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Pointwise convergence of sequential Schrödinger means

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Abstract

We study pointwise convergence of the fractional Schrödinger means along sequences t_n that converge to zero. Our main result is that bounds on the maximal function $\sup_n |e^{it_n(-\Delta)^{\alpha/2}} f|$ can be deduced from those on $\sup_{0 < t \leq 1} |e^{it(-\Delta)^{\alpha/2}} f|$, when $\{t_n\}$ is contained in the Lorentz space $\ell^{r, \infty}$. Consequently, our results provide seemingly optimal results in higher dimensions, which extend the recent work of Dimou and Seeger, and Li, Wang, and Yan to higher dimensions. Our approach based on a localization argument also works for other dispersive equations and provides alternative proofs of previous results on sequential convergence.

MSC: 42B25; 35Q41; 35S10

Keywords: Pointwise convergence; Schrödinger operator

1 Introduction

Let $\alpha > 0$. We consider the fractional Schrödinger operator

$$e^{it(-\Delta)^{\alpha/2}} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|^\alpha)} \widehat{f}(\xi) d\xi. \tag{1.1}$$

A classical problem posed by Carleson [5] is to determine the optimal regularity s for which

$$\lim_{t \rightarrow 0} e^{it(-\Delta)^{\alpha/2}} f = f \quad \text{a.e. } \forall f \in H^s, \tag{1.2}$$

where H^s denotes the inhomogeneous Sobolev spaces of order s with its norm $\|f\|_{H^s(\mathbb{R}^d)} = \|(1 + |\cdot|^2)^{s/2} \widehat{f}\|_{L^2(\mathbb{R}^d)}$. The case when $\alpha = 2$ has been extensively studied previously. When $d = 1$, it was shown by the work of Carleson [5] and Dahlberg and Kenig [10] that (1.2) holds true if and only if $s \geq 1/4$. In higher dimensions, the problem turned out to be more difficult. Progress was made by the contributions of numerous authors. Sjölin [32] and Vega [39] independently obtained (1.2) for $s > 1/2$. In particular, further improvement on the required regularity was made by Moyua, Vargas, and Vega [26] and Tao and Vargas [38] when $d = 2$, and convergence for $s > (2d - 1)/4d$ was shown by Lee [19] for $d = 2$ and Bourgain [3] in higher dimensions. Bourgain [4] showed that (1.2) holds only if $s \geq$

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$d/2(d + 1)$. The lower bound was shown to be sufficient for (1.2) by Du, Guth, and Li [12] for $d = 2$, and Du and Zhang [14] for $d \geq 3$ except for the endpoint case $s = d/2(d + 1)$ (also, see [23] for earlier results and references therein).

In general, (1.2) continues to be true for $\alpha > 1$ if $s > d/2(d + 1)$ (see, e.g., [7, 32], also see [25]). If $\alpha = 1$, it is easy to show that (1.2) holds if and only if $s > 1/2$ in all dimensions ([9, 41]). When $0 < \alpha < 1$, it is known that (1.2) holds if $s > \alpha/4$ and the convergence fails if $s < \alpha/4$ in \mathbb{R}^1 . However, only partial results are known in higher dimensions, i.e., (1.2) holds true for $s > \alpha/2$ and fails for $s < \alpha/4$ when $d \geq 2$ (see [40, 41]).

1.1 Convergence along sequences

Recently, pointwise convergence along sequences was considered by several authors [11, 34–37]. More precisely, the problem under consideration is to determine the regularity exponent s such that, for a given sequence $\{t_n\}$ satisfying $\lim_{n \rightarrow \infty} t_n = 0$,

$$\lim_{n \rightarrow \infty} e^{it_n(-\Delta)^{\alpha/2}} f(x) = f(x) \quad \text{a.e. } x, \forall f \in H^s. \tag{1.3}$$

Naturally, one may expect that the more rapidly the sequence $\{t_n\}$ converges to zero, the less regularity is required to guarantee almost-everywhere convergence. To quantify how fast the sequence converges to zero, the sequences in $\ell^r(\mathbb{N})$ and $\ell^{r,\infty}(\mathbb{N})$ were considered, where $\ell^{r,\infty}(\mathbb{N})$ denotes the Lorentz space

$$\ell^{r,\infty}(\mathbb{N}) := \left\{ t_n : \sup_{\delta > 0} \delta^r \#\{n \in \mathbb{N} : |t_n| \geq \delta\} < \infty \right\}$$

for $r < \infty$. Note that $\{n^{-b}\} \in \ell^{r,\infty}(\mathbb{N})$ if and only if $b \geq 1/r$. In particular, Dimou and Seeger [11] studied the almost-everywhere convergence problem in \mathbb{R}^1 using $\ell^{r,\infty}(\mathbb{N})$. They proved that (1.3) holds for all $f \in H^s$ if and only if $s \geq \min\{\frac{r\alpha}{4r+2}, \frac{1}{4}\}$ (when $\alpha > 1$), $s \geq \frac{r\alpha}{4r+2}$ (when $0 < \alpha < 1$), and $s \geq \frac{r}{2(r+1)}$ (when $\alpha = 1$) for a strictly decreasing convex sequence $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$. There are also results in higher dimensions by Sjölin [34] and Sjölin and Strömberg [35–37]. Recently, Li, Wang, and Yan [21], relying on the bilinear approach in [19], obtained some partial results for the case $d = \alpha = 2$ without assuming that $\{t_n\}$ decreases.

1.2 Maximal estimates

In the study of pointwise convergence the associated maximal functions play important roles. By a standard argument (1.2) follows if we have

$$\left\| \sup_{0 < t \leq 1} |e^{it(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1))} \leq C \|f\|_{H^s}, \tag{1.4}$$

where $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$. Likewise, (1.3) follows from the estimate

$$\left\| \sup_{t_n} |e^{it_n(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1))} \leq C \|f\|_{H^s}, \tag{1.5}$$

which is, in fact, essentially equivalent to (1.3) by Stein’s maximal principle. Our first result shows that the maximal estimate (1.5) can be deduced from (1.4) when $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$.

