=|m_\zeta-m_1|^2+(m_\zeta)^2.$$

It implies that

$$\mathcal{H}_{p_{\mathfrak{b}}}(Sm_\zeta,Sm_1)=|m_{\zeta+1}-m_1|^2+(m_{\zeta+1})^2.\tag{2.25}$$

Also,

$$\mathbb{M}(m_\zeta, m_1) = |m_\zeta - m_1|^2 + (m_\zeta)^2 \le |m_\zeta - m_1|^2 + (m_{\zeta+2})^2. \tag{2.26}$$

One writes

$$\begin{aligned}
&\frac{\mathcal{H}_{p_\mathfrak{b}}(Sm_\zeta, Sm_1)}{\mathbb{M}(m_\zeta, m_1)} e^{(\mathcal{H}_{p_\mathfrak{b}}(Sm_\zeta, Sm_1) - \mathbb{M}(m_\zeta, m_1))} \\
&\le \frac{|m_{\zeta+1} - m_1|^2 + (m_{\zeta+1})^2}{|m_\zeta - m_1|^2 + (m_{\zeta+2})^2} e^{(|m_{\zeta+1}-m_1|^2 + (m_{\zeta+1})^2) - (|m_\zeta - m_1|^2 + (m_{\zeta+2})^2)} \\
&= \frac{|(\frac{1}{2}) - (\frac{1}{2})^{\zeta+1}|^2 + (1 - (\frac{1}{2})^{\zeta+1})^2}{|(\frac{1}{2}) - (\frac{1}{2})^{\zeta}|^2 + (1 - (\frac{1}{2})^{\zeta+2})^2} e^{|(\frac{1}{2}) - (\frac{1}{2})^{\zeta+1}|^2 + (1 - (\frac{1}{2})^{\zeta+1})^2 - |(\frac{1}{2}) - (\frac{1}{2})^{\zeta}|^2 - (1 - (\frac{1}{2})^{\zeta+2})^2} \\
&\le e^{(2(\frac{1}{2})^{2\zeta+2} + (\frac{1}{2})^{\zeta} - [(\frac{1}{2})^{2\zeta} + (\frac{1}{2})^{2\zeta+4} + 3(\frac{1}{2})^{\zeta+1} + 2(\frac{1}{2})^{\zeta+2}])} \\
&< e^{-\tau},
\end{aligned}$$

for some $\tau > 0$.

Case II

If $\mathcal{H}_{p_\mathfrak{b}}(Sm_\zeta, Sm_\nu) > 0$ with $\zeta \ge \nu > 1$, we have

$$\mathcal{H}_{p_\mathfrak{b}}(Sm_\zeta, Sm_\nu) = |m_{\zeta+1} - m_{\nu+1}|^2 + (m_{\zeta+1})^2,$$

and

$$\mathbb{M}(m_\zeta, m_\nu) = |m_\zeta - m_\nu|^2 + (m_\zeta)^2 \le |m_\zeta - m_\nu|^2 + (m_{\zeta+2})^2.$$

From (2.24), we have

$$\begin{aligned}
&\frac{\mathcal{H}_{p_\mathfrak{b}}(Sm_\zeta, Sm_\nu)}{\mathbb{M}(m_\zeta, m_\nu)} e^{(\mathcal{H}_{p_\mathfrak{b}}(Sm_\zeta, Sm_\nu) - \mathbb{M}(m_\zeta, m_\nu))} \\
&\le \frac{|m_{\zeta+1} - m_{\nu+1}|^2 + (m_{\zeta+1})^2}{|m_\zeta - m_\nu|^2 + (m_{\zeta+2})^2} e^{(|m_{\zeta+1}-m_{\nu+1}|^2 + (m_{\zeta+1})^2) - (|m_\zeta - m_\nu|^2 + (m_{\zeta+2})^2)} \\
&= \frac{|(\frac{1}{2})^{\nu+1} - (\frac{1}{2})^{\zeta+1}|^2 + (1 - (\frac{1}{2})^{\zeta+1})^2}{|(\frac{1}{2})^{\nu} - (\frac{1}{2})^{\zeta}|^2 + (1 - (\frac{1}{2})^{\zeta+2})^2} e^{|(\frac{1}{2})^{\nu+1} - (\frac{1}{2})^{\zeta+1}|^2 + (1 - (\frac{1}{2})^{\zeta+1})^2 - |(\frac{1}{2})^{\nu} - (\frac{1}{2})^{\zeta}|^2 - (1 - (\frac{1}{2})^{\zeta+2})^2} \\
&\le e^{(|(\frac{1}{2})^{\nu+1} - (\frac{1}{2})^{\zeta+1}|^2 + (1 - (\frac{1}{2})^{\zeta+1})^2) - (|(\frac{1}{2})^{\nu} - (\frac{1}{2})^{\zeta}|^2 + (1 - (\frac{1}{2})^{\zeta+2})^2)} \\
&< e^{-\tau},
\end{aligned}$$

