Salisu et al. *Journal of Inequalities and Applications* (2022) 2022:143
https://doi.org/10.1186/s13660-022-02870-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Strong convergence theorems for fixed point of multi-valued mappings in Hadamard spaces

Sani Salisu[1,2], Poom Kumam[1,3*] and Songpon Sriwongsa[1]

*Correspondence:
poom.kum@kmutt.ac.th
[1]Center of Excellence in Theoretical and Computational Science (TaCS-CoE) & KMUTT Fixed Point Research Laboratory, Room SCL 802, Fixed Point Laboratory, Science Laboratory Building, Departments of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
[3]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan
Full list of author information is available at the end of the article

Abstract

With the help of CN-inequality, we study fixed point of multi-valued mappings with closed bounded images and establish some strong convergence theorems involving a countable family of demicontractive mappings in Hadamard spaces. Furthermore, we use the established theorems to deduce some theorems involving a family of minimization problems, variational inequality problems, and monotone inclusion problems. We finally give examples to illustrate the results. The results obtained herein generalise some recent results in the literature.

MSC: 47H09; 47H10; 47J25

Keywords: Common fixed point; Demiclosedness-type property; Hadamard space; Multi-valued demicontractive mappings; Strong convergence; Δ-convergence

1 Introduction

Let (H,ρ) be a metric space and D be a nonempty subset of H. A point $u \in D$ is said to be a *fixed point* of a single-valued map $g : D \to H$ if $g(u) = u$. For a multi-valued map $T : D \to 2^H$, a point $v \in D$ is said to be a *fixed point* of T if $u \in Tu$. In what follows, we denote the fixed points set of T by $F(T)$, that is, $F(T) = \{x \in D : x \in Tx\}$. The problem of finding fixed points of certain mapping(s) is known as the Fixed Point Problem (FPP). It is known that many problems that arise from engineering, biology, economics, and mathematics can be reduced to FPP (see, e.g., [1, 2]). Moreover, for certain nonlinear mappings, FPP solves Minimization Problem (MP), Variational Inequality Problem (VIP), and Monotone Inclusion Problem (MIP), which play significant role in optimization, semigroup theory, graph theory, and differential equations and have applications in control theory, chaos theory, nonlinear programming, image restorations, and radiation therapy (for more details, see, e.g., [3, 4]).

The study of fixed point problems (for both single-valued and multi-valued) has attracted the attentions of many researchers. They usually focused on the existence, uniqueness, and/or approximations of fixed points of certain nonlinear mappings in various settings. For instance, Banach proved that a single-valued contractive mapping g defined on a complete metric space always had a unique fixed point and the sequence $u_n = g(u_{n-1})$

converges to the fixed point [5]. Banach's result is indeed remarkable because it not only shows the existence and uniqueness of fixed point of contractive mapping but also establishes an iterative scheme that approximate the point. However, it is known that (see, e.g., [1, 2]) contractive mappings are not always common in applications, and many problems result in more general nonlinear mappings, where identifying the fixed point, even if it exists, is a difficult task. As a result, many iterative schemes for approximating fixed point(s) of various generalizations of contractive mappings (for single-valued) are developed in linear spaces, namely in Banach and Hilbert spaces. For example, *Mann iteration*, *Ishikawa iteration*, and *Noor iteration* were developed in [6, 7], and [8], respectively. The convergence results obtained from the named iterative schemes are mostly weak convergence. However, in an infinite dimensional space, strong convergence is more desirable. As a result, several modified iterative schemes using various techniques are developed to obtain strong convergence results [9–11].

In [12], Nadler proved a multi-valued version of Banach's result in a complete metric space [12]. Since then, the study of fixed point theory for multi-valued mappings has extensively attracted the attention of many scholars. This is, perhaps, because it incorporates single-valued fixed point theory and has additional applications from Game Theory, Market Economy, and Non-smooth Deferential Equation/Inclusions. Moreover, iterative schemes for single-valued mappings are extended to multi-valued setting (see, e.g., [10, 13–17]).

Scholars recently used CN-inequality to extend many results in fixed point theory from linear setting (e.g., Hilbert spaces) to nonlinear setting (CAT(0) spaces), as the inequality allows CAT(0) to incorporate linear spaces (e.g., pre-Hilbert space) and nonlinear spaces (e.g., $\mathbb{R}$-tree, Hadamard Manifold) [18]. Moreover, CAT(0) spaces capture a wide range of problems in addition to that of Hilbert spaces. For example, non-convex sets in the classical sense can be seen as convex set in CAT(0); non-convex functional in linear spaces can be convex functional in CAT(0) spaces; constrained optimization problems can be an unconstrained problem in some sense [19, 20]. Additionally, some problems that cannot be fitted in Hilbert spaces, such as the asymptotic behavior of the Calabi flow in Kahler geometry, have been properly analyzed in CAT(0) spaces [21]. For further existing applications of these spaces and fixed point theory in various mathematical fields, see [17, 22–26].