Theorem 1.1 *Let $d \geq 1, \alpha > 0, s_* > 0$ and $0 < r < \infty$. Suppose (1.4) holds for $s \geq s_*$. Then, if $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$, (1.5) holds provided*

$$s \geq \min \left\{ \frac{r\alpha}{r \min\{\alpha, 1\} + 2s_*} s_*, s_* \right\}. \tag{1.6}$$

Thanks to Theorem 1.1 we can improve the previous results and obtain seemingly optimal results for the convergence of sequential Schrödinger means in higher dimensions. For $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$ and $d = 1$, the known estimates (1.4) ([5, 32, 40]) and Theorem 1.1 give (1.5) for $s \geq \min(r\alpha/(4r + 2), 1/4)$ when $\alpha > 1$, and for $s > r\alpha/(4r + 2)$ when $0 < \alpha < 1$. This recovers the result (sufficiency part except the endpoint case when $0 < \alpha < 1$) in [11] without the assumption that $\{t_n\}$ decreases.

In higher dimensions $d \geq 2$, by combining Theorem 1.1 and recent progress on the maximal bounds, i.e., (1.4) for $\alpha > 1$ and $s > \frac{d}{2(d+1)}$ [7, 12, 14], we have the estimate (1.5) for

$$s > \min \left\{ \frac{\alpha dr}{2(d+1)r + 2d}, \frac{d}{2(d+1)} \right\} \tag{1.7}$$

whenever $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$. As a consequence, we have the following result on pointwise convergence.

Corollary 1.2 *Let $d \geq 2, \alpha > 1$ and $0 < r < \infty$. For any sequence $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$, (1.3) holds for all $f \in H^s(\mathbb{R}^d)$ if (1.7) holds.*

This improves the previous results in [21, 36]. We expect that the regularity exponent given in (1.7) is sharp up to the endpoint case. However, we are not able to verify this for the moment.

Remark 1.3 As mentioned before, when $0 < \alpha < 1$ and $d \geq 2$, it is known that (1.2) holds if $s > \alpha/2$ ([41]). Thus, Theorem 1.1 yields (1.3) for $s > \frac{r\alpha}{2(r+1)}$. The implication in Theorem 1.1 also works for more general operators (see Remark 2.3). In particular, one can also recover the result of Li, Wang, and Yan [22] for the nonelliptic Schrödinger operator by combining Theorem 1.1 with the results in [29].

Remark 1.4 For the wave operator, i.e., $\alpha = 1$, (1.3) holds true if and only if $s \geq \frac{r}{2(r+1)}$ for $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$. When $d = 1$, this was shown in [11]. In higher dimensions, one can show this using Theorem 1.1 (also Corollary 3.2). The sharpness can be obtained by following the argument in [11]. We remark that (1.3) is closely related to L^p boundedness of the spherical maximal operator given by taking the supremum over more general sets (see [1, 30, 31]).

1.3 Localization argument

The proof of Theorem 1.1 relies on a localization argument. We briefly explain our approach. From Littlewood–Paley decomposition, the proof of (1.3) can be reduced to showing

$$\left\| \sup_{t_n} |e^{it_n(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1))} \leq CR^s \|f\|_{L^2(\mathbb{R}^d)}, \tag{1.8}$$

where \widehat{f} is supported in $\mathbb{A}_R := \{\xi : R \leq |\xi| \leq 2R\}$ (see Sect. 3). In the previous work [11, 34–36] the estimate (1.8) was obtained by relying on the kernel estimates. In contrast, we deduce (1.8) directly from (1.4). Clearly, (1.4) gives

$$\left\| \sup_{0 < t \leq 1} |e^{it(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1))} \leq CR^s \|f\|_{L^2(\mathbb{R}^d)} \tag{1.9}$$

for $R \geq 1$ whenever \widehat{f} is supported in \mathbb{A}_R . Using the estimate and a localization argument, we first obtain from (1.9) a temporally localized maximal estimate

$$\left\| \sup_{t \in I} |e^{it(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1))} \leq C' (1 + R^\alpha |I|)^{\max\{s, \frac{s}{\alpha}\}} \|f\|_{L^2(\mathbb{R}^d)} \tag{1.10}$$

for $R \geq 1$ and any subinterval $I \subset [0, 1]$ with $|I| \leq R^{1-\alpha}$ when \widehat{f} is supported in \mathbb{A}_R . Moreover, the converse implication from (1.10) to (1.9) is also true as long as $R^{-\alpha} < |I| \leq R^{1-\alpha}$ (see Lemma 2.2 for detail). Once we have (1.10), we can obtain (1.5) by following the argument in [11].

If the exponent s in the estimate (1.9) is sharp, then the same is true for the estimate (1.10). For instance, when $\alpha = 2$, (1.9) holds for $s > d/2(d + 1)$, which is optimal up to the endpoint case, and hence so does (1.10) for the same s . When $\alpha > 1$ and $|I| \geq R^{1-\alpha}$, one can see the exponent s in (1.10) can not be smaller than that in (1.9) using the localization lemma in [19] (cf. [8, 20, 28]).

To show the implication from (1.9) to (1.10), we adapt the idea of the temporal localization lemma in [8, 19]. We establish a spatial localization lemma (Lemma 2.4), which plays a crucial role in proving Theorem 1.1. More precisely, we show that the local-in-spatial estimate (1.9) can be extended to the global-in-spatial estimate with the same regularity exponent. After a suitable scaling, we obtain the temporal localized estimate (1.10) from the global-in-spatial estimate.

