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Nodal solution for critical Kirchhoff-type equation with fast increasing weight in $\mathbb{R}^2$

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Abstract

In this paper, we investigate the existence of a least-energy sign-changing solutions for the following Kirchhoff-type equation:

$$-\left(1+b\int_{\mathbb{R}^2}K(x)|\nabla u|^2\,dx\right)\operatorname{div}(K(x)\nabla u)=K(x)f(u),\quad x\in\mathbb{R}^2,$$

where f has exponential subcritical or exponential critical growth in the sense of the Trudinger–Moser inequality. By using the constrained variational methods, combining the deformation lemma and Miranda's theorem, we prove the existence of a least-energy sign-changing solution. Moreover, we also prove that this sign-changing solution has exactly two nodal domains.

Keywords: Critical exponential; Constraint variational; Trudinger–Moser inequality; Nodal solution; Miranda's theorem; Deformation lemma

1 Introduction and main results

In this present paper, we consider the existence of the least energy sign-changing solutions for the following equation:

$$-\left(1+b\int_{\mathbb{R}^2}K(x)|\nabla u|^2\,dx\right)\operatorname{div}\bigl(K(x)\nabla u\bigr)=K(x)f(u),\quad x\in\mathbb{R}^2,\tag{1.1}$$

where $K(x)=\exp(|x|^2/4)$, b is a positive constant, and we assume that f satisfies:

(f_0) $f(t)\in C^1(\mathbb{R},\mathbb{R})$;

(f_1) $f(t)=o(|t|)$ as $|t|\to 0$;

(f_2) $\lim_{|t|\to\infty}\frac{F(t)}{t^4}=\infty$, where $F(t)=\int_0^t f(s)\,ds$;

(f_3) $\frac{f(t)}{t^3}$ is an increasing function on $\mathbb{R}\backslash\{0\}$;

(f_4) There exist

$$p>4\quad\text{and}\quad\varrho_0>\left[4m_p\left(\frac{p-2}{p-4}\right)\frac{\alpha_0}{\pi}\right]^{\frac{p-2}{2}},$$

such that

$$tf(t) \geq \varrho_0 |t|^p,$$

for all $t \in \mathbb{R}$, where $\alpha_0 > 0$ and m_p is attained in a ground state nodal energy of Eq. (1.1) when $f(u) = |u|^{p-2}u$.

As we all know, we call problems of type (1.1) nonlocal problems because there is an integral over $\mathbb{R}^2$. Such problems were first posed by G. Kirchhoff in [1] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings.

Similar nonlocal problems also model several physical and biological systems, where u describes a process which depends on the average of itself, for example, the population density, see [2] and the references therein. After J.L. Lions [3] proposed the functional analysis method of the equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u), \tag{1.2}$$

where $a,\ b > 0$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain, the steady-state form of the problem (1.2) has received a lot of attention. At the same time, many more results were obtained; we refer to [4–9] for bounded domains. In [6] the authors obtained sign-changing solutions to the nonlocal quasilinear elliptic boundary value problem using variational methods and invariant sets of descent flow in the subcritical case.

For the entire space $\mathbb{R}^N (N \geq 3)$, we know that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ $(2 \leq q < 2^*)$ is not compact. In order to overcome the lack of compactness, many researchers introduced the potential function $V(x)$, to study the Kirchhof-type equation of the following form:

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\Delta u + V(x)u = f(x,u), \tag{1.3}$$

restoring spatial compactness by making different assumptions about $V(x)$. In [10], the author showed that problem (1.3) has sign-changing solutions, if we assume $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geq a_1 > 0$ and, for each $A > 0$, $\operatorname{meas}\{x \in \mathbb{R}^3 : V(x) \leq A\} < \infty$, with a_1 being a constant and meas denoting the Lebesgue measure in $\mathbb{R}^3$. In [11], the author got a positive solution to the problem (1.3), considering $V(x)$ as a locally Hölder continuous function, and assuming there is a constant α such that $V(x) \geq \alpha > 0$ for all $x \in \mathbb{R}^3$ and $\inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x)$, where Λ is an open bounded set. There are many diverse results for equations of type (1.3) in $\mathbb{R}^N$; we refer to [12–15] and the references therein. In fact, by observation, we can see that our problem can be viewed as a generalization of the constant-coefficient Kirchhoff equation, when $K(x) = 1$, it is exactly the Kirchhoff equation as in (1.3). At the same time, we use the properties of function $K(x)$ to avoid using potential function $V(x)$ to overcome the problem of lost space embedding compactness.

It is well known that the critical growth for nonlinear terms also leads to the loss of compactness for the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, where the critical Sobolev exponent is $2^* = 2N/(N-2)$ $(N > 3)$. When $N = 2$, the critical exponential growth is related to

Trudinger–Moser inequality, which appears in the pioneer work [16, 17], that is,

$$\sup_{\|u\|_{H_0^1(\Omega)}\le 1}\int_\Omega e^{\alpha u^2}\le C(\alpha)$$

for all $\alpha \le 4\pi$ and $\Omega \subset \mathbb{R}^2$. Motivated by this inequality, de Figueiredo et al. [18] introduced the notion of subcritical and critical growth in the plane, i.e.,

(f_5) $f \in C(\mathbb{R},\mathbb{R})$ and there exists $\alpha_0 \ge 0$ such that

$$\lim_{|t|\to\infty}\frac{f(t)}{e^{\alpha|t|^2}}=\begin{cases}0, & \alpha>\alpha_0,\\ \infty, & \alpha<\alpha_0.\end{cases}$$

If the above holds for all $\alpha > 0$, we say that f has exponential subcritical growth at $+\infty$, and if there exists $\alpha_0 > 0$ as above then f has exponential critical growth at $+\infty$. When dealing with the entire space, we need a new version of the Trudinger–Moser inequality. It asserts that

$$\sup_{\|u\|_{H_0^1(\mathbb{R}^2)}\le 1}\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)\le C(\alpha)$$

for all $\alpha \le 4\pi$; see [19, 20] and the references therein.

To obtain our results, we consider using the variational method in a weighted Sobolev space consisting of rapidly decaying functions at infinity, where the embedding of $\mathbb{R}^2$ is recovered in the weighted Sobolev space. This idea was first proposed by M. Escobedo and O. Kavian in [21], mainly used to find a self-similar solution of the heat equation in $\mathbb{R}^N$, more precisely, they define the weighting function

$$K(x):=\exp\left(\frac{|x|^2}{4}\right),\quad \text{for } x\in\mathbb{R}^N.$$

For scholars interested in weighted Sobolev spaces, we recommend [22–28]. In [27], the author proves that the weighted semilinear elliptic problem has a sign-changing solution in the critical case, where the nonlinear term f satisfies the standard Ambrosetti–Rabinowitz superlinearity condition (namely, there exists $\theta > 2$ such that $tf(t) \ge \theta F(t) > 0$). In our paper, we directly use the Trudinger–Moser inequality in the weighted space considered in [29]; see Lemma 2.1.

Now, we introduce our work space. Consider $C_c^\infty(\mathbb{R}^2)$, the space of infinitely differentiable functions with compact support, and denote by X the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\|=\left(\int_{\mathbb{R}^2}K(x)|\nabla u|^2\,dx\right)^{\frac{1}{2}},$$

which is induced by the inner product

$$\langle u,v\rangle=\int_{\mathbb{R}^2}K(x)\nabla u\nabla v\,dx.$$

Define the weighted spaces for each $s \geq 2$ as

$$L_K^s(\mathbb{R}^2) = \left\{ u \text{ measurable in } \mathbb{R}^2 : \int_{\mathbb{R}^2} K(x)|u|^s \, dx < \infty \right\}.$$

By the results from [21, 22, 28] and Lemma 2.1 of [29], the space X is complete and the embedding X $\hookrightarrow L_K^s(\mathbb{R}^2)$ is continuous and compact for all $s \in [2, \infty)$. Note that $X \nsubseteq L_K^\infty(\mathbb{R}^2)$, thus we use the Trudinger–Moser inequality in $\mathbb{R}^2$ as a substitution of the Sobolev inequality.