Regarding the fixed points of nonlinear mappings in a CAT(0) space, Kirk studied fixed point theory and proved that a nonexpansive mapping on a convex closed and bounded set of a Hadamard space possesses a fixed point [27]. After that, Dhompongsa and Panyanak obtained some Δ-convergence theorems for the so-called Mann and Ishikawa iterations involving a single mapping in CAT(0) spaces [28]. In 2014, Chidume et al. proved strong and Δ-convergence theorems using the Krasnoselskii-type scheme to approximate a finite family of demicontractive maps in CAT(0) spaces. For recent development concerning fixed point theory in a geodesic/CAT(0) space, the reader is referred to [29–33] and the references therein.

On the other hand, inspired by the Noor iteration, Phuengrattana and Suantai introduced SP iteration and proved that for a continuous function g on a closed interval E with a fixed point, SP is bounded if and only if it converges to a fixed point of g [34]. The authors further analyzed that if g is a nondecreasing function, the convergence due to SP is faster than that of Mann, Ishikawa, and Noor iterations. Very recently, Chaolamjiak et al. [11] incorporated SP iteration with an inertial term to approximate a common fixed point of three multi-valued quasi-nonexpansive mappings and established the following theorem, among others:

Theorem 1.1 *Let D be a convex closed subset of a real Hilbert space. Let $T_i, i = 1,2,3$ be three quasi-nonexpansive multi-valued mappings on D with closed bounded images with $\mathcal{F} := \bigcap_i^3 F(T_i) \neq \emptyset$ and each $T_i p = \{p\}$ for all $p \in \mathcal{F}$. Suppose that each $I - T_i$ is demiclosed at 0 and consider a sequence $\{u_n\}$ defined by*

$$\begin{cases} u_0, u_1 \in C \quad \text{chosen arbitrary}, \\ w_n^{(0)} = (1+\theta_n)u_n - \theta_n u_{n-1}, \\ w_n^{(1)} \in \beta_n^{(1)} w_n^{(0)} + (1-\beta_n^{(1)}) T_1 w_n^{(0)}, \\ w_n^{(2)} \in \beta_n^{(2)} w_n^{(1)} + (1-\beta_n^{(2)}) T_2 w_n^{(1)}, \\ u_{n+1} \in \beta_n^{(3)} w_n^{(2)} + (1-\beta_n^{(3)}) T_3 w_n^{(2)}, \end{cases} \tag{1}$$

where $\{\beta_n^{(i)}\} \subset (0,1)$, $i = 1,2,3$. If

$$\sum_{n=1}^{+\infty} \theta_n \|u_n - u_{n-1}\| < +\infty \quad \text{and} \quad 0 < \liminf_{n\to+\infty} \beta_n^{(i)} < \limsup_{n\to+\infty} \beta_n^{(i)} < 1, \quad i = 1,2,3, \tag{2}$$

then the sequence $\{u_n\}$ converges weakly to an element of $\mathcal{F}$.

In addition, the authors incorporated hybrid CQ shrinking projection to the iterative scheme in (1) to obtain strong convergence results with the assumption that (2) holds.

Knowing that strong convergence is more desirable in infinite dimensional spaces, and projection is difficult to compute in most cases, it is natural to ask if one can have strong convergence without using projection, also if the such result could hold for any number of families, not just for 3, or even hold for a wider class of mappings than quasi-nonexpansive. Considering the recent line of research, it is also worth asking if the results can be extended to more general spaces than real Hilbert spaces, such as Hadamard Manifold, $\mathbb{R}$-tree, and so on. We as well ask if some assumptions in (2) can be relaxed to weaker assumptions.

This work gives an affirmative answer to the naturally raised questions, thereby establishing strong convergence theorems for a countable family of multi-valued demicontractive mappings in the setting of Hadamard spaces. Moreover, as an application, this work establishes additional convergence theorems for solving a collection of minimization problems, variational inequality problems for inverse strongly monotone mappings, and monotone inclusion problems.

The paper is organized as follows: In Sect. 2, preliminaries consisting of lemmas, definitions, and some characterizations, which are essential for the convergence analysis of the proposed scheme, are stated in the setting of Hadamard spaces. The proposed scheme and its convergence analysis are presented in Sect. 3. Finally, in Sect. 4, the applications and computational illustrations to show the implementation of our method are given.

2 Preliminaries

In this section, we state some definitions and basic facts that will be useful in our convergence analysis in Sect. 3. We start by recalling some basic and required ingredients consisting of definitions, characterizations, and lemmas in Hadamard spaces, which can be found in [18, 22].

Let (H,ρ) be a metric space and u, v be two points in H. A map $\tau : [0,r] \subset \mathbb{R} \to H$ is a *geodesic path* from u to v if the followings hold.

(i) $\tau(0) = u$ and $\tau(r) = v$;

(ii) $\rho(\tau(t_1),\tau(t_2)) = |t_1 - t_2|$ for every t_1, $t_2 \in [0,r]$.