1.4 Extension to fractal measure

Maximal estimates relative to general measures (instead of the Lebesgue measure) have been used to obtain a more precise description on the pointwise behavior of the Schrödinger mean $e^{it(-\Delta)^{\alpha/2}} f$. For a given sequence $\{t_n\}$ converging to zero, we consider

$$D^{\alpha,d}(f, \{t_n\}) = \{x \in \mathbb{R}^d : e^{it_n(-\Delta)^{\alpha/2}} f(x) \not\rightarrow f(x) \text{ as } t_n \rightarrow 0\}$$

and set

$$\mathfrak{D}^{\alpha,d}(s, r) = \sup_{f \in H^s, \{t_n\} \in \ell^r, \infty} \dim_H D^{\alpha,d}(f, \{t_n\}),$$

where \dim_H denotes the Hausdorff dimension. One can compare $\mathfrak{D}^{\alpha,d}(s, r)$ with the dimension of the divergence set

$$\mathfrak{D}^{\alpha,d}(s) = \sup_{f \in H^s} \dim_H \{x \in \mathbb{R}^d : e^{it(-\Delta)^{\alpha/2}} f(x) \not\rightarrow f(x) \text{ as } t \rightarrow 0\}.$$

The bounds on $\mathfrak{D}^{\alpha,d}(s)$ can be obtained by the maximal estimate relative to general measures (see, for example, [2, 14, 17]), to which the fractal Strichartz estimates studied in [6, 13, 18] are closely related (also see [15, 24, 42]).

The implication in Theorem 1.1 also extends to the maximal estimates relative to general fractal measures, so we can make use of the known estimates for the L^2 -fractal maximal estimates and the fractal Strichartz estimates to obtain the upper bounds on $\mathfrak{D}^{\alpha,d}(s,r)$, $0 < r < \infty$. We discuss this in detail in Sect. 3.2.

1.4.1 Organization of the paper

In Sect. 2, we deduce from (1.4) temporally localized maximal estimates in Lemma 2.2 (relative to the general measure) that are to be used to prove Theorem 1.1. We prove Theorem 1.1 and discuss upper bounds on the dimension of divergence sets in Sect. 3.

1.4.2 Notations

Throughout this paper, a generic constant $C > 0$ depends only on dimension d , which may change from line to line. If a constant depends on some other values (e.g., ϵ), we denote it by C_ϵ . The notation $A \lesssim B$ denotes $A \leq CB$ for a constant $C > 0$, and we denote by $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We often denote $L^2(\mathbb{R}^d)$ by L^2 , and similarly $H^s(\mathbb{R}^d)$ by H^s .

2 Temporally localized maximal estimates

In this section, we prove that the estimates (1.9) and (1.10) are equivalent. For later use, we consider the equivalence in a more general setting, that is to say, in the form of estimates relative to fractal measures (Lemma 2.2). To do this, we recall the following.

Definition 2.1 Let $0 < \gamma \leq d$ and let μ be a nonnegative Borel measure. We say μ is γ -dimensional if there is a constant C_μ such that

$$\mu(B(x,r)) \leq C_\mu r^\gamma, \quad \forall (x,r) \in \mathbb{R}^d \times \mathbb{R}_+. \tag{2.1}$$

By $\langle \mu \rangle_\gamma$ we denote the infimum of such a constant C_μ .

We first deduce a temporally localized maximal estimate from the estimate

$$\left\| \sup_{0 < t \leq 1} |e^{it(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1),d\mu)} \leq CR^{\frac{d-\gamma}{2}+s} \langle \mu \rangle_\gamma^{\frac{1}{\gamma}} \|f\|_{L^2}, \tag{2.2}$$

which holds whenever \widehat{f} is supported on \mathbb{A}_R .

Lemma 2.2 Let $p \geq 2$, $R \geq 1$, $\alpha > 0$, $0 < \gamma \leq d$. Let $I \subset [0, 1]$ be an interval such that $|I| \leq \min\{R^{1-\alpha}, 1\}$. Suppose that (2.2) holds for some s whenever \widehat{f} is supported in \mathbb{A}_R and μ is a γ -dimensional measure in \mathbb{R}^d . Then, for any γ -dimensional measure $\widetilde{\mu}$ in \mathbb{R}^d , there is a constant $C' > 0$ such that

$$\left\| \sup_{t \in I} |e^{it(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1),d\widetilde{\mu})} \leq C' \langle \widetilde{\mu} \rangle_\gamma^{\frac{1}{\gamma}} R^{\frac{d-\gamma}{2}} (1 + R^\alpha |I|)^{\max\{s, \frac{s}{\alpha}\}} \|f\|_{L^2} \tag{2.3}$$

holds whenever \widehat{f} is supported on \mathbb{A}_R . Conversely, if (2.3) holds for \widehat{f} supported on \mathbb{A}_R , $\widetilde{\mu}$ is γ -dimensional, and interval $I \subset [0, 1]$ satisfies $R^{-\alpha} < |I| \leq \min\{R^{1-\alpha}, 1\}$, then there exists $C > 0$ such that (2.2) holds whenever \widehat{f} is supported on \mathbb{A}_R and μ is γ -dimensional.

Remark 2.3 By a simple modification of our argument, Lemma 2.2 can be extended to a class of evolution operators $e^{itP(D)}$ as long as

$$|\partial_\xi^\beta P(\xi)| \lesssim |\xi|^{\alpha-|\beta|}, \quad \forall \beta$$

and $|\nabla P(\xi)| \gtrsim |\xi|^{\alpha-1}$ hold (see [8]). Hence, an analog of Theorem 1.1 holds true for $e^{itP(D)}$. A typical example of such an operator is the nonelliptic Schrödinger operator $e^{it(\partial_{x_1}^2 - \partial_{x_2}^2 \pm \partial_{x_3}^2 \pm \dots \pm \partial_{x_d}^2)}$.

The rest of the section is devoted to the proof of Lemma 2.2, for which we first consider spatial localization.

2.1 Spatial localization

By adapting the argument in [8, 19], we prove a spatial-localization lemma exploiting rapid decay of the kernel.