From (f_1), for all $\varepsilon > 0$, there exists $\delta > 0$ such that, when $|t| < \delta$, we have

$$\left| f(t) \right| \leq \varepsilon |t|. \tag{1.4}$$

Let $\alpha > \alpha_0$ be given by (f_5) and $q \geq 2$. By using the critical growth of f, we obtain

$$\lim_{t \to +\infty} \frac{|f(t)|}{|t|^{q-1}(e^{\alpha t^2} - 1)} = 0. \tag{1.5}$$

Therefore, for all $\varepsilon > 0$, $t \in \mathbb{R}$, there exists C_ε such that

$$\max\left\{ \left| f(t)t \right|, \left| F(t) \right| \right\} \leq \varepsilon |t|^2 + C_\varepsilon |t|^q \left(e^{\alpha t^2} - 1 \right). \tag{1.6}$$

The problem (1.1) corresponds to the energy functional $I : X \to \mathbb{R}$ which can be constructed as

$$I_b(u) = \frac{1}{2} \int_{\mathbb{R}^2} K(x)|\nabla u|^2 \, dx + \frac{b}{4} \left(\int_{\mathbb{R}^2} K(x)|\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^2} K(x)F(u) \, dx. \tag{1.7}$$

By assumptions on f and a standard argument, we can affirm that I_b is a well-defined C^1 functional, and its derivative can be computed as

$$\left\langle I_b'(u), \varphi \right\rangle = \left(1 + b\|u\|^2 \right) \int_{\mathbb{R}^2} K(x) \nabla u \nabla \varphi \, dx - \int_{\mathbb{R}^2} K(x) f(u) \varphi \, dx, \tag{1.8}$$

for all $\varphi \in X$. Furthermore, u is a sign-changing solution of system (1.1) if and only if u is a critical point of I_b and $u^\pm \neq 0$, where

$$u^+ := \max(u, 0), \qquad u^- := \min(u, 0).$$

Motivated by [5, 10], in order to find a sign-changing solution of equation (1.1), we make the following decompositions for $u \in X$:

$$I_b(u) = I_b\left(u^+\right) + I_b\left(u^-\right) + \frac{b}{2} \int_{\mathbb{R}^2} K(x) \left| \nabla u^+ \right|^2 dx \int_{\mathbb{R}^2} K(x) \left| \nabla u^- \right|^2 dx, \tag{1.9}$$

$$\left\langle I_b'(u), u^+ \right\rangle = \left\langle I_b'\left(u^+\right), u^+ \right\rangle + b \int_{\mathbb{R}^2} K(x) \left| \nabla u^+ \right|^2 dx \int_{\mathbb{R}^2} K(x) \left| \nabla u^- \right|^2 dx, \tag{1.10}$$

$$\left\langle I_b'(u), u^- \right\rangle = \left\langle I_b'\left(u^-\right), u^- \right\rangle + b \int_{\mathbb{R}^2} K(x) \left| \nabla u^+ \right|^2 dx \int_{\mathbb{R}^2} K(x) \left| \nabla u^- \right|^2 dx. \tag{1.11}$$

Meanwhile, we consider the Nehari manifold and Nehari nodal set associated to (1.7) defined respectively by

$$\mathcal{N} = \left\{u \in X\backslash\{0\} : \left\langle I_b'(u), u\right\rangle = 0\right\}$$

and

$$\mathcal{M} = \left\{u \in X; u^{\pm} \neq 0 : \left\langle I_b'(u), u^{+}\right\rangle = \left\langle I_b'(u), u^{-}\right\rangle = 0\right\}.$$

In this paper, we have the following result.

Theorem 1.1 (Subcritical case) *Assuming* (f_5) *with* $\alpha_0 = 0$ *and* (f_0)–(f_3) *hold, equation* (1.1) *has a least-energy sign-changing solution, which has precisely two nodal domains.*

Theorem 1.2 (Critical case) *Assuming* (f_5) *with* $\alpha_0 > 0$ *and* (f_0)–(f_4) *hold, equation* (1.1) *has a least-energy sign-changing solution, which has precisely two nodal domains.*

We organize this paper as follows. In Sect. 2 we give some useful preliminary lemmas which pave the way for getting a least-energy sign-changing solution. Then Sect. 3 is devoted to proving Theorems 1.1 and 1.2.

2 Some preliminary lemmas

According to [29], the following version of the Trudinger–Moser inequality holds:

Lemma 2.1 *For any* $r \geq 0$, $u \in X$, *we have* $K(x)|u|^{r+2} \in L^1(\mathbb{R}^2)$. *If* $\|u\| \leq M$, $\varsigma M^2 < 4\pi$, *then there exists* $C = C(M, r, \varsigma) > 0$ *such that*

$$\int_{\mathbb{R}^2} K(x)|u|^{2+r}\left[\exp\left(\varsigma u^2\right) - 1\right] dx \leq C(M, r, \varsigma)\|u\|^{r}. \tag{2.1}$$

Proof See [29, Theorem 1.1 and Corollary 1.2]. □

Next, we prove that the set $\mathcal{M}$ is nonempty. In this proof, we adopt in part the idea of Zhong and Tang [30].

Lemma 2.2 *Suppose that* f *satisfies* (f_0)–(f_3). *For any* $u \in X$ *with* $u^{\pm} \neq 0$, *there exists a unique pair of numbers* $s_u, t_u > 0$ *such that* $s_u u^{+} + t_u u^{-} \in \mathcal{M}$ *and* $I_b(s_u u^{+} + t_u u^{-}) = \max_{s,t \geq 0} I_b(su^{+} + tu^{-})$.

Proof Fix $u \in X$ with $u^{\pm} \neq 0$. We first verify the existence of (s_u, t_u). Write

$$\begin{aligned} I_b\left(su^{+} + tu^{-}\right) &= \frac{1}{2}\left\|su^{+} + tu^{-}\right\|^2 + \frac{b}{4}\left\|su^{+} + tu^{-}\right\|^4 - \int_{\mathbb{R}^2} K(x)F\left(su^{+} + tu^{-}\right) dx \\ &= \frac{1}{2}s^2\left\|u^{+}\right\|^2 + \frac{b}{4}s^4\left\|u^{+}\right\|^4 - \int_{\mathbb{R}^2} K(x)F\left(su^{+}\right) dx + \frac{b}{2}s^2t^2\left\|u^{+}\right\|^2\left\|u^{-}\right\|^2 \\ &\quad + \frac{1}{2}t^2\left\|u^{-}\right\|^2 + \frac{b}{4}t^4\left\|u^{-}\right\|^4 - \int_{\mathbb{R}^2} K(x)F\left(tu^{-}\right) dx. \end{aligned}$$

Let $\Phi(s,t) = I_b(su^+ + tu^-)$ and use Φ^u to represent the gradient at (s,t), i.e., $\Phi^u = (\Phi_s'(s,t), \Phi_t'(s,t)) = (I_b'(su^+ + tu^-)u^+, I_b'(su^+ + tu^-)u^-)$, and then

$$\begin{cases} \Phi_s'(s,t) = s\|u^+\|^2 + bs^3\|u^+\|^4 + bst^2\|u^+\|^2\|u^-\|^2 - \int_{\mathbb{R}^2} K(x)f(su^+)(u^+)\,dx, \\ \Phi_t'(s,t) = t\|u^-\|^2 + bt^3\|u^-\|^4 + bs^2t\|u^+\|^2\|u^-\|^2 - \int_{\mathbb{R}^2} K(x)f(tu^-)(u^-)\,dx. \end{cases} \tag{2.2}$$