The image $\tau([0,r])$ of τ is a *geodesic segment* joining u and v, when $\tau([0,r])$ is unique, it is denoted by $[u,v]$. A metric space (H,ρ) is a geodesic space if every two elements u, v in H are joined by a geodesic segment and is said to be a uniquely geodesic space if every two points u, v are joined by a unique geodesic segment $[u,v]$ in H. A geodesic triangle $\Delta(u,v,w)$ in H is a set of three points u, v, w (called the vertices of Δ) together with three geodesic segments connecting each pair. For a uniquely geodesic space the triangle Δ is simply

$$\Delta(v,u,w) = [v,w] \cup [w,u] \cup [u,v], \tag{3}$$

where u, v, w are the vertices, and $[u,v]$, $[v,w]$, $[w,u]$ are the edges of Δ. A comparison triangle of a geodesic triangle $\Delta(u,v,w)$ is a triangle in the Euclidean space $(\mathbb{R}^2, \|\cdot\|_2)$ denoted by $\bar{\Delta}(\bar{u},\bar{v},\bar{w})$ satisfying

$$\rho(u,v) = \|\bar{u} - \bar{v}\|_2; \qquad \rho(v,w) = \|\bar{v} - \bar{w}\|_2; \qquad \rho(u,w) = \|\bar{u} - \bar{w}\|_2.$$

For any point $z \in \Delta(u,v,w)$, if z lies in the segment connecting u and v, then a comparison point of z in a comparison triangle $\bar{\Delta}(\bar{u},\bar{v},\bar{w})$ is the point $\bar{z} \in [\bar{u},\bar{v}] \subset \bar{\Delta}(\bar{u},\bar{v},\bar{w})$ with $\rho(u,z) = \|\bar{u} - \bar{z}\|_2$. For points in the triangle $\Delta(u,v,w)$, their corresponding comparison points are defined in a similar way with that of z.

A metric space (H,ρ) is called a CAT(0) space if it is a geodesic space, and every geodesic triangle Δ in H is as thin as its comparison triangle in $(\mathbb{R}^2, \|\cdot\|_2)$, in the sense that for a geodesic triangle $\Delta(u,v,w)$ with a comparison triangle $\bar{\Delta}(\bar{u},\bar{v},\bar{w})$,

$$\rho(x,z) \le \|\bar{x} - \bar{z}\|_2, \tag{4}$$

where x and z are arbitrary points in $\Delta(u,v,w)$ with corresponding comparison points $\bar{x}$ and $\bar{z}$. Equivalently, a geodesic space (H,ρ) is a CAT(0) space if and only if it satisfies the CN-inequality of Bruhat and Tits [35] as follows: Let $w,v \in H$ and z be a midpoint of a geodesic segment connecting w and v, then

$$d^2(z,y) \le \frac{1}{2}d^2(w,y) + \frac{1}{2}d^2(v,y) - \frac{1}{4}d^2(w,v), \tag{5}$$

for every $y \in H$. This CN-inequality guaranteed that CAT(0) spaces are unique geodesic spaces and subsequently yield to the notation $\oplus$, which we will defined later. Example of CAT(0) spaces includes pre-Hilbert spaces [22], Hilbert balls [36], Hyperbolic metrics [37], Euclidean buildings [38], $\mathbb{R}$-trees [23], and Hadamard manifolds. A complete CAT(0) space is called a *Hadamard space*.

In the sequel, unless otherwise stated, we denote a Hadamard space by (H,ρ), nonempty subset of H by E, convex closed nonempty subset of H by D and denote the family of nonempty closed bounded subsets of H by $\mathcal{CB}(H)$, and set

$$\rho_H(A,B) := \max\Big\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\Big\},$$

where A, B are any two nonempty closed bounded subsets of H and $\operatorname{dist}(u,D) := \inf\{\rho(u,v) : v \in D\}$ for any $u \in H$. A multi-valued map $T : E \to \mathcal{CB}(H)$ is said to be

(i) *contractive* if there exists $a \in [0,1)$ such that

$$\rho_H(Tu,Tw) \le a\rho(u,w) \quad \text{for all } u,w \in E;$$

(ii) *nonexpansive* if

$$\rho_H(Tu,Tw) \le \rho(u,w) \quad \text{for all } u,w \in E;$$

(iii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\rho_H(Tu,Tp) \le \rho(u,p) \quad \text{for all } u \in E \text{ and } p \in F(T);$$

(iv) *demicontractive* if $F(T) \neq \emptyset$, and there exists $k \in [0,1)$ such that

$$\rho_H^2(Tu,Tp) \le \rho^2(u,p) + \operatorname{dist}^2(u,Tu) \quad \text{for all } u \in E \text{ and } p \in F(T);$$

(v) *hybrid* if

$$3\rho_H^2(Tu,Tw) \le \rho^2(u,w) + \operatorname{dist}^2(u,Tw) + \operatorname{dist}^2(w,Tu), \quad \text{for all } u,w \in D.$$

It is well known that if $F(T) \neq \emptyset$, then

$$\text{(i)} \implies \text{(ii)} \implies \text{(v)} \implies \text{(iii)} \implies \text{(iv)}.$$

Let $\{u_n\}$ be a bounded sequence in H, and let $\rho(\cdot,\{u_n\}) : H \to [0,\infty)$ be defined by

$$\rho\big(u,\{u_n\}\big) := \limsup_{n\to\infty}\rho(u,u_n), \quad u \in H.$$

If $\rho(\{u_n\})$ is given by

$$\rho\big(\{u_n\}\big) := \inf_{u\in H}\rho\big(u,\{u_n\}\big),$$

then the asymptotic center $A(\{u_n\})$ of $\{u_n\}$ is the set

$$A\big(\{u_n\}\big) := \big\{u \in H : \rho\big(u,\{u_n\}\big) = \rho\big(\{u_n\}\big)\big\}.$$

According to [39, Proposition 7], $A(\{u_n\})$ is a singleton set.