which is true for all $\zeta, \nu \in \mathbb{N}$ provided that $\zeta \ge \nu > 1$, where $\tau > 0$. Thus, all the required conditions of Theorem 2.7 are satisfied. Here, the mapping S has a fixed point (m_1 and m_ζ are fixed points).

3 An application

Here, we apply our main result to find a solution to an integral equation of Fredholm type. Take $I = [0, 1]$. Denote by $\mathcal{M} = \mathcal{C}(I, \mathbb{R}^2)$ the space of all continuous functions defined from

I to $\mathbb{R}^2$. We endow $\mathcal{M}$ with the usual sup-norm. We consider a partial $\mathfrak{b}$ metric on $\mathcal{M}$ defined by

$$p_{\mathfrak{b}}(\phi,\psi) = \|\phi-\psi\|_\infty = \sup_{m\in I}\left\{e^{-mp}\left|\phi(m)-\psi(m)\right|^q\right\} \quad p,q>1,$$

for all $\phi,\psi\in\mathcal{M}$. It is easy to verify that $(\mathcal{M},p_{\mathfrak{b}})$ is a complete P$\mathfrak{b}$MS. Consider the Fredholm integral inclusion

$$\phi(\zeta)\in f(\zeta)+\int_0^1 k_\phi\big(\zeta,x^*,\phi(x^*)\big)\,dx^*, \tag{3.1}$$

such that for every $\mathcal{K}_\phi : I\times I\times\mathbb{R}^2\to K(\mathcal{M})$ there exists

$$k_\phi\big(\zeta,x^*,\phi^*\big)\in\mathcal{K}_\phi\big(\zeta,x^*,\phi^*\big).$$

Define a multivalued mapping $S:\mathcal{M}\to K(\mathcal{M})$ as

$$S\big(\phi(\zeta)\big)=\left\{\phi^*(\zeta):\phi^*(\zeta)\in\omega(\zeta)+\int_0^1\mathcal{K}_\phi\big(\zeta,x^*,\phi(x^*)\big)\,dx^*\right\}. \tag{3.2}$$

Theorem 3.1 *Suppose that the following conditions hold*:

(1) $\mathcal{K}_\phi : I\times I\times\mathbb{R}^2\to\mathbb{R}^2$ *and* $f:I\to\mathbb{R}^2$ *are continuous*;
(2) *there exists* $\phi_0\in\mathcal{M}$ *such that* $\phi_k\in S\phi_{k-1}$;
(3) *there exists a continuous function* $\mathfrak{f}:I\times I\to I$ *such that*

$$\left|k_\phi\big(\zeta,x^*,\phi(x^*)\big)-k_\psi\big(\zeta,x^*,\psi(x^*)\big)\right|^q\le\sup_{x^*\in I}\mathfrak{f}\big(\phi(x^*),\psi(x^*)\big)\left|\phi(x^*)-\psi(x^*)\right|^q,$$

for each $\zeta,x^*\in I$ *and* $\mathfrak{f}(\phi(x^*),\psi(x^*))\le\gamma$.

Then the integral inclusion (3.1) *has a solution*.

Proof Let $(\mathcal{M},p_{\mathfrak{b}})$ be a complete P$\mathfrak{b}$MS. We choose

$$\mathcal{F}(\zeta)=\ln(\zeta),$$

for all $\zeta\in(0,\infty)$. So after going through a natural logarithm, our condition will be

$$\mathcal{H}_{p_{\mathfrak{b}}}\big(S\big(\phi(\zeta),S\psi(\zeta)\big)\big)\le e^{-\tau}M(\phi,\psi),$$

with $\alpha(\phi,\psi)=1$. Next, to show that S satisfies this condition, let $p>1$ such that