Combining (f_1)–(f_3), it is easy to verify $\Phi_s'(s,s) > 0$, $\Phi_t'(s,s) > 0$ for $s > 0$ small enough and $\Phi_s'(t,t) < 0$, $\Phi_t'(t,t) < 0$ for $t > 0$ large enough. Then there exists $0 \le r \le R$ such that

$$\Phi_s'(r,r) > 0, \qquad \Phi_t'(r,r) > 0; \qquad \Phi_s'(R,R) < 0, \qquad \Phi_t'(R,R) < 0. \tag{2.3}$$

By the monotonicity with respect to $s > 0$ (resp. $t > 0$) if $t > 0$ (resp. $s > 0$) is fixed, one has

$$\begin{aligned} &\Phi_s'(r,t) > 0, \qquad \Phi_s'(R,t) < 0, \quad \text{for all } t \in [r,R], \\ &\Phi_t'(s,r) > 0, \qquad \Phi_t'(s,R) < 0, \quad \text{for all } s \in [r,R]. \end{aligned}$$

It follows from the Miranda's theorem [31] that there exists a pair $(s_u, t_u) \in [r,R] \times [r,R]$ such that

$$\Phi_s'(s_u,t_u) = 0, \qquad \Phi_t'(s_u,t_u) = 0,$$

which implies that $s_u u^+ + t_u u^- \in \mathcal{M}$, i.e., $\mathcal{M} \neq \emptyset$.

Next, we will prove that the positive number pair (s_u, t_u) is unique. We suppose that there are two pairs of positive numbers (s_{u_1}, t_{u_1}), (s_{u_2}, t_{u_2}) satisfying $\Phi_s'(s_{u_i}, t_{u_i}) = 0$, $i = 1, 2$. Without loss of generality, we assume $s_{u_1} < s_{u_2}$ and that there exists a unique s_u such that $\Phi_s'(s_u, t_u) = 0$. From $\Phi_s'(s_{u_i}, t_{u_i}) = 0$ we derive that

$$s_{u_1}\|u^+\|^2 + bs_{u_1}^3\|u^+\|^4 + bs_{u_1}t_u^2\|u^+\|^2\|u^-\|^2 = \int_{\mathbb{R}^2} K(x)f(s_{u_1}u^+)(u^+)\,dx \tag{2.4}$$

and

$$s_{u_2}\|u^+\|^2 + bs_{u_2}^3\|u^+\|^4 + bs_{u_2}t_u^2\|u^+\|^2\|u^-\|^2 = \int_{\mathbb{R}^2} K(x)f(s_{u_2}u^+)(u^+)\,dx, \tag{2.5}$$

and then, combing (2.4) with (2.5), we have

$$\begin{aligned} &\left(\frac{1}{s_{u_1}^2} - \frac{1}{s_{u_2}^2}\right)\|u^+\|^2 + b\left(\frac{1}{s_{u_1}^2} - \frac{1}{s_{u_2}^2}\right)t_u^2\|u^+\|^2\|u^-\|^2 \\ &\quad = \int_{\mathbb{R}^2} K(x)\left(\frac{f(s_{u_1}u^+)}{(s_{u_1}u^+)^3} - \frac{f(s_{u_2}u^+)}{(s_{u_2}u^+)^3}\right)(u^+)^4\,dx. \end{aligned} \tag{2.6}$$

We know that the left-hand side of the latter equality is positive due to assumption $s_{u_1} < s_{u_2}$. At the same time, using hypothesis (f_3), we can see that the right-hand side is negative, which leads to a contradiction. Therefore, we have $s_{u_1} = s_{u_2}$, so s_u is unique. The proof of t_u uniqueness is similar.

The existence of an extreme value of $\Phi(s,t)$ at (s_u,t_u) is verified by using the sufficient condition for the existence of an extreme value of a binary function:

$$\begin{cases} \Phi''_{ss}(s,t) = \|u^+\|^2 + 3bs^2\|u^+\|^4 + bt^2\|u^+\|^2\|u^-\|^2 - \int_{\mathbb{R}^2} K(x)f'(su^+)(u^+)^2\,dx, \\ \Phi''_{st}(s,t) = 2bst\|u^+\|^2\|u^-\|^2 = \Phi''_{ts}(s,t), \\ \Phi''_{tt}(s,t) = \|u^-\|^2 + 3bt^2\|u^-\|^4 + bs^2\|u^+\|^2\|u^-\|^2 - \int_{\mathbb{R}^2} K(x)f'(tu^-)(u^-)^2\,dx. \end{cases} \tag{2.7}$$

Substituting point (s_u,t_u) into (2.7), we have

$$\begin{cases} \Phi''_{ss}(s_u,t_u) = \|u^+\|^2 + 3bs_u^2\|u^+\|^4 + bt_u^2\|u^+\|^2\|u^-\|^2 \\ \qquad\qquad - \int_{\mathbb{R}^2} K(x)f'(s_uu^+)(u^+)^2\,dx, \\ \Phi''_{st}(s_u,t_u) = 2bs_ut_u\|u^+\|^2\|u^-\|^2 = \Phi''_{ts}(s_u,t_u), \\ \Phi''_{tt}(s_u,t_u) = \|u^-\|^2 + 3bt_u^2\|u^-\|^4 + bs_u^2\|u^+\|^2\|u^-\|^2 \\ \qquad\qquad - \int_{\mathbb{R}^2} K(x)f'(t_uu^-)(u^-)^2\,dx, \end{cases} \tag{2.8}$$

then, combing $\Phi'_s(s_u,t_u) = 0$ with hypothesis (f_3), we obtain that

$$\begin{aligned} \Phi''_{ss}(s_u,t_u) &= \|u^+\|^2 + 3bs_u^2\|u^+\|^4 + bt_u^2\|u^+\|^2\|u^-\|^2 - \int_{\mathbb{R}^2} K(x)f'(s_uu^+)(u^+)^2\,dx \\ &< \|u^+\|^2 + 3bs_u^2\|u^+\|^4 + bt_u^2\|u^+\|^2\|u^-\|^2 - 3\int_{\mathbb{R}^2} K(x)f(s_uu^+)\frac{1}{s_u}(u^+)\,dx \\ &= -2\|u^+\|^2 - 2t_u^2b\|u^+\|^2\|u^-\|^2 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \Phi''_{tt}(s_u,t_u) &= \|u^-\|^2 + 3bt_u^2\|u^-\|^4 + bs_u^2\|u^+\|^2\|u^-\|^2 - \int_{\mathbb{R}^2} K(x)f'(s_uu^-)(u^-)^2\,dx \\ &< \|u^-\|^2 + 3bt_u^2\|u^-\|^4 + bs_u^2\|u^+\|^2\|u^-\|^2 - 3\int_{\mathbb{R}^2} K(x)f(s_uu^-)\frac{1}{s_u}(u^-)\,dx \\ &= -2\|u^-\|^2 - 2s_u^2b\|u^+\|^2\|u^-\|^2 \end{aligned} \tag{2.10}$$

hold. Since, obviously,

$$\Phi''_{ss}(s_u,t_u) < 0,$$

from (2.8)–(2.10) we get

$$\begin{aligned} &\Phi''_{ss}(s_u,t_u)\Phi''_{tt}(s_u,t_u) - \Phi''^2_{st}(s_u,t_u) \\ &\quad > \left(2\|u^+\|^2 + 2t_u^2b\|u^+\|^2\|u^-\|^2\right)\left(2\|u^-\|^2 + 2s_u^2b\|u^+\|^2\|u^-\|^2\right) \\ &\qquad - \left(2bs_ut_u\|u^+\|^2\|u^-\|^2\right)^2 \\ &\quad > 0. \end{aligned} \tag{2.11}$$