Lemma 2.1 ([40]) *The asymptotic centre $A(\{u_n\})$ of any bounded sequence $\{u_n\}$ in D is in D.*

A bounded sequence $\{u_n\}$ Δ-converges to a point u in H if u is the unique asymptotic centre for every subsequence $\{u_{n_k}\}$ of $\{u_n\}$. In other words, if the $\limsup_{k\to\infty}\rho(u_{n_k},u)\leq \limsup_{k\to\infty}\rho(u_{n_k},y)$ for every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and for every $y\in X$ [41]. In this case, we write $\Delta\text{-}\lim_{n\to\infty}u_n=u$ and call u the Δ-limit of $\{u_n\}$. Recall that $\{u_n\}$ converges strongly to a point u in H if $\lim_{n\to\infty}d(u_n,u)=0$, and we write $\lim_{n\to\infty}u_n=u$ and call u the strong limit of $\{u_n\}$. We say that a multi-valued map $T:D\subseteq H\to\mathcal{CB}(H)$, has *demiclosedness-type property* if for any sequence $\{u_n\}\subseteq D$ and $u\in D$,

$$\left.\begin{aligned}&\Delta\text{-}\lim_{n\to\infty}u_n=u\\&\lim_{n\to\infty}\operatorname{dist}(u_n,Tu_n)=0\end{aligned}\right\}\quad\Longrightarrow\quad u\in Tu.$$

Lemma 2.2 ([41, Proposition 3.6]) *Every bounded sequence $\{u_n\}$ in D has a Δ-convergent subsequence $\{u_{n_k}\}$ in D.*

Lemma 2.3 ([41, Proposition 3.7]) *For a single-valued nonexpansive map $g:D\to H$, the conditions $\{u_n\}$ Δ-converges to u and $\rho(u_n,gu_n)\to 0$ imply $u\in D$ and $Tu=u$.*

In the next section, we provide a multi-valued version of Lemma 2.3.

Berg and Nokilaev denoted $(u,w)\in H\times H$ by $\overrightarrow{uw}$ and defined a quasilinearization map $\langle\cdot,\cdot\rangle:(H\times H)\times(H\times H)\to\mathbb{R}$ by

$$\langle\overrightarrow{uw},\overrightarrow{vz}\rangle=\frac{1}{2}\big(\rho^2(u,z)+\rho^2(w,v)-\rho^2(u,v)-\rho^2(w,z)\big),\quad(u,v,w,z\in H).$$

Lemma 2.4 ([42, Theorem 2.6]) *A bounded sequence $\{u_n\}$ Δ-converges to a point u in H if and only if $\limsup_{n\to\infty}\langle\overrightarrow{u_nu},\overrightarrow{wu}\rangle\leq 0$ for all $w\in H$.*

Lemma 2.5 ([43, Lemma 3.1]) *Let $\{\ell_n\}$ be a sequence in $\mathbb{R}$ such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $\ell_{n_j}<\ell_{n_j+1}$ for every $j\in\mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$ and for a sufficiently large number $k\in\mathbb{N}$,*

$$\ell_{m_k}\leq\ell_{m_k+1}\quad\textit{and}\quad\ell_k\leq\ell_{m_k+1}.$$

In fact, $m_k=\max\{i\leq k:\ell_i<\ell_{i+1}\}$.

Lemma 2.6 ([9, Lemma 2.5]) *Let $\{\ell_n\}$ be a sequence in $[0,+\infty)\subset\mathbb{R}$ with*

$$\ell_{n+1}\leq(1-\sigma_n)\ell_n+\sigma_n\phi_n+\gamma_n,\quad n\geq 1,\tag{6}$$

where $\{\sigma_n\}$, $\{\phi_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\sigma_n\}\subset[0,1]$, $\sum_{n=1}^{\infty}\sigma_n=\infty$,

(ii) $\limsup_{n\to\infty}\phi_n\leq 0$, *and*

(iii) $\{\gamma_n\}\subset[0,\infty)$, $\sum_{n=1}^{\infty}\gamma_n<\infty$.

Then $\lim_{n\to\infty}\ell_n=0$.

Lemma 2.7 ([28, Lemma 2.1 (iv)]) *Let u, v in H. Then for each $\alpha\in[0,1]$, there exists a unique point $w\in[u,v]$ such that*

$$\rho(u,w)=\alpha\rho(u,v)\quad\textit{and}\quad\rho(v,w)=(1-\alpha)\rho(u,v).$$

In the sequel, w is denoted by $(1-\alpha)u \oplus \alpha v$.

Lemma 2.8 ([28, Lemma 2.4]) *Let u_1, u_2 be points in H and $a \in [0,1]$. Then*

$$\rho\big((1-a)u_1 \oplus au_2, u_3\big) \le (1-a)\rho(u_1,u_3) + a\rho(u_2,u_3),$$

for every $u_3 \in H$.