Lemma 2.4 *Let $\alpha > 0$, $r \geq 1$, and $0 < \gamma \leq d$. Let μ be a γ -dimensional measure in \mathbb{R}^d . For $R \geq 1$, we set $I_R = [0, R]$. Suppose that*

$$\|e^{it(-\Delta)^{\alpha/2}} f\|_{L_x^2(B(0,R), d\mu; L_t^r(I_R))} \leq CR^s \langle \mu \rangle_\gamma^{\frac{1}{2}} \|f\|_{L^2} \tag{2.4}$$

holds for some $s \in \mathbb{R}$ whenever \widehat{f} is supported in \mathbb{A}_1 . Then, there exists a constant $C_1 > 0$ such that

$$\|e^{it(-\Delta)^{\alpha/2}} f\|_{L_x^2(\mathbb{R}^d, d\mu; L_t^r(I_R))} \leq C_1 R^s \langle \mu \rangle_\gamma^{\frac{1}{2}} \|f\|_{L^2} \tag{2.5}$$

holds whenever \widehat{f} is supported in \mathbb{A}_1 .

Proof Let P be a projection operator defined by $\mathcal{F}(Pg)(\xi) = \beta(|\xi|)\widehat{g}(\xi)$, where $\beta \in C_c(2^{-1}, 2^2)$ and $\beta = 1$ on $[1, 2]$. Let $\{B\}$ be a collection of finitely overlapping balls of radius R that cover \mathbb{R}^d . Denote $\widetilde{B} = B(a, 10\alpha 2^{2|\alpha-1|}R)$ if $B = B(a, R)$. Then, we note that $\|F\|_{L^2(\mathbb{R}^d)}^2 \lesssim \sum_B \|F\|_{L^2(B)}^2$.

Since $Pf = f$, by Minkowski's inequality we have

$$\|e^{it(-\Delta)^{\alpha/2}} f\|_{L_x^2(\mathbb{R}^d, d\mu; L_t^r(I_R))}^2 \lesssim (\mathcal{L}_1 + \mathcal{L}_2),$$

where

$$\mathcal{L}_1 = \sum_B \|e^{it(-\Delta)^{\alpha/2}} P(\chi_{\widetilde{B}} f)\|_{L_x^2(B, d\mu; L_t^r(I_R))}^2,$$

$$\mathcal{L}_2 = \sum_B \|e^{it(-\Delta)^{\alpha/2}} P(\chi_{\widetilde{B}^c} f)\|_{L_x^2(B, d\mu; L_t^r(I_R))}^2.$$

Note that $e^{it(-\Delta)^{\alpha/2}} Pf = K(\cdot, t) * f$, where K is given by

$$K(x, t) = \int e^{i(x \cdot \xi + t|\xi|^\alpha)} \beta(|\xi|) d\xi.$$

By integration by parts, it is easy to see that $|K(x, t)| \leq C_N R^{-N} (1 + |x|)^{-N}$ for any $N \geq 1$ if $|x| > 10\alpha 2^{2|\alpha-1|}R$ and $|t| \leq R$. Thus, we have

$$\|e^{it(-\Delta)^{\alpha/2}}(\chi_{\tilde{B}^c} f)(x)\|_{L^2_t(\mathbb{R})} \leq C_N R^{-N} (1 + |\cdot|)^{-N} * |f|(x)$$

for any $N \geq 0$ whenever $x \in B$. Taking N large enough, we obtain

$$\mathcal{L}_2 \leq CR^{2s} \|(1 + |\cdot|)^{-(d+\gamma)} * |f|\|_{L^2_x(\mathbb{R}^d, d\mu)}^2.$$

By Schur’s test, it follows that $\|(1 + |\cdot|)^{-(d+\gamma)} * |f|\|_{L^2_x(B, d\mu)} \leq C \langle \mu \rangle_\gamma^{\frac{1}{2}} \|f\|_2$. Therefore, we only need to consider \mathcal{L}_1 .

Applying (2.4) on each B , we obtain

$$\mathcal{L}_1 \lesssim R^{2s} \langle \mu \rangle_\gamma \sum_B \|\chi_{\tilde{B}} f\|_{L^2}^2 \leq CR^{2s} \langle \mu \rangle_\gamma \|f\|_{L^2}^2.$$

The last inequality follows since the balls \tilde{B} overlap finitely. This completes the proof. \square

2.2 Proof of Lemma 2.2

To prove Lemma 2.2, we invoke an elementary lemma.

Lemma 2.5 ([17]) *Let μ be a γ -dimensional measure in \mathbb{R}^d . If \widehat{F} is supported on $B(0, R)$, then $\|F\|_{L^2(d\mu)} \leq CR^{\frac{d-\gamma}{2}} \langle \mu \rangle_\gamma^{\frac{1}{2}} \|F\|_{L^2(\mathbb{R}^d)}$.*

By translation and Plancherel’s theorem, we may assume that $I = [0, \delta]$ with $\delta \leq \min\{R^{1-\alpha}, 1\}$. We may further assume $R^{-\alpha} < \delta$ since (2.3) follows by the Sobolev embedding and Lemma 2.5 if $\delta \leq R^{-\alpha}$.

For a given γ -dimensional measure μ , we denote by μ_R the measure defined by the relation¹

$$\int F(x) d\mu_R(x) = R^\gamma \int F(Rx) d\mu(x), \quad F \in C_0(\mathbb{R}^d). \tag{2.6}$$

It is easy to see that μ_R is a γ -dimensional measure in \mathbb{R}^d such that

$$\mu_R(B(x, r)) \leq C \langle \mu \rangle_\gamma r^\gamma$$

for some $C > 0$. Changing variables $(x, t) \rightarrow (R^{-1}x, R^{-\alpha}t)$ and $\xi \rightarrow R\xi$, we see that (2.2) is equivalent to

$$\left\| \sup_{t \in [0, R^\alpha]} |e^{it(-\Delta)^{\alpha/2}} f_R| \right\|_{L^2(B(0, R), d\mu_R)} \leq CR^s \langle \mu \rangle_\gamma^{\frac{1}{2}} \|f_R\|_{L^2}, \tag{2.7}$$

where $\widehat{f_R}(\xi) = R^{\frac{d}{2}} \widehat{f}(R\xi)$. Note that $\|f_R\|_2 = \|f\|_2$ and $\widehat{f_R}$ is supported on \mathbb{A}_1 . Let us denote $R' = \min\{R, R^\alpha\}$. We claim that the estimate (2.7) is equivalent to the seemingly weaker