Thus we can get the maximum value of $\Phi(s,t)$ at (s_u,t_u). The proof is complete. □

Lemma 2.3 *Assume that* (f_0)–(f_3) *and* (f_5) *hold, as well as* $u \in X$ *and* $u^{\pm} \neq 0$. *Then we have*:

(i) *If* $\Phi_s'(1,1) \leq 0$, $\Phi_t'(1,1) \leq 0$, *there is a unique positive number pair* (s_u, t_u) *obtained in Lemma* 2.2, *satisfying* $0 < s_u, t_u \leq 1$, *such that* $s_u u^+ + t_u u^- \in \mathcal{M}$.

(ii) *If* $\Phi_s'(1,1) \geq 0$, $\Phi_t'(1,1) \geq 0$, *there is a unique positive number pair* (s_u, t_u) *obtained in Lemma* 2.2, *satisfying* $s_u, t_u \geq 1$, *such that* $s_u u^+ + t_u u^- \in \mathcal{M}$.

Proof (i) Assuming that $s_u \geq t_u > 0$, in view of $s_u u^+ + t_u u^- \in \mathcal{M}$, we have

$$\begin{aligned} s_u\|u^+\|^2 + bs_u^3\|u^+\|^4 + bs_u^3\|u^+\|^2\|u^-\|^2 &\geq s_u\|u^+\|^2 + bs_u^3\|u^+\|^4 + bs_u t_u^2\|u^+\|^2\|u^-\|^2 \\ &= \int_{\mathbb{R}^2} K(x) f(s_u u^+)(u^+)\,dx. \end{aligned} \tag{2.12}$$

From the hypothesis $\Phi_s'(1,1) \leq 0$, we have

$$\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 \leq \int_{\mathbb{R}^2} K(x) f(u^+)(u^+)\,dx. \tag{2.13}$$

Combing (2.12) with (2.13), we get

$$\left(\frac{1}{s_u^2} - 1\right)\|u^+\|^2 \geq \int_{\mathbb{R}^2} K(x)\left[\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f(u^+)}{(u^+)^3}\right](u^+)^4\,dx. \tag{2.14}$$

If $s_u > 1$, then the left-hand side of this inequality is negative, but from (f_3) the right-hand side is positive, so (2.14) yields a contradiction. Therefore we conclude $s_u \leq 1$. Using a similar method, we can prove that $t_u \leq 1$.

(ii) Similarly, assuming that $0 < s_u \leq t_u$ and using the fact that $s_u u^+ + t_u u^- \in \mathcal{M}$, we get

$$\begin{aligned} s_u\|u^+\|^2 + bs_u^3\|u^+\|^4 + bs_u^3\|u^+\|^2\|u^-\|^2 &\leq s_u\|u^+\|^2 + bs_u^3\|u^+\|^4 + bs_u t_u^2\|u^+\|^2\|u^-\|^2 \\ &= \int_{\mathbb{R}^2} K(x) f(s_u u^+)(u^+)\,dx. \end{aligned} \tag{2.15}$$

From the assumption $\Phi_s'(1,1) \geq 0$, we have

$$\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 \geq \int_{\mathbb{R}^2} K(x) f(u^+)(u^+)\,dx. \tag{2.16}$$

Now combing (2.15) with (2.16), we get

$$\left(\frac{1}{s_u^2} - 1\right)\|u^+\|^2 \leq \int_{\mathbb{R}^2} K(x)\left[\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f(u^+)}{(u^+)^3}\right](u^+)^4\,dx. \tag{2.17}$$

If $s_u < 1$, then the two sides of (2.17) are contradictory, therefore we conclude $s_u \geq 1$. Using a similar method, we can prove that $t_u \geq 1$. □

Lemma 2.4 *Assume that* (f_0)–(f_3) *and* (f_5) *hold. Then there exists* $\rho > 0$ *such that* $\|u\| \geq \rho$ *for all* $u \in \mathcal{M}$. *Furthermore*, $m := \inf\{I_b(u) : u \in \mathcal{M}\} > 0$.

Proof Suppose, to the contrary, that there exists $\{u_n\} \subset \mathcal{M}$ such that $\|u_n\| \to 0$. Using (1.6), we obtain

$$\begin{aligned}
\|u_n\|^2 &\le \|u_n\|^2 + b\|u_n\|^4 \\
&\le \varepsilon \int_{\mathbb{R}^2} K(x)|u_n|^2\,dx + C_\varepsilon \int_{\mathbb{R}^2} K(x)|u_n|^q\big(e^{\alpha u_n^2} - 1\big)\,dx \\
&= \varepsilon \int_{\mathbb{R}^2} K(x)|u_n|^2\,dx + C_\varepsilon \int_{\mathbb{R}^2} K(x)|u_n|^q\big(e^{\alpha\|u_n\|^2(\frac{u_n}{\|u_n\|})^2} - 1\big)\,dx.
\end{aligned} \tag{2.18}$$

Using Sobolev embedding theorem and Hölder's inequality with $s', s > 1$, we get

$$\|u_n\|^2 \le \varepsilon S_2^{-1}\|u_n\|^2 + C_\varepsilon\left(\int_{\mathbb{R}^2} K(x)|u_n|^{qs'}\,dx\right)^{\frac{1}{s'}}\left(\int_{\mathbb{R}^2} K(x)\big(e^{\alpha\|u_n\|^2(\frac{u_n}{\|u_n\|})^2} - 1\big)^s\,dx\right)^{\frac{1}{s}},$$

which after a rearrangement yields

$$\big(1 - \varepsilon S_2^{-1}\big)\|u_n\|^2 \le C_\varepsilon\left(\int_{\mathbb{R}^2} K(x)|u_n|^{qs'}\,dx\right)^{\frac{1}{s'}}\left(\int_{\mathbb{R}^2} K(x)\big(e^{\alpha\|u_n\|^2(\frac{u_n}{\|u_n\|})^2} - 1\big)^s\,dx\right)^{\frac{1}{s}}.$$

Arguing as in the proof of Lemma 3.4 in [32], there exists $\tilde{C}_\varepsilon$ such that

$$\big(1 - \varepsilon S_2^{-1}\big)\|u_n\|^2 \le \tilde{C}_\varepsilon\left(\int_{\mathbb{R}^2} K(x)|u_n|^{qs'}\,dx\right)^{\frac{1}{s'}}\left(\int_{\mathbb{R}^2} K(x)\big(e^{s\alpha\|u_n\|^2(\frac{u_n}{\|u_n\|})^2} - 1\big)\,dx\right)^{\frac{1}{s}}.$$

Let $v_n := \frac{u_n}{\|u_n\|}$, then $\|v_n\|^2 = 1$. Since $\|u_n\| \to 0$, there exists $\beta < 4\pi$ such that $s\alpha\|u_n\|^2 < \beta$ holds. For $q > 2$, using Lemma 2.1 and the embedding theorem, there exists a constant $\tilde{C}_\varepsilon$ such that

$$\big(1 - \varepsilon S_2^{-1}\big)\|u_n\|^2 \le M\tilde{C}_\varepsilon\left(\int_{\mathbb{R}^2} K(x)|u_n|^{qs'}\,dx\right)^{\frac{1}{s'}} \le M\tilde{C}_\varepsilon S_{qs'}^{-\frac{q}{2}}\|u_n\|^q.$$

By simplifying we get

$$\frac{(1 - \varepsilon S_2^{-1})}{M\tilde{C}_\varepsilon S_{qs'}^{-\frac{q}{2}}} \le \|u_n\|^{q-2}.$$

By arbitrariness of ε, there is a constant $\rho = [\frac{(1-\varepsilon S_2^{-1})}{M\tilde{C}_\varepsilon S_{qs'}^{-\frac{q}{2}}}]^{\frac{1}{q-2}} > 0$ such that $\|u_n\| \ge \rho > 0$.