Lemma 2.9 ([28, Lemma 2.5]) *Let u_1, u_2, u_3 be points in H and $a \in [0,1]$. Then*

$$\rho^2\big((1-a)u_1 \oplus au_2, u_3\big) \le (1-a)\rho^2(u_1,u_3) + a\rho^2(u_2,u_3) - a(1-a)\rho^2(u_1,u_2),$$

for every $u_3 \in H$.

Lemma 2.10 ([44, Lemma 2.7]) *Let u_1, u_2 be points in H and $a \in [0,1]$. Then*

$$\rho^2\big((1-a)u_1 \oplus au_2, u_3\big) \le (1-a)^2\rho^2(u_1,u_3) + a^2\rho^2(u_2,u_3) + 2a(1-a)\langle\overrightarrow{u_1u_3}, \overrightarrow{u_2u_3}\rangle,$$

for every $u_3 \in H$.

3 Main results

Recall that (H,ρ) denotes a Hadamard space, and D denotes a nonempty convex closed subset of H. We start by proving a multi-valued version of the result in Lemma 2.3.

Theorem 3.1 *A multi-valued nonexpansive map $T : D \to \mathcal{CB}(D)$ has demiclosedness-type property.*

Proof Let $\{u_n\} \subset D$ such that

$$\Delta\text{-}\lim_{n\to\infty} = u, \tag{7}$$

$$\lim_{n\to\infty} \operatorname{dist}(u_n, Tu_n) = 0. \tag{8}$$

Then by Lemma 2.1, $u \in D$. Now let $y \in Tu$. We show that $y = u$. Assume, on the contrary, that $y \ne u$. Then by uniqueness of asymptotic center, we have

$$\limsup_{n\to\infty} \rho(u_n,u) < \limsup_{n\to\infty} \rho(u_n,y). \tag{9}$$

However,

$$\begin{aligned}\rho(u_n,y) &\le \rho(u_n,z_n) + \rho(z_n,y)\\ &\le \rho(u_n,z_n) + \rho_H(Tu_n,Tu), \quad \text{for every } z_n \in Tu_n.\end{aligned}$$

This implies that

$$\begin{aligned}\rho(u_n,y) &\le \operatorname{dist}(u_n,Tu_n) + \rho_H(Tu_n,Tu)\\ &\le \operatorname{dist}(u_n,Tu_n) + \rho(u_n,u).\end{aligned}$$

By (8), we have

$$\limsup_{n\to\infty}\rho(u_n,y) < \limsup_{n\to\infty}\rho(u_n,u)$$

that contradicts (9). □

Lemma 3.2 *For any bounded sequence $\{u_n\}$ in D, there exists a point u that is a Δ-limit of some subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and*

$$\limsup_{n\to\infty}\langle\overrightarrow{u_n u},\overrightarrow{yu}\rangle \le 0, \quad \textit{for all } y\in H.$$

Proof Let $y,z\in H$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle\overrightarrow{u_n z},\overrightarrow{yz}\rangle = \lim_{k\to\infty}\langle\overrightarrow{u_{n_k} z},\overrightarrow{yz}\rangle. \tag{10}$$

Also, by the boundedness of $\{u_{n_k}\}$ and Lemma 2.2, there exists a subsequence $\{u_{n_j}\}$ of $\{u_{n_k}\}$ such that $\{u_{n_j}\}$ Δ-converges to some point u and by Lemma 2.1, $u\in D$. By Lemma 2.4 and (10), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle\overrightarrow{u_n u},\overrightarrow{yu}\rangle &= \lim_{k\to\infty}\langle\overrightarrow{u_{n_k} u},\overrightarrow{yu}\rangle \\
&= \lim_{j\to\infty}\langle\overrightarrow{u_{n_j} u},\overrightarrow{yu}\rangle \\
&= \limsup_{j\to\infty}\langle\overrightarrow{u_{n_j} u},\overrightarrow{yu}\rangle \\
&\le 0.
\end{aligned} \tag{11}$$

This completes the proof. □

We now consider countably infinite family of demicontractive mappings with common fixed point.

Lemma 3.3 *Let $T_i: D\to \mathcal{CB}(D)$, $i\in\mathbb{N}$ be a family of multi-valued demicontractive mappings with constants $\{k_i\}\subset(0,1)$, $\mathcal{F} := \bigcap_{i\in\mathbb{N}} F(T_i)\neq\emptyset$ and each $T_i p = \{p\}$ for all $p\in\mathcal{F}$. Suppose that $\{u_n\}$ is a sequence generated by*

$$\begin{cases}
v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \quad u_1\in D,\\
v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) w_n^{(i-1)}, \quad w_n^{(i-1)}\in T_i v_n^{(i-1)}, i = 1,\dots,n-1,\\
u_{n+1} = \beta_n^{(n)} v_n^{(n-1)} \oplus (1-\beta_n^{(n)}) w_n^{(n-1)}, \quad w_n^{(n-1)}\in T_n v_n^{(n-1)}, n\ge 1,
\end{cases} \tag{12}$$

with $\{\alpha_n\}\subset[0,1]$, $\{\beta_n^{(i)}\}\subset[k_i,1]$ and $\alpha_n\to 0$ as $n\to\infty$. Then