¹ μ_R is a positive Borel measure by the Riesz representation theorem.

estimate

$$\left\| \sup_{t \in [0, R']} |e^{it(-\Delta)^{\alpha/2}} g| \right\|_{L^2(B(0, R'), d\mu_R)} \leq CR^s \langle \mu \rangle_{\gamma}^{\frac{1}{2}} \|g\|_{L^2} \tag{2.8}$$

for $R \geq 1$ whenever \widehat{g} is supported on \mathbb{A}_1 . To show this, we only need to prove that (2.8) implies (2.7) since the converse is trivially true. When $\alpha > 1$, the implication from (2.8) to (2.7) was shown in [8] (also, see [19]) when μ_R is the Lebesgue measure and α is an integer. It is easy to see that the argument in [8] works for the general γ -dimensional measure μ_R . When $0 < \alpha \leq 1$, using Lemma 2.4 with R replaced by R^α , we obtain (2.7) from (2.8). This proves the claim.

We now show that (2.2) and (2.3) are equivalent. Recall that we are assuming that $R^{-\alpha} < \delta$. Changing variables $(x, t) \rightarrow (R^{-1}x, R^{-\alpha}t)$, $\xi \rightarrow R\xi$ in (2.3) as above, we see that (2.3) is equivalent to

$$\left\| \sup_{t \in [0, R^\alpha \delta]} |e^{it(-\Delta)^{\alpha/2}} f_R| \right\|_{L^2(B(0, R), d\widetilde{\mu}_R)} \leq C'(R^\alpha \delta)^{\max\{s, \frac{s}{\alpha}\}} \langle \widetilde{\mu} \rangle_{\gamma}^{\frac{1}{2}} \|f_R\|_{L^2}. \tag{2.9}$$

By Lemma 2.4 with R replaced by $R^\alpha \delta$, (2.9) follows from

$$\left\| \sup_{t \in [0, R^\alpha \delta]} |e^{it(-\Delta)^{\alpha/2}} f_R| \right\|_{L^2(B(0, R^\alpha \delta), d\widetilde{\mu}_R)} \leq C'(R^\alpha \delta)^{\max\{s, \frac{s}{\alpha}\}} \langle \widetilde{\mu} \rangle_{\gamma}^{\frac{1}{2}} \|f\|_{L^2}. \tag{2.10}$$

Thus, (2.10) and (2.9) are trivially equivalent. Therefore, to show the equivalence of (2.2) and (2.3), it is enough to prove that of (2.8) and (2.10). Indeed, it is clear that (2.10) follows from (2.8) by replacing R' in (2.8) with $R^\alpha \delta$. Conversely, if we replace $R^\alpha \delta$ in (2.10) with R' , we obtain (2.8) as long as $\delta > R^{-\alpha}$.

3 Maximal estimate for sequential Schrödinger means

In this section, we prove Theorem 1.1 and obtain results regarding upper bounds on the dimension of the divergence set of $e^{it_n(-\Delta)^{\alpha/2}} f$. The results are a consequence of extension of the maximal estimates to a general measure, see Sect. 3.2.

3.1 L^2 -Maximal estimates

Making use of Lemma 2.2, we deduce the maximal estimates for the sequential Schrödinger mean from the estimate (2.2).

Proposition 3.1 *Let $R \geq 1, \alpha > 0$, and $0 < r < \infty$. Suppose that (2.2) holds for some $s = s_* > 0$ whenever μ is a γ -dimensional measure in \mathbb{R}^d and $\text{supp} \widehat{f} \subset \mathbb{A}_R$. Let*

$$\widetilde{s}_* = \frac{r\alpha}{r \min\{\alpha, 1\} + 2s_*} s_*. \tag{3.1}$$

Then, if $\{t_n\} \in \ell^{r, \infty}$, there is a constant $C > 0$ such that

$$\left\| \sup_n |e^{it_n(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0, 1), d\mu)} \leq CR^{\frac{d-\gamma}{2} + s} \langle \mu \rangle_{\gamma}^{\frac{1}{2}} \|f\|_{L^2} \tag{3.2}$$

holds for $s \geq \min\{s_, \widetilde{s}_*\}$ whenever μ is γ -dimensional and $\text{supp} \widehat{f} \subset \mathbb{A}_R$.*

When $\alpha > 1$, Proposition 3.1 is meaningful only for $r < 2s_*/(\alpha - 1)$.

Proof of Proposition 3.1 We may assume $\tilde{s}_* \leq s_*$ since (3.2) trivially holds for $s \geq s_*$ by the maximal estimate (2.2). For $0 < \delta < 1$, let us set

$$N(\delta) = \{n \in \mathbb{N} : t_n < \delta\}.$$

Since $\{t_n\} \in \ell^{r,\infty}(\mathbb{N})$, there is a uniform constant $C_0 > 0$ such that

$$|N(\delta)^c| \leq C_0 \delta^{-r}. \tag{3.3}$$

Then, it follows that

$$\left\| \sup_n |e^{it_n(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1),\mu)} \leq \mathcal{I} + \mathcal{II},$$

where

$$\begin{aligned} \mathcal{I} &= \left\| \sup_{n \in N(\delta)} |e^{it_n(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1),d\mu)}, \\ \mathcal{II} &= \left\| \sup_{n \in N(\delta)^c} |e^{it_n(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1),d\mu)}. \end{aligned}$$