Now assume that $\{u_n\} \subset \mathcal{M}$ is a minimizing sequence for m. Using hypothesis (f_3), we get

$$m = \lim_{n\to\infty}\inf\left[I(u_n) - \frac{1}{4}\big\langle I'(u_n), u_n\big\rangle\right] \ge \frac{1}{4}\lim_{n\to\infty}\inf\|u_n\|^2 \ge \frac{1}{4}\rho^2 > 0, \tag{2.19}$$

which completes the proof. □

Because $\{u_n\}$ is bounded in X, there exists $u \in$ X such that $u_n^{\pm} \rightharpoonup u^{\pm}$ in X. Since $\{u_n\} \subset \mathcal{M}$, one has $\langle I'(u_n), u_n^{\pm}\rangle = 0$, i.e.,

$$\begin{aligned}&\int_{\mathbb{R}^2} K(x)\left|\nabla u_n^{\pm}\right|^2 dx + b\int_{\mathbb{R}^2} K(x)|\nabla u|^2 dx \int_{\mathbb{R}^2} K(x)\left|\nabla u_n^{\pm}\right|^2 dx \\ &\quad = \int_{\mathbb{R}^2} K(x) f\left(u_n^{\pm}\right)\left(u_n^{\pm}\right) dx.\end{aligned} \tag{2.20}$$

Since $\|u_n^{\pm}\| \geq \rho > 0$, using (1.6), we have

$$\begin{aligned}\rho^2 \leq \left\|u_n^{\pm}\right\|^2 &\leq \int_{\mathbb{R}^2} K(x) f\left(u_n^{\pm}\right)\left(u_n^{\pm}\right) dx \\ &\leq \varepsilon \int_{\mathbb{R}^2} K(x)\left|u_n^{\pm}\right|^2 dx + C_\varepsilon \int_{\mathbb{R}^2} K(x)\left|u_n^{\pm}\right|^q \left(e^{\alpha (u_n^{\pm})^2} - 1\right) dx.\end{aligned} \tag{2.21}$$

By the boundedness of $\{u_n\}$ in X, there exists C_1 such that

$$\begin{aligned}\rho^2 &\leq \varepsilon C_1 + C_\varepsilon \int_{\mathbb{R}^2} K(x)\left|u_n^{\pm}\right|^q \left(e^{\alpha (u_n^{\pm})^2} - 1\right) dx \\ &\leq \varepsilon C_1 + \tilde{C}_\varepsilon \left(\int_{\mathbb{R}^2} K(x)\left|u_n^{\pm}\right|^{qs'}\right)^{\frac{1}{s'}} \left(\int_{\mathbb{R}^2} K(x)\left(e^{\alpha s (u_n^{\pm})^2} - 1\right) dx\right)^{\frac{1}{s}},\end{aligned} \tag{2.22}$$

from which we get

$$\rho^2 - \varepsilon C_1 \leq M\tilde{C}_\varepsilon \left(\int_{\mathbb{R}^2} K(x)\left|u_n^{\pm}\right|^{qs'} dx\right)^{\frac{1}{s'}}. \tag{2.23}$$

Choosing $\varepsilon = \frac{\rho^2}{2C_1}$, we have

$$0 < \frac{\rho^2}{2M\tilde{C}_\varepsilon} \leq \left(\int_{\mathbb{R}^2} K(x)\left|u_n^{\pm}\right|^{qs'} dx\right)^{\frac{1}{s'}}. \tag{2.24}$$

Since $qs' > 2$, we conclude that $u_n^{\pm} \to u^{\pm}$ in $L^{qs'}(\mathbb{R}^2)$. So, we have

$$\left(\int_{\mathbb{R}^2} K(x)\left|u^{\pm}\right|^{qs'} dx\right)^{\frac{1}{s'}} \geq \frac{\rho^2}{2M\tilde{C}_\varepsilon} > 0. \tag{2.25}$$

Therefore $u^{\pm} \neq 0$.

3 Proof of theorems

In this section, we will prove our main results. We first deal with the subcritical case ($\alpha_0 = 0$), it is related to the convergence of involved functions f and F, see (f_0)–(f_5).

Lemma 3.1 *Let $\{u_n^{\pm}\} \subset \mathcal{M}$ be a minimizing sequence for m. Then there exists $u^{\pm} \in X$ such that*

$$\int_{\mathbb{R}^2} K(x) f\left(u_n^{\pm}\right)\left(u_n^{\pm}\right) dx \to \int_{\mathbb{R}^2} K(x) f\left(u^{\pm}\right)\left(u^{\pm}\right) dx \tag{3.1}$$

and

$$\int_{\mathbb{R}^2} K(x)F\bigl(u_n^{\pm}\bigr)\,dx \to \int_{\mathbb{R}^2} K(x)F\bigl(u^{\pm}\bigr)\,dx. \tag{3.2}$$

Proof According to Lemma 2.4, there exists $M_1 > 0$ such that

$$\bigl\|u_n^{\pm}\bigr\|^2 \le M_1, \quad \forall n \in \mathbb{N}, \tag{3.3}$$

and there exists a function $u \in X$ such that $u_n^{\pm}(x) \to u^{\pm}(x)$ and $f(u_n^{\pm}(x))(u_n^{\pm}(x)) \to f(u^{\pm}(x))(u^{\pm}(x))$ a.e. in $\mathbb{R}^2$. In order to prove the first limit, the generalized Lebesgue convergence theorem is used here. Letting $g : \mathbb{R} \to \mathbb{R}$ and $g \in L^1(\mathbb{R}^2)$, and using (1.6), we have that

$$K(x)f\bigl(u_n^{\pm}(x)\bigr)\bigl(u_n^{\pm}(x)\bigr) \le \varepsilon K(x)\bigl|u_n^{\pm}(x)\bigr|^2 + C_\varepsilon K(x)\bigl|u_n^{\pm}(x)\bigr|^q\bigl(e^{\alpha(u_n^{\pm}(x))^2} - 1\bigr) := g\bigl(u_n^{\pm}(x)\bigr).$$

We will prove that $g(u_n^{\pm})$ is convergent in $L^1(\mathbb{R}^2)$. First, note that

$$\int_{\mathbb{R}^2} K(x)\bigl|u_n^{\pm}\bigr|^2\,dx \to \int_{\mathbb{R}^2} K(x)\bigl|u^{\pm}\bigr|^2\,dx.$$

Choosing $s', s > 1$ such that $\frac{1}{s} + \frac{1}{s'} = 1$, we have

$$K(x)^{\frac{1}{s'}}\bigl|u_n^{\pm}\bigr|^q \to K(x)^{\frac{1}{s'}}\bigl|u^{\pm}\bigr|^q \quad \text{in } L^{s'}\bigl(\mathbb{R}^2\bigr). \tag{3.4}$$