(i) *$\{u_n\}$ is bounded, and*

(ii) *$\limsup_{n\to\infty}(\rho(u_n,p)^2 - \rho(u_{n+1},p)^2) = 0$, for all $p\in\mathcal{F}$.*

Proof Let $p \in \mathcal{F}$ and let $n \in \mathbb{N}$. By Lemma 2.9, scheme (12), and the assumptions on $T_i's$, we have

$$\begin{aligned}
\rho^2\big(v_n^{(i)},p\big) &\le \beta_n^{(i)}\rho^2\big(v_n^{(i-1)},p\big) + \big(1-\beta_n^{(i)}\big)\rho^2\big(w_n^{(i-1)},p\big) \\
&\quad - \beta_n^{(i)}\big(1-\beta_n^{(i)}\big)\rho^2\big(v_n^{i-1},w_n^{(i-1)}\big) \\
&\le \beta_n^{(i)}\rho^2\big(v_n^{(i-1)},p\big) + \big(1-\beta_n^{(i)}\big)\operatorname{dist}^2\big(w_n^{(i-1)},T_ip\big) \\
&\quad - \beta_n^{(i)}\big(1-\beta_n^{(i)}\big)\rho^2\big(v_n^{(i-1)},w_n^{(i-1)}\big) \\
&\le \beta_n^{(i)}\rho^2\big(v_n^{(i-1)},p\big) + \big(1-\beta_n^{(i)}\big)\rho_H^2\big(T_iv_n^{(i-1)},T_ip\big) \\
&\quad - \beta_n^{(i)}\big(1-\beta_n^{(i)}\big)\rho^2\big(v_n^{(i-1)},w_n^{(i-1)}\big) \\
&\le \beta_n^{(i)}\rho^2\big(v_n^{(i-1)},p\big) + \big(1-\beta_n^{(i)}\big)\big[\rho^2\big(v_n^{(i-1)},p\big) + k_i\rho^2\big(v_n^{(i-1)},w_n^{(i-1)}\big)\big] \\
&\quad - \beta_n^{(i)}\big(1-\beta_n^{(i)}\big)\rho^2\big(v_n^{(i-1)},w_n^{(i-1)}\big) \\
&\le \rho^2\big(v_n^{(i-1)},p\big) + \big(1-\beta_n^{(i)}\big)\big(k_i-\beta_n^{(i)}\big)\rho^2\big(v_n^{(i-1)},w_n^{(i-1)}\big) \\
&= \rho^2\big(v_n^{(i-1)},p\big) - \big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)}-k_i\big)\rho^2\big(v_n^{(i-1)},w_n^{(i-1)}\big),
\end{aligned}$$

for each $i \in \{1,\dots,n-1\}$. Also,

$$\begin{aligned}
\rho^2(u_{n+1},p) &\le \beta_n^{(n)}\rho^2\big(v_n^{(n-1)},p\big) + \big(1-\beta_n^{(n)}\big)\rho^2\big(w_n^{(n-1)},p\big) \\
&\quad - \beta_n^{(n)}\big(1-\beta_n^{(n)}\big)\rho^2\big(v_n^{n-1},w_n^{(n-1)}\big) \\
&\le \beta_n^{(n)}\rho^2\big(v_n^{(n-1)},p\big) + \big(1-\beta_n^{(n)}\big)\operatorname{dist}^2\big(w_n^{(n-1)},T_np\big) \\
&\quad - \beta_n^{(n)}\big(1-\beta_n^{(n)}\big)\rho^2\big(v_n^{(n-1)},w_n^{(n-1)}\big) \\
&\le \beta_n^{(n)}\rho^2\big(v_n^{(n-1)},p\big) + \big(1-\beta_n^{(n)}\big)\rho_H^2\big(T_nv_n^{(n-1)},T_np\big) \\
&\quad - \beta_n^{(n)}\big(1-\beta_n^{(n)}\big)\rho^2\big(v_n^{(n-1)},w_n^{(n-1)}\big) \\
&\le \beta_n^{(n)}\rho^2\big(v_n^{(n-1)},p\big) + \big(1-\beta_n^{(n)}\big)\big[\rho^2\big(v_n^{(n-1)},p\big) + k_n\rho^2\big(v_n^{(n-1)},w_n^{(n-1)}\big)\big] \\
&\quad - \beta_n^{(n)}\big(1-\beta_n^{(n)}\big)\rho^2\big(v_n^{(n-1)},w_n^{(n-1)}\big) \\
&\le \rho^2\big(v_n^{(n-1)},p\big) + \big(1-\beta_n^{(n)}\big)\big(k_i-\beta_n^{(n)}\big)\rho^2\big(v_n^{(n-1)},w_n^{(n-1)}\big) \\
&= \rho^2\big(v_n^{(n-1)},p\big) - \big(1-\beta_n^{(n)}\big)\big(\beta_n^{(n)}-k_i\big)\rho^2\big(v_n^{(n-1)},w_n^{(n-1)}\big) \\
&\le \rho^2\big(v_n^{(n-2)},p\big) - \big(1-\beta_n^{(n-1)}\big)\big(\beta_n^{(n-1)}-k_{n-1}\big)\rho^2\big(v_n^{(n-2)},w_n^{(n-2)}\big) \\
&\quad - \big(1-\beta_n^{(n)}\big)\big(\beta_n^{(n)}-k_n\big)\rho^2\big(v_n^{(n-1)},w_n^{(n-1)}\big).
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
\rho^2(u_{n+1},p) &\le \rho^2\big(v_n^{(n-2)},p\big) - \big(1-\beta_n^{(n-1)}\big)\big(\beta_n^{(n-1)}-k_{n-1}\big)\rho^2\big(v_n^{(n-2)},w_n^{(n-2)}\big) \\
&\quad - \big(1-\beta_n^{(n)}\big)\big(\beta_n^{(n)}-k_n\big)\rho^2\big(v_n^{(n-1)},w_n^{(n-1)}\big) \\
&\le \rho^2\big(v_n^{(n-3)},p\big) - \sum_{i=n-2}^{n}\big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)}-k_i\big)\rho^2\big(v_n^{(i-1)},w_n^{(i-1)}\big) \qquad (13)
\end{aligned}$$