We consider \mathcal{I} first. Since $\sup_{n \in N(\delta)} |e^{it_n(-\Delta)^{\alpha/2}} f| \leq \sup_{0 < t \leq \delta} |e^{it(-\Delta)^{\alpha/2}} f|$, by Lemma 2.2 we have

$$\mathcal{I} \leq CR^{\frac{d-\gamma}{2}} (R^\alpha \delta)^{\max\{s_*, \frac{s_*}{\alpha}\}} \langle \mu \rangle^{\frac{1}{\gamma}} \|f\|_{L^2} \tag{3.4}$$

provided that $\text{supp} \hat{f} \subset \mathbb{A}_R$ and $R^{-\alpha} \leq \delta \leq \min\{R^{-\alpha+1}, 1\}$. To deal with \mathcal{II} , we first note that

$$\|e^{it_n(-\Delta)^{\alpha/2}} f\|_{L^2(B(0,1),d\mu)} \lesssim R^{\frac{d-\gamma}{2}} \langle \mu \rangle^{\frac{1}{\gamma}} \|f\|_2,$$

which follows by Lemma 2.5 and Plancherel’s theorem. Hence, by the embedding $\ell^2 \subset \ell^\infty$, combining the above estimate and (3.3), we obtain

$$\mathcal{II} \lesssim \left(\sum_{n \in N(\delta)^c} \|e^{it_n(-\Delta)^{\alpha/2}} f\|_{L^2(B(0,1),d\mu)} \right)^{1/2} \lesssim C_0^{\frac{1}{2}} \delta^{-\frac{r}{2}} R^{\frac{d-\gamma}{2}} \langle \mu \rangle^{\frac{1}{\gamma}} \|f\|_{L^2}.$$

Now, we prove (3.2) by optimizing the estimates with a suitable choice of δ . When $\alpha \geq 1$, we take $\delta = R^{-2s_*\alpha/(r+2s_*)}$, which gives (3.2) for $s \geq rs_*\alpha/(r + 2s_*)$. When $0 < \alpha < 1$, we choose $\delta = R^{-2s_*\alpha/(r\alpha+2s_*)}$ and obtain (3.2) for $s \geq rs_*\alpha/(r\alpha + 2s_*)$. In both cases, one can easily check $R^{-\alpha} \leq \delta \leq \min\{R^{-\alpha+1}, 1\}$ for r and s_* satisfying $\tilde{s}_* \leq s_*$. Indeed, if $\alpha > 1$, then $\delta \leq R^{-\alpha+1}$ since $2s_* + r \geq r\alpha$. When $0 < \alpha \leq 1$, we have $\delta \leq 1$ since $s_* > 0$. □

Theorem 1.1 is an immediate consequence of the following.

Corollary 3.2 *Let $0 < \gamma \leq d$ and $0 < r < \infty$. Suppose*

$$\left\| \sup_{0 < t < 1} |e^{it(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1),d\mu)} \leq C \langle \mu \rangle^{\frac{1}{\gamma}} \|f\|_{H^{\frac{d-\gamma}{2} + s_*}} \tag{3.5}$$

holds for some $0 < s_$ whenever μ is a γ -dimensional measure in \mathbb{R}^d . Then, if $\{t_n\} \in \ell^{r,\infty}$, there is a constant $C' > 0$ such that*

$$\left\| \sup_{n \in \mathbb{N}} |e^{it_n(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1),d\mu)} \leq C' \langle \mu \rangle^{\frac{1}{\gamma}} \|f\|_{H^{\frac{d-\gamma}{2} + s}} \tag{3.6}$$

holds for $s \geq \min\{s_, \tilde{s}_*\}$, where \tilde{s}_* is given by (3.1).*

The estimate (3.5) clearly implies (2.2). However, to prove Corollary 3.2, we need to remove the frequency localization in the estimate (3.2) so that the right-hand side of (3.2) is replaced by $C \langle \mu \rangle^{\frac{1}{\gamma}} \|f\|_{H^{\frac{d-\gamma}{2} + s}}$. This can be achieved by adapting the argument in [11].

Proof of Corollary 3.2 As before, we may assume $\tilde{s}_* \leq s_*$. It suffices to show (3.6) for $s \geq \tilde{s}_*$. Let us choose a smooth function $\beta \in C_0^\infty((1/2, 2))$ such that $\sum_k \beta(2^{-k} \cdot) = 1$ and set $\beta_0 = \sum_{k \leq 0} \beta(2^{-k} \cdot)$. Let $P_k, k \geq 0$, be the projection operator defined by $\widehat{P_k f}(\xi) = \beta(2^{-k} |\xi|) \widehat{f}(\xi)$, $k \geq 1$, and $\widehat{P_0 f}(\xi) = \beta_0(|\xi|) \widehat{f}(\xi)$.

For $\ell \geq 0$, we set

$$N_\ell = \{n \in \mathbb{N} : 2^{-2(\ell+1)\tilde{s}_*/r} < t_n \leq 2^{-2\ell\tilde{s}_*/r}\}.$$

For each $\ell \geq 0$, we write $f = \sum_{0 \leq k < \ell} P_k f + \sum_{k \geq 0} P_{\ell+k} f$. Hence, we have

$$\left\| \sup_{n \in \mathbb{N}} |e^{it_n(-\Delta)^{\alpha/2}} f| \right\|_{L^2(B(0,1),d\mu)} \leq \text{I} + \text{II},$$

where

$$\begin{aligned} \text{I} &= \sup_{\ell \geq 0} \left\| \sup_{n \in N_\ell} |e^{it_n(-\Delta)^{\alpha/2}} \left(\sum_{0 \leq k < \ell} P_k f \right)| \right\|_{L^2(B(0,1),d\mu)}, \\ \text{II} &= \sup_{\ell \geq 0} \left\| \sup_{n \in N_\ell} \left| \sum_{k \geq 0} e^{it_n(-\Delta)^{\alpha/2}} P_{\ell+k} f \right| \right\|_{L^2(B(0,1),d\mu)}. \end{aligned}$$