$$\frac{1}{p}+\frac{1}{q}=1,$$

then for $\phi^*\in S(\phi)$, we have

$$\begin{aligned}p_{\mathfrak{b}}\big(\big(\phi^*(\zeta),S\big(\psi(\zeta)\big)\big)\big)&\le p_{\mathfrak{b}}\big(\phi^*(\zeta),\big(\psi^*(\zeta)\big)\big)\\&=\sup_{\zeta\in I}e^{-\zeta\gamma}\left|\phi^*(\zeta)-\psi^*(\zeta)\right|^q\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\zeta\in I} e^{-\zeta\gamma}\left|\int_0^1 k_\phi\big(\zeta,x^*,\phi(x^*)\big) - k_\psi\big(\zeta,x^*,\psi(x^*)\big)\right|^q dx^* \\
&\leq \sup_{\zeta\in I} e^{-\zeta\gamma}\left[\left(\int_0^1 |1|^p\,dx^*\right)^{\frac{1}{p}} \int_0^1 \left(\left|k_\phi\big(\zeta,x^*,\phi(x^*)\big) - k_\psi\big(\zeta,x^*,\psi(x^*)\big)\right|^q\right)^{\frac{1}{q}}\right]^q dx^* \\
&= \sup_{\zeta\in I} e^{-\zeta\gamma}\int_0^1 \left|k_\phi\big(\zeta,x^*,\phi(x^*)\big) - k_\psi\big(\zeta,x^*,\psi(x^*)\big)\right|^q dx^* \\
&= \sup_{\zeta\in I} e^{-\zeta\gamma}\int_0^1 \left|e^{-x^*\gamma+x^*\gamma}k_\phi\big(\zeta,x^*,\phi(x^*)\big) - k_\psi\big(\zeta,x^*,\psi(x^*)\big)\right|^q dx^* \\
&\leq \sup_{\zeta\in I} e^{-\zeta\gamma}\int_0^1 e^{x^*\gamma}\mathfrak{f}\big(\phi(x^*),\psi(x^*)\big)\sup_{x^*\in I} e^{-x^*\gamma}\left|\phi(x^*) - \psi(x^*)\right|^q dx^* \\
&= \gamma\left\|\phi(x^*) - \psi(x^*)\right\|_\infty \sup_{\zeta\in I} e^{-\zeta\gamma}\int_0^1 e^{x^*\gamma}\,dx^* \\
&= p_{\mathfrak{b}}\big(\phi(x^*),\psi(x^*)\big)(1)\big(e^\gamma - 1\big) \\
&\leq p_{\mathfrak{b}}\big(\phi(x^*),\psi(x^*)\big)e^\gamma \\
&\leq e^\gamma\mathbb{M}\big(\phi(x^*),\psi(x^*)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbb{M}\big(\phi(x^*),\psi(x^*)\big) \\
&\quad= \max\Big\{p_{\mathfrak{b}}\big(\phi(x^*),\psi(x^*)\big), p_{\mathfrak{b}}\big(\phi(x^*),S(\phi(x^*))\big), p_{\mathfrak{b}}\big(\psi(x^*),S(\psi(x^*))\big), \\
&\qquad \frac{p_{\mathfrak{b}}(\phi(x^*),S(\psi(x^*))) + p_{\mathfrak{b}}(\psi(x^*),S(\phi(x^*)))}{2\mathfrak{b}}\Big\}.
\end{aligned}
$$

Also, as ϕ^* is arbitrary, we have

$$\delta_{p_{\mathfrak{b}}}\big(S(\phi),S(\psi)\big) \leq e^\gamma\mathbb{M}(\phi,\psi).$$

Similarly, one finds

$$\delta_{p_{\mathfrak{b}}}\big(S(\psi),S(\phi)\big) \leq e^\gamma\mathbb{M}(\psi,\phi).$$

Then

$$\mathcal{H}_{p_{\mathfrak{b}}}\big(S(\phi),S(\psi)\big) \leq e^\gamma\mathbb{M}(\phi,\psi).$$

That is, $\mathcal{H}_{p_{\mathfrak{b}}}(S(\phi),S(\psi)) \leq e^{-\tau}M(\phi,\psi)$.

Our desired contraction condition is then satisfied by choosing $-\tau = \gamma$. Thus, all conditions of Theorem 2.5 are satisfied, and so the integral inclusion (3.1) has a solution, and 0 is a fixed point of S. □

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