$$\begin{aligned} &\vdots \\ &\le \rho^2\big(v_n^{(0)},p\big) - \sum_{i=1}^{n}\big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)}-k_i\big)\rho^2\big(v_n^{(i-1)},w_n^{(i-1)}\big). \end{aligned}$$

Therefore, we have

$$\rho(u_{n+1},p) \le \rho\big(v_n^{(0)},p\big) \quad \text{for every } n \in \mathbb{N}.$$

Moreover, by Lemma 2.8, we have

$$\begin{aligned} \rho(u_{n+1},p) &\le (1-\alpha_n)\rho(u_n,p) + a_n\rho(u_1,p) \\ &\le \max\big\{\rho(u_n,p),\rho(u_1,p)\big\} \\ &\le \max\big\{\rho(u_{n-1},p),\rho(u_1,p)\big\} \\ &\ \ \vdots \\ &\le \rho(u_1,p). \end{aligned}$$

This proves (i). To show (ii), let $p \in \mathcal{F}$. We consider the following two cases:

Case I: Assume that $\{\rho^2(u_n,p)\}$ is a monotonically nonincreasing sequence, that is,

$$\rho^2(u_{n+1},p) \le \rho^2(u_n,p), \quad n \in \mathbb{N}.$$

Then by the boundedness of $\{u_n\}$, we have that $\{\rho^2(u_n,p)\}$ converges and, consequently, (ii) hold.

Case II: Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\rho^2(u_{n_j},p) \le \rho^2(u_{n_j+1},p)$ for every $j \in \mathbb{N}$. Then, by Lemma 2.5, there exists a subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\rho^2(u_{m_k},p) < \rho^2(u_{m_k+1},p).$$

Thus,

$$\begin{aligned} 0 &\le \lim_{k\to\infty}\big(\rho^2(u_{m_k+1},p) - \rho^2(u_{m_k},p)\big) \\ &\le \limsup_{n\to\infty}\big(\rho^2(u_{n+1},p) - \rho^2(u_n,p)\big) \\ &\le \limsup_{n\to\infty}\big(\rho^2(u_n,p) + \alpha_n\rho(u_1,u) - \rho^2(u_n)\big) \\ &\le \limsup_{n\to\infty}\big(\alpha_n\rho(u_1,u)\big) \\ &= 0. \end{aligned}$$

Therefore, by Case I and II, the proof is complete. □

Lemma 3.4 *Let $T_i : D \to \mathcal{CB}(D)$, $i \in \mathbb{N}$ be a family of multi-valued Lipschitzian demicontractive mappings with constants $\{k_i\} \subset (0,1)$, $\mathcal{F} := \bigcap_{i\in\mathbb{N}} F(T_i) \neq \emptyset$ and each $T_ip = \{p\}$ for* *all* $p \in \mathcal{F}$. *Let* $\{u_n\}$ *be defined by iterative process* (12) *with* $\{\beta_n^{(i)}\} \subset [k_i, 1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (k_i, 1)$ *and* $\alpha_n \to 0$. *Then* $\lim_{n\to\infty} \operatorname{dist}(u_n, T_i u_n) = 0$ *for all* $i \in \mathbb{N}$.

Proof From (13) and scheme (12), we have

$$\begin{aligned} \rho(u_{n+1},p)^2 &\le (1-\alpha_n)\rho(u_n,p)^2 + \alpha_n \rho(u_1,p) \\ &\quad - \sum_{i=1}^{n} \big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)} - k_i\big)\rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big)^2 \\ &\le \rho(u_n,p)^2 + \alpha_n \rho(u_1,p) - \sum_{i=1}^{n} \big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)} - k_i\big)\rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big)^2. \end{aligned}$$

Let $i \in \mathbb{N}$. Then for $n \ge i$, we have

$$\begin{aligned} \big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)} - k_i\big)\rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big)^2 &\le \sum_{i=1}^{n} \big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)} - k_i\big)\rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big)^2 \\ &\le \rho(u_n,p)^2 - \rho(u_{n+1},p)^2 + \alpha_n \rho(u_1,p). \end{aligned} \tag{14}$$