We consider II first. Since $\{t_n\} \in \ell^{r,\infty}$, it follows that $|N_\ell| \lesssim 2^{2\tilde{s}_*\ell}$. As before, by the embedding $\ell^2 \subset \ell^\infty$ and then applying Lemma 2.5 and Plancherel’s theorem, we obtain

$$\sup_{\ell \geq 0} \left\| \sup_{n \in N_\ell} \left| \sum_{k \geq 0} e^{it_n(-\Delta)^{\alpha/2}} P_{\ell+k} f \right| \right\|_{L^2(B(0,1),d\mu)} \lesssim \langle \mu \rangle^{\frac{1}{\gamma}} \sum_{k \geq 0} \left(\sum_{\ell \geq 0} 2^{(d-\gamma)(\ell+k) + 2\tilde{s}_*\ell} \|P_{\ell+k} f\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Thus, we obtain

$$\text{II} \lesssim \langle \mu \rangle^{\frac{1}{\gamma}} \sum_{k \geq 0} 2^{-\tilde{s}_*k} \|f\|_{H^{\frac{d-\gamma}{2} + \tilde{s}_*}} \lesssim \langle \mu \rangle^{\frac{1}{\gamma}} \|f\|_{H^{\frac{d-\gamma}{2} + \tilde{s}_*}}. \tag{3.7}$$

We now turn to I. Note that $2\tilde{s}_* < r\alpha$ by definition of (3.1), and decompose

$$\sum_{0 \leq k < \ell} P_k f = \sum_{0 \leq k < \frac{2\tilde{s}_*}{r\alpha} \ell} P_k f + \sum_{0 < k \leq \frac{r\alpha - 2\tilde{s}_*}{r\alpha} \ell} P_{\ell-k} f.$$

By the Minkowski inequality, we have $I \leq I_a + I_b$, where

$$I_a = \sum_{k \geq 0} \left\| \sup_{\ell > \frac{r\alpha}{2\tilde{s}_*} k} \sup_{n \in N_\ell} |e^{it_n(-\Delta)^{\alpha/2}} P_k f| \right\|_{L^2(B(0,1), d\mu)},$$

$$I_b = \sup_{\ell \geq 0} \left\| \sup_{n \in N_\ell} |e^{it_n(-\Delta)^{\alpha/2}} \left(\sum_{0 < k \leq \frac{r\alpha - 2\tilde{s}_*}{r\alpha} \ell} P_{\ell-k} f \right)| \right\|_{L^2(B(0,1), d\mu)}.$$

Regarding I_a , note that $\bigcup_{\ell > \frac{r\alpha}{2\tilde{s}_*} k} N_\ell \subset [0, 2^{-\alpha k}]$. By (3.5) and Lemma 2.2 with $R = 2^k$ and $I = [0, 2^{-\alpha k}]$, we obtain

$$I_a \lesssim \langle \mu \rangle_{\frac{1}{\gamma}}^{\frac{1}{2}} \sum_{k \geq 0} 2^{\frac{d-\gamma}{2} k} \|P_k f\|_2 \lesssim \langle \mu \rangle_{\frac{1}{\gamma}}^{\frac{1}{2}} \|f\|_{H^{\frac{d-\gamma}{2} + s}} \tag{3.8}$$

for $s > 0$.

To deal with I_b , we note that $t_n \in J_\ell := [0, 2^{-2\ell\tilde{s}_*/r}]$ if $n \in N_\ell$, and $2^{(\ell-k)\alpha} |J_\ell| \geq 1$ since $k \leq \frac{r\alpha - 2\tilde{s}_*}{r\alpha} \ell$. By Lemma 2.2 with $R = 2^{\ell-k}$ and $I = J_\ell$, it follows that

$$I_b \lesssim \langle \mu \rangle_{\frac{1}{\gamma}}^{\frac{1}{2}} \sup_{\ell \geq 0} \sum_{0 \leq k \leq \frac{r\alpha - 2\tilde{s}_*}{r\alpha} \ell} (2^{\ell-k})^{\frac{d-\gamma}{2}} (2^{(\ell-k)\alpha - 2\ell\tilde{s}_*/r})^{\max\{s_*, \frac{s_*}{\alpha}\}} \|P_{\ell-k} f\|_{L^2}.$$

Using (3.1) and the fact that $\min\{\alpha, 1\} \times \max\{s_*, \frac{s_*}{\alpha}\} = s_*$, one can easily see that $(2^{(\ell-k)\alpha - 2\ell\tilde{s}_*/r})^{\max\{s_*, \frac{s_*}{\alpha}\}} = 2^{-k\alpha \max\{s_*, \frac{s_*}{\alpha}\}} 2^{\ell\tilde{s}_*}$. Hence, by the embedding $\ell^2 \subset \ell^\infty$ and Minkowski's inequality, we obtain

$$I_b \lesssim \langle \mu \rangle_{\frac{1}{\gamma}}^{\frac{1}{2}} \sum_{k \geq 0} \left(\sum_{\ell > \frac{r\alpha}{r\alpha - 2\tilde{s}_*} k} (2^{\ell-k})^{d-\gamma} 2^{2\ell\tilde{s}_*} 2^{-2k \max\{\alpha s_*, s_*\}} \|P_{\ell-k} f\|_{L^2}^2 \right)^{1/2}$$

$$\lesssim \langle \mu \rangle_{\frac{1}{\gamma}}^{\frac{1}{2}} \sum_{k \geq 0} 2^{(\tilde{s}_* - \max\{\alpha s_*, s_*\})k} \|f\|_{H^{\frac{d-\gamma}{2} + \tilde{s}_*}}.$$

Thus, we have $I_b \lesssim \langle \mu \rangle_{\frac{1}{\gamma}}^{\frac{1}{2}} \|f\|_{H^{\frac{d-\gamma}{2} + \tilde{s}_*}}$. Combining this and the estimates (3.7) and (3.8), we obtain (3.6). □

3.2 Dimension of divergence set

From the implication in Corollary 3.2, we can obtain upper bounds on the divergence set $\mathcal{D}^{\alpha,d}(s, r)$ making use of the known estimates for the maximal Schrödinger operator $f \rightarrow \sup_{0 < t \leq 1} |e^{it(-\Delta)^{\alpha/2}} f|$. We start by recalling the following lemma ([17, 27]).