Using (3.3) and choosing $\alpha < \frac{4\pi}{sM_1^2}$, we conclude by Lemma 2.1 that

$$\begin{aligned}\int_{\mathbb{R}^2} K(x)\bigl(e^{\alpha s(u_n^{\pm}(x))^2} - 1\bigr)\,dx &\le \int_{\mathbb{R}^2} K(x)\bigl(e^{\alpha s M_1^2(\frac{u_n^{\pm}(x)}{\|u_n^{\pm}\|})^2} - 1\bigr)\,dx \\ &\le \int_{\mathbb{R}^2} K(x)\bigl(e^{4\pi(\frac{u_n^{\pm}(x)}{\|u_n^{\pm}\|})^2} - 1\bigr)\,dx \le M_2.\end{aligned} \tag{3.5}$$

Because

$$K(x)e^{\alpha s|u_n^{\pm}(x)|^2} \to K(x)e^{\alpha s|u^{\pm}(x)|^2} \quad \text{a.e. in } \mathbb{R}^2,$$

we can use Lemma 4.8 of [33] and conclude that

$$K(x)e^{\alpha s|u_n^{\pm}|^2} \rightharpoonup K(x)e^{\alpha s|u^{\pm}|^2}. \tag{3.6}$$

Using (3.4) and (3.6), as well as Lemma 4.8 of [33] again, we conclude

$$\int_{\mathbb{R}^2} K(x)f\bigl(u_n^{\pm}\bigr)\bigl(u_n^{\pm}\bigr)\,dx \to \int_{\mathbb{R}^2} K(x)f\bigl(u^{\pm}\bigr)\bigl(u^{\pm}\bigr)\,dx.$$

Analogously, $\int_{\mathbb{R}^2} K(x)F(u_n^{\pm})\,dx \to \int_{\mathbb{R}^2} K(x)F(u^{\pm})\,dx$.

Using the lower semicontinuity of convex functions, one has

$$\bigl\|u^{\pm}\bigr\|^2 \le \liminf_{n\to\infty}\bigl\|u_n^{\pm}\bigr\|^2. \tag{3.7}$$

Using (3.1), (3.2), and Lemma 2.3, there exists $(s_u, t_u) \in (0,1] \times (0,1]$ such that

$$\bar{u} := s_u u^+ + t_u u^-.$$

By (f_3), we have

$$\begin{aligned}
m \le I_b(\bar{u}) &= I_b(\bar{u}) - \frac{1}{4}\langle I_b'(\bar{u}), \bar{u}\rangle \\
&= \frac{1}{4}\|\bar{u}\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(\bar{u})\bar{u} - 4F(\bar{u})\big]\,dx \\
&= \frac{1}{4}\|s_u u^+\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(s_u u^+)(s_u u^+) - 4F(s_u u^+)\big]\,dx \\
&\quad + \frac{1}{4}\|t_u u^-\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(t_u u^-)(t_u u^-) - 4F(t_u u^-)\big]\,dx \\
&\le \frac{1}{4}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(u)u - 4F(u)\big]\,dx \\
&\le \liminf_{n\to\infty}\left[\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(u_n)u_n - 4F(u_n)\big]\,dx\right] \\
&= \liminf_{n\to\infty}\left(I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle\right) = m.
\end{aligned} \tag{3.8}$$

Thus we conclude that $s_u = t_u = 1$. So $\bar{u} = u$, $I_b(u) = m$. □

Lemma 3.2 *Assuming (f_0)–(f_3) and (f_5) hold, and $u \in \mathcal{M}$, one has $\Phi(s,t) < \Phi(1,1) = I_b(u)$ for all $(s,t) \in C(\mathbb{R}^+, \mathbb{R}^+)\backslash\{(1,1)\}$. Furthermore,* $\det(\Phi^u)'(1,1) > 0$.

Proof Letting $u \in \mathcal{M}$ and noting that $\langle I_b'(u), u^\pm\rangle = \langle I_b'(u^+ + u^-), u^\pm\rangle = 0$, we get that $(1,1)$ is a critical point of Φ, i.e.,

$$\Phi^u(1,1) = \left(\frac{\partial \Phi}{\partial s}(1,1), \frac{\partial \Phi}{\partial t}(1,1)\right) = (0,0).$$

According to Lemma 2.2, we know that $\Phi(s,t)$ reaches its maximum at (s_u, t_u), so from (3.8) we conclude that $s_u, t_u = 1$. To verify $\det(\Phi^u)'(1,1) > 0$, first note that

$$(\Phi^u)'(s,t) = \begin{pmatrix} g_1'(s) & 0 \\ 0 & g_2'(t) \end{pmatrix},$$

where

$$\begin{aligned}
g_1(s) &:= \Phi_1^u(su^+)u^+ = s\|u^+\|^2 + bs^3\|u^+\|^4 - \int_{\mathbb{R}^2} K(x)f(su^+)u^+, \\
g_2(s) &:= \Phi_2^u(tu^-)u^- = t\|u^-\|^2 + bt^3\|u^-\|^4 - \int_{\mathbb{R}^2} K(x)f(tu^-)u^-.
\end{aligned}$$

Because $u^+ \in \mathcal{N}$, it follows from the definition of $g_1(s)$ and (f_3) that

$$\begin{aligned} g_1'(1) &= \|u^+\|^2 + 3b\|u^+\|^4 - \int_{\mathbb{R}^2} K(x)f'(u^+)(u^+)^2 \\ &= -2\|u^+\|^2 + \int_{\mathbb{R}^2} K(x)\big[3f(u^+)u^+ - f'(u^+)(u^+)^2\big]\,dx < 0. \end{aligned} \tag{3.9}$$

Similarly, $g_2'(1) < 0$, and therefore we conclude that

$$\det(\Phi^u)'(1,1) > 0.$$

□

Lemma 3.3 *Assume* (f_0)–(f_3) *and* (f_5) *hold. If* $u \in \mathcal{M}$ *and*

$$I_b(u) = m := \inf_{v\in\mathcal{M}} I(v),$$

then $I_b'(u) = 0$.

Proof Suppose to the contrary that the conclusion is not valid. Then there are $\delta, \lambda > 0$ such that $\|I_b'(u)\| > \lambda$ whenever $\|u - v\| < 3\delta$. Let $D \subset \mathbb{R}^2$ be such that $(1,1) \in D$, and define a continuous mapping $g : D \to X$ by $g(s,t) = su^+ + tu^-$. From Lemma 3.2, we conclude that

$$\alpha := \max_{(s,t)\in\partial D} I_b \circ g < m. \tag{3.10}$$

For $0 < \varepsilon < \min\{(m-\alpha)/2, \lambda\delta/8\}$ and $S := B_\delta(v)$, using Lemma 2.3 of [34], there exists $\eta \in C([0,1] \times X, X)$ verifying:

(a_1) $\eta(1,u) = u$, $u \notin I_b^{-1}([m-2\varepsilon, m+2\varepsilon])$;
(a_2) $\eta(1, I_b^{m+\varepsilon} \cap S) \subset I_b^{m-\varepsilon}$;
(a_3) $I_b(\eta(1,u)) \le I_b(u)$, $\forall u \in X$.