Lemma 3.3 *Suppose*

$$\|e^{it(-\Delta)^{\alpha/2}} f\|_{L^2(B(0,1), d\nu)} \leq C' \langle \nu \rangle_{\frac{1}{\gamma}}^{\frac{1}{2}} R^{\frac{d-\gamma}{2} + s} \|f\|_{L^2(\mathbb{R}^d)}, \tag{3.9}$$

holds for some $s \in \mathbb{R}$ whenever $\text{supp} \widehat{f} \subset \mathbb{A}_R$ and ν is a γ -dimensional measure in \mathbb{R}^{d+1} . Then, there exists $C > 0$ such that (2.2) holds whenever μ is a γ -dimensional measure in \mathbb{R}^d .

In what follows, we summarize the currently available maximal estimates (2.2) that can be obtained by the best-known fractal Strichartz estimates (3.9) combined with Lemma 3.3.

Proposition 3.4 *Let $d \geq 1$, $\alpha \in (0, 1) \cup (1, \infty)$, and μ be a γ -dimensional measure in \mathbb{R}^d . Then, (2.2) holds for $s > s_\alpha(\gamma, d)$ where*

$$s_\alpha(\gamma, d) = \begin{cases} \min\{\max\{0, \frac{\gamma}{2} - \frac{d}{4}\}, \frac{\gamma}{2(d+1)}\}, & \text{when } \alpha > 1, \\ \min\{\max\{0, \alpha(\frac{\gamma}{2} - \frac{d}{4})\}, \frac{\alpha}{2}\}, & \text{when } 0 < \alpha < 1. \end{cases}$$

When $\alpha > 1$, the estimates (2.2) and (3.9) for $s > s_\alpha(\gamma, d)$ were already obtained in [2, 14, 24]). The estimate for $0 < \alpha < 1$ also can be shown by the standard argument in [15, 33] and Lemma 3.3. However, since the latter case is less well known, we provide the proof for the convenience of the reader.

Proof of Proposition 3.4 for $0 < \alpha < 1$ By Lemma 3.3 we only need to prove (3.9) for $s > s_\alpha(\gamma, d)$.

Let σ be the surface measure on $\{(\xi, |\xi|^\alpha) : 1/2 \leq |\xi| \leq 2\}$. In [15, 33], it was shown that

$$\int |\widehat{v}(R\eta)|^2 d\sigma(\eta) \lesssim I_\gamma(v)R^{-\beta} \tag{3.10}$$

holds for $\beta = \max\{\min\{\gamma, \frac{d}{2}\}, \gamma - 1\}$ whenever ν is a γ -dimensional measure in \mathbb{R}^{d+1} . Here, $I_\gamma(v)$ denotes the γ -dimensional energy of ν . Let ν_λ be the rescaled measure defined by the relation (2.6) with d replaced by $d + 1$. Then, it is easy to see (see, e.g., [17, 18]) that (3.10) implies the estimate

$$\|e^{it(-\Delta)^{\alpha/2}} g\|_{L^2(B(0,\lambda), d\nu_\lambda)} \leq C\lambda^s \langle \nu \rangle_\gamma^{\frac{1}{2}} \|g\|_2 \tag{3.11}$$

for $s > (\gamma - \beta)/2$ whenever ν is γ -dimensional and \widehat{g} is supported on \mathbb{A}_1 ([16, 42]). Therefore, we have (3.11) for $s > s_\alpha(\gamma, d)/\alpha$.

Now, we take $\lambda = R^\alpha$ in (3.11). Then, applying Lemma 2.4 with R replaced by R^α , we have

$$\|e^{it(-\Delta)^{\alpha/2}} g\|_{L^2(B(0,R) \times [0,R^\alpha], d\nu_{R^\alpha})} \leq CR^s \langle \nu \rangle_\gamma^{\frac{1}{2}} \|g\|_2$$

for $s > s_\alpha(\gamma, d)$. By rescaling $\xi \rightarrow R^{-1}\xi$ and $(x, t) \rightarrow (Rx, R^\alpha t)$, we see that (3.9) holds for $s > s_\alpha(\gamma, d)$. □

We recall the maximal estimate (2.2) for the wave operator shown in [2, 17]. (See also [8, 18] for the fractal Strichartz estimates (3.9)).

Proposition 3.5 *Let I be a subinterval in $[0, 1]$. Then, (2.2) holds with $\alpha = 1$ for $s > s_1(\gamma, d)$ where*

$$s_1(\gamma, d) = \begin{cases} 0, & 0 < \gamma \leq \frac{1}{2}, \\ \frac{\gamma}{2} - \frac{1}{4}, & \frac{1}{2} < \gamma \leq 1, \\ \frac{\gamma}{4}, & 1 < \gamma \leq 2, \end{cases} \text{ for } d = 2;$$

$$s_1(\gamma, d) = \begin{cases} 0, & 0 < \gamma \leq \frac{d-1}{2}, \\ \frac{-d+1+2\gamma}{8}, & \frac{d-1}{2} < \gamma \leq \frac{d+1}{2}, \\ \frac{\gamma-1}{2(d-1)}, & \frac{d+1}{2} < \gamma \leq d, \end{cases} \text{ for } d \geq 3.$$

By Proposition 3.1, the estimates in Propositions 3.4 and 3.5 give the corresponding estimates for $\sup_n |e^{it_n(-\Delta)^{\alpha/2}} f|$ with $\{t_n\} \in \ell^{r,\infty}$ relative to γ -dimensional measures. Then, by a standard argument (see [2]), one can obtain upper bounds on the Hausdorff dimension of the divergence sets. We summarize the results as follows:

Corollary 3.6 *Let $\alpha > 0$, $d \geq 1$, $r \in (0, \infty)$, and $0 < \gamma \leq d$. Let $s_* = s_\alpha(\gamma, d)$, which is given in Propositions 3.4 and 3.5. Then, $\mathfrak{D}^{\alpha,d}(s, r) \leq \gamma$ if $s > 2^{-1}(d - \gamma) + \min\{s_*, \tilde{s}_*\}$.*

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Author contributions

CC and HK carried out preparatory investigation. Formal analysis was done by YK and SL. The manuscript was prepared by CC, HK and YK. SL was responsible for the main funding. CC, HK and YK worked together to conclude the result. All authors read and approved the final manuscript.

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