By Lemma 3.2, (a_2), and (a_3), it follows that

$$\max_{(s,t)\in D} I_b\big(\eta\big(1, g(s,t)\big)\big) < m. \tag{3.11}$$

It follows from the definition of Φ^u and $u \in \mathcal{M}$ that $\Phi^u(s,t) = 0$ if and only if $(s,t) = (1,1) \in D$. Therefore, from the Brouwer degree theory and Lemma 3.2, we get

$$\deg\big(\Phi^u, D, 0\big) = \operatorname{sgn}\det\big(\Phi^u\big)'(1,1) = 1. \tag{3.12}$$

Let $h(s,t) := \eta(1, g(s,t))$ and

$$\Psi(s,t) := \big(s^{-1}I_b'\big(h(s,t)\big)h(s,t)^+,\ t^{-1}I_b'\big(h(s,t)\big)h(s,t)^-\big). \tag{3.13}$$

By the choice of $\varepsilon > 0$, (3.10), and (a_1), we have $g = h$ in ∂D. Thus, the definition of Φ^u and (3.13) imply $\Phi^u = \Psi$ in ∂D, from which we get

$$\det(\Psi, D, 0) = \det\big(\Phi^u, D, 0\big) = 1.$$

So, there exists $(s,t) \in D$ such that $h(s,t) \in \mathcal{M}$, which is in contradiction with (3.11). Thus we get $I_b'(u) = 0$. □

Proof of Theorem 1.1 Letting $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence for I_b under the constraint set $\mathcal{M}$, we know that the sequence $\{u_n\}$ is bounded in X by Lemma 2.4. Also there exists $u \in X$ such that $u_n \rightharpoonup u$ in X. Combining (2.25), (3.8), and Lemma 3.3, we have $I_b(u) = m$, $I_b'(u) = 0$, and $u^{\pm} \neq 0$. Therefore, when $\alpha_0 = 0$, Eq. (1.1) has a least-energy sign-changing solution u.

Next, it is proved that u has two nodal domains through contradictory assumptions. First, by Fatou's lemma, one can easily observe that

$$\langle I_b'(u), u^{\pm}\rangle \leq \liminf_{n\to\infty}\langle I_b'(u_n), u_n^{\pm}\rangle = 0.$$

Now, we assume

$$u = u_1 + u_2 + u_3 \tag{3.14}$$

with $u_i \neq 0$, $u_1 > 0$, $u_2 < 0$, $u_3 \geq 0$, $\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset$, $i \neq j$ $(i,j = 1,2,3)$, and

$$\langle I_b'(u), u_i\rangle = 0, \quad i = 1,2,3.$$

Let $v := u_1 + u_2$, $v^+ = u_1$ and $v^- = u_2$, as well as $v^{\pm} \neq 0$. Then, by Lemma 2.3(i), there exists $(s_v, t_v) \in (0,1] \times (0,1]$ such that

$$s_v v^+ + t_v v^- = s_v u_1 + t_v u_2 \in \mathcal{M}, \qquad I_b(s_v u_1 + t_v u_2) \geq m. \tag{3.15}$$

Through direct calculation, we have

$$\begin{aligned}
I_b\big(s_v v^+ + t_v v^-\big) &= I_b\big(s_v v^+\big) + I_b\big(t_v v^-\big) + \frac{b s_v^2 t_v^2}{2}\|v^+\|^2\|v^-\|^2 \\
&= \frac{s_v^2}{4}\|u_1\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(s_v u_1)s_v u_1 - 4F(s_v u_1)\big]\,dx \\
&\quad + \frac{t_v^2}{4}\|u_2\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(t_v u_2)t_v u_2 - 4F(t_v u_2)\big]\,dx \\
&\leq \frac{1}{4}\|u_1\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(u_1)u_1 - 4F(u_1)\big]\,dx \\
&\quad + \frac{1}{4}\|u_2\|^2 + \frac{1}{4}\int_{\mathbb{R}^2} K(x)\big[f(u_2)u_2 - 4F(u_2)\big]\,dx \\
&= I_b(u_1) + I_b(u_2) + \frac{b}{2}\|u_1\|^2\|u_2\|^2 + \frac{b}{4}\|u_1\|^2\|u_3\|^2 \\
&\quad + \frac{b}{4}\|u_2\|^2\|u_3\|^2.
\end{aligned} \tag{3.16}$$

In addition,

$$\begin{aligned} 0 &= \frac{1}{4}\langle I_b'(u), u_3\rangle \\ &= \frac{1}{4}\|u_3\|^2 + \frac{b}{4}\|u\|^2\|u_3\|^2 - \frac{1}{4}\int_{\mathbb{R}^2} K(x)f(u_3)u_3\,dx \\ &< I_b(u_3) + \frac{b}{4}\|u_1\|^2\|u_3\|^2 + \frac{b}{4}\|u_2\|^2\|u_3\|^2. \end{aligned} \tag{3.17}$$

From (3.15)–(3.17), we get the following contradiction:

$$\begin{aligned} m &\le I_b(s_v u_1 + t_v u_2) \\ &< I_b(u_1) + I_b(u_2) + I_b(u_3) + \frac{b}{2}\|u_1\|^2\|u_2\|^2 + \frac{b}{2}\|u_1\|^2\|u_3\|^2 + \frac{b}{2}\|u_2\|^2\|u_3\|^2 \\ &= I_b(u) = m. \end{aligned} \tag{3.18}$$

So $u_3 = 0$, and u exactly does have two nodal domains. □

In order to prove Theorem 1.2, we first introduce an auxiliary equation

$$-\left(1 + b\int_{\mathbb{R}^2} K(x)|\nabla u|^2\,dx\right)\operatorname{div}\bigl(K(x)\nabla u\bigr) = K(x)|u|^{p-2}u, \tag{3.19}$$

where $p > 4$ is given by (f_4). The energy functional corresponding to equation (3.19) is

$$I_p(u) = \frac{1}{2}\int_{\mathbb{R}^2} K(x)|\nabla u|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^2} K(x)|\nabla u|^2\,dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^2} K(x)|u|^p\,dx. \tag{3.20}$$

The corresponding Nehari manifold and Nehari nodal set are

$$\mathcal{N}_p = \bigl\{u \in X\backslash\{0\}; u \neq 0 : \langle I_p'(u), u\rangle = 0\bigr\} \tag{3.21}$$

and

$$\mathcal{M}_p = \bigl\{u \in X; u^{\pm} \neq 0 : \langle I_p'(u), u^+\rangle = \langle I_p'(u), u^-\rangle = 0\bigr\}. \tag{3.22}$$

When $p > 4$, the embedding $X \hookrightarrow L_K^p(\mathbb{R}^2)$ is compact. We use the previous proof to establish the existence of $w_p \in X$ satisfying $I_p(w_p) = m_p$, $I_p'(w_p) = 0$, and such that

$$m_p = \inf_{u\in\mathcal{M}_p} I_p(u) > 0 \tag{3.23}$$

holds.

For the critical case, we need to control m below the threshold to restore compactness, and now we estimate the value of m.

Let $\{u_n\} \subset \mathcal{M}_p$ be a minimizing sequence for $I_p(u_n) \to m_p$.

Lemma 3.4 *For $b > 0$, we have $0 < m < \frac{\pi}{2\alpha_0}$.*

Proof Let $w = w^+ + w^-$ and $w^\pm \neq 0$ be the sign-changing solution of (3.19). Then we have

$$\langle I_p'(w), w^+\rangle = \langle I_p'(w), w^-\rangle = \langle I_p'(w), w\rangle = 0 \tag{3.24}$$

and

$$m_p = I_p(w) = I_p(w) - \frac{1}{4}\langle I_p'(w), w\rangle \geq \frac{p-4}{4p}|w|_p^p. \tag{3.25}$$

Using (f_4) and (3.24), we have $\langle I_b'(w), w^\pm\rangle \leq 0$, while using Lemmas 2.3 and 2.4, there is a unique number pair $(s,t) \in (0,1] \times (0,1]$ such that $sw^+ + tw^- \in \mathcal{M}$. Combing (f_4), (3.24), (3.25), for $(s,t) \in (0,1] \times (0,1]$, we obtain

$$\begin{aligned}
m &\leq I_b\left(sw^+ + tw^-\right) \\
&\leq \frac{s^2}{2}\left\|w^+\right\|^2 + \frac{t^2}{2}\left\|w^-\right\|^2 + \frac{bs^4}{4}\left\|w^+\right\|^4 + \frac{bt^4}{4}\left\|w^-\right\|^4 \\
&\quad + \frac{bs^2t^2}{2}\left\|w^+\right\|^2\left\|w^-\right\|^2 - \frac{\varrho_0 s^p}{p}\left|w^+\right|_p^p - \frac{\varrho_0 t^p}{p}\left|w^-\right|_p^p \\
&= \frac{s^2}{2}\left[\int_{\mathbb{R}^2} K(x)\left|w^+\right|^p dx - b\left\|w^+\right\|^4 - b\left\|w^+\right\|^2\left\|w^-\right\|^2\right] + \frac{bs^4}{4}\left\|w^+\right\|^4 \\
&\quad + \frac{t^2}{2}\left[\int_{\mathbb{R}^2} K(x)\left|w^-\right|^p dx - b\left\|w^-\right\|^4 - b\left\|w^-\right\|^2\left\|w^-\right\|^2\right] + \frac{bt^4}{4}\left\|w^-\right\|^4 \\
&\quad + \frac{bs^2t^2}{2}\left\|w^+\right\|^2\left\|w^-\right\|^2 - \frac{\varrho_0 s^p}{p}\left|w^+\right|_p^p - \frac{\varrho_0 t^p}{p}\left|w^-\right|_p^p \\
&\leq \max_{\xi>0}\left(\frac{\xi^2}{2} - \frac{\varrho_0\xi^p}{p}\right)|w|_p^p - \frac{bs^2}{4}\left\|w^+\right\|^4\left(2-s^2\right) - \frac{bt^2}{4}\left\|w^-\right\|^4\left(2-t^2\right) \\
&\quad - \frac{s^2+t^2-s^2t^2}{2}b\left\|w^+\right\|^2\left\|w^-\right\|^2 \\
&\leq \max_{\xi>0}\left(\frac{\xi^2}{2} - \frac{\varrho_0\xi^p}{p}\right)|w|_p^p = \frac{p-2}{2p}\varrho_0^{-\frac{2}{p-2}}|w|_p^p \\
&\leq \frac{2(p-2)}{p-4}\varrho_0^{-\frac{2}{p-2}}m_p.
\end{aligned} \tag{3.26}$$

□

Lemma 3.5 *Suppose $\{u_n\} \subset \mathcal{M}$ is a minimizing sequence for m. Then*

$$\limsup_{n\to\infty}\|u_n\|^2 < \frac{2\pi}{\alpha_0}.$$

Proof From the assumption, we have $I_b(u_n) \to m$, $\langle I_b'(u_n), u_n\rangle = 0$, when $n \to +\infty$. From (f_3), we have

$$m + o(1) = I_b(u_n) - \frac{1}{4}\langle I_b'(u_n), u_n\rangle \geq \frac{1}{4}\|u_n\|^2.$$

From Lemma 2.4, we have

$$\limsup_{n\to\infty}\|u_n\|^2 \le 4m \le \frac{8(p-2)}{p-4}\varrho_0^{-\frac{2}{p-2}} m_p.$$

Using (f_4), we get $\limsup_{n\to\infty}\|u_n\|^2 < \frac{2\pi}{\alpha_0}$. □

Lemma 3.6 *Assume $\{u_n\}\subset \mathcal{M}$ is a minimizing sequence for m. Then*

$$\int_{\mathbb{R}^2} K(x)f\big(u_n^\pm\big)\big(u_n^\pm\big)\,dx \to \int_{\mathbb{R}^2} K(x)f\big(u^\pm\big)\big(u^\pm\big)\,dx$$

and

$$\int_{\mathbb{R}^2} K(x)F\big(u_n^\pm\big)\,dx \to \int_{\mathbb{R}^2} K(x)F\big(u^\pm\big)\,dx.$$

Proof We only prove the first limit here, as the second is obtained similarly. By Lemma 3.5, we have $\limsup_{n\to\infty}\|u_n\|^2 \le \frac{2\pi}{\alpha_0}$ and, up to a subsequence, $u_n^\pm(x)\to u^\pm(x)$ and

$$f\big(u_n^\pm(x)\big)\big(u_n^\pm(x)\big) \to f\big(u^\pm(x)\big)\big(u^\pm(x)\big) \quad \text{a.e. in } \mathbb{R}^2.$$

Arguing as in the proof of Lemma 3.1, introducing $g:\mathbb{R}\to\mathbb{R}$, $g\in L^1(\mathbb{R}^2)$, and using (1.6), we have

$$K(x)f\big(u_n^\pm(x)\big)\big(u_n^\pm(x)\big) \le \varepsilon K(x)\big|u_n^\pm(x)\big|^2 + C_\varepsilon K(x)\big|u_n^\pm(x)\big|^q\big(e^{\alpha(u_n^\pm(x))^2}-1\big) := g\big(u_n^\pm(x)\big).$$

We will prove that $g(u_n^\pm)$ converges in $L^1(\mathbb{R}^2)$. First, note that

$$\int_{\mathbb{R}^2} K(x)\big|u_n^\pm\big|^2\,dx \to \int_{\mathbb{R}^2} K(x)\big|u^\pm\big|^2\,dx.$$

Considering $s',s>1$ such that $\frac{1}{s}+\frac{1}{s'}=1$ and $s\to 1^+$, we obtain

$$K(x)^{\frac{1}{s'}}\big|u_n^\pm\big|^q \to K(x)^{\frac{1}{s'}}\big|u^\pm\big|^q \quad \text{in } L^{s'}\big(\mathbb{R}^2\big). \tag{3.27}$$

Now, choosing $\alpha>\alpha_0$ and close to α_0, using Lemma 2.1, there exists $M_2>0$ such that

$$\begin{aligned}\int_{\mathbb{R}^2} K(x)\big(e^{\alpha s(u_n^\pm(x))^2}-1\big)\,dx &= \int_{\mathbb{R}^2} K(x)\big(e^{\alpha s\|u_n^\pm\|^2(\frac{u_n^\pm(x)}{\|u_n^\pm\|})^2}-1\big)\,dx\\ &\le \int_{\mathbb{R}^2} K(x)\big(e^{4\pi(\frac{u_n^\pm(x)}{\|u_n^\pm\|})^2}-1\big)\,dx \le M_2.\end{aligned} \tag{3.28}$$

Since

$$K(x)e^{\alpha s|u_n^\pm(x)|^2} \to K(x)e^{\alpha s|u^\pm(x)|^2} \quad \text{a.e. in } \mathbb{R}^2,$$

we use Lemma 4.8 of [33] and conclude that

$$K(x)e^{\alpha s|u_n^\pm|^2} \rightharpoonup K(x)e^{\alpha s|u^\pm|^2} \quad \text{in } L^s\big(\mathbb{R}^2\big). \tag{3.29}$$

Using (3.27), (3.29), and Lemma 4.8 of [33] again, we conclude

$$\int_{\mathbb{R}^2} K(x)f\left(u_n^{\pm}\right)\left(u_n^{\pm}\right)dx \to \int_{\mathbb{R}^2} K(x)f\left(u^{\pm}\right)\left(u^{\pm}\right)dx. \qquad \square$$

Proof of Theorem 1.2 The proof is similar to that of Theorem 1.1. We conclude that in the critical case, I_b has a least-energy sign-changing solution which has precisely two nodal domains. $\square$

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