Thus, by Lemma 3.3(ii) and the assumption on $\{\alpha_n\}$, we have

$$\limsup_{n\to\infty} \big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)} - k_i\big)\rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big)^2 = 0 \quad \text{for every } i \in \{1,2,\dots,m\},$$

which implies

$$\lim_{n\to\infty} \big(1-\beta_n^{(i)}\big)\big(\beta_n^{(i)} - k_i\big)\rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big)^2 = 0, \quad \text{for every } i \in \mathbb{N}.$$

Consequently, by the assumption on $\{\beta_n^{(i)}\}$, we have

$$\lim_{n\to\infty} \rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big) = 0, \quad \text{for every } i \in \mathbb{N}. \tag{15}$$

Now, let $i \in \mathbb{N}$. Then,

$$\begin{aligned} \rho\big(u_n, w_n^{(i-1)}\big) &\le \rho\big(v_n^{(0)}, v_n^{(1)}\big) + \rho\big(v_n^{(1)}, v_n^{(2)}\big) + \cdots + \rho\big(v_n^{(i-2)}, v_n^{(i-1)}\big) \\ &\quad + \rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big) \\ &\le \rho\big(v_n^{(0)}, w_n^{(0)}\big) + \rho\big(v_n^{(1)}, v_n^{(2)}\big) + \cdots + \rho\big(v_n^{(i-2)}, v_n^{(i-1)}\big) \\ &\quad + \rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big) \\ &\le \rho\big(v_n^{(0)}, w_n^{(0)}\big) + \rho\big(v_n^{(1)}, w_n^{(1)}\big) + \cdots + \rho\big(v_n^{(i-2)}, v_n^{(i-1)}\big) \\ &\quad + \rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big) \\ &\;\;\vdots \\ &\le \rho\big(v_n^{(0)}, w_n^{(0)}\big) + \rho\big(v_n^{(1)}, w_n^{(1)}\big) + \cdots + \rho\big(v_n^{(i-2)}, w_n^{(i-2)}\big) \\ &\quad + \rho\big(v_n^{(i-1)}, w_n^{(i-1)}\big) \\ &\le \sum_{k=1}^{i} \rho\big(v_n^{(k-1)}, w_n^{(k-1)}\big). \end{aligned}$$

This implies that

$$\lim_{n} \rho\big(u_n, w_n^{(i-1)}\big) = 0, \quad \text{for each } i \geq 1. \tag{16}$$

Now using the assumption that $\{T_i\}$ are Lipschitzian maps, we get

$$\begin{aligned}
\operatorname{dist}(u_n, T_i u_n) &\leq \rho\big(u_n, w_n^{(i-1)}\big) + \operatorname{dist}\big(w_n^{(i-1)}, T_i u_n\big) \\
&\leq \rho\big(u_n, w_n^{(i-1)}\big) + \rho_H\big(T_i v_n^{(i-1)}, T_i u_n\big) \\
&\leq \rho\big(u_n, w_n^{(i-1)}\big) + L_i \rho\big(v_n^{(i-1)}, u_n\big) \\
&\leq \rho\big(u_n, w_n^{(i-1)}\big) + L_i\big[\rho\big(v_n^{(i-1)}, w_n^{i-1}\big) + \rho\big(w_n^{i-1}, u_n\big)\big] \\
&\leq (1 + L_i)\rho\big(u_n, w_n^{(i-1)}\big) + L_i \rho\big(v_n^{(i-1)}, w_n^{i-1}\big).
\end{aligned}$$

Therefore, by the consequence of (15) and (16), the proof is complete. □

Theorem 3.5 *Let $T_i : D \to \mathcal{CB}(D)$, $i \in \mathbb{N}$ be a family of multi-valued Lipschitzian demicontractive mappings satisfying the demiclosedness-type property with constants $\{k_i\} \subset (0,1)$, $\mathcal{F} := \bigcap_{i\in\mathbb{N}} F(T_i) \neq \emptyset$ and each $T_i p = \{p\}$ for all $p \in \mathcal{F}$. Then the sequence $\{u_n\}$ generated by iterative scheme (12) with*

(i) $\alpha_n \to 0$ *as* $n \to \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*

(ii) $\{\beta_n^{(i)}\} \subset [k_i, 1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (k_i, 1)$ *for all* $i \in \mathbb{N}$,

strongly converges to a point in $\mathcal{F}$.

Proof From Lemma 3.3(i), $\{u_n\}$ is bounded. By Lemma 3.2, there exist $u \in D$ and subsequence $\{u_{n_j}\}$ of $\{u_n\}$ with $u = \Delta \lim_{j\to\infty} u_{n_j}$ and

$$\limsup_{n\to\infty} \langle \overrightarrow{u_n u}, \overrightarrow{y u} \rangle \leq 0, \quad \text{for every } y \in H. \tag{17}$$

By Lemma 3.4, we have $\operatorname{dist}(u_{n_j}, T_i u_{n_j}) \to 0$ for every $i \in \mathbb{N}$. Using the fact that each T_i has demiclosedness-type property for each $i \in \mathbb{N}$, we have $u \in \mathcal{F}$. By (13) and Lemma 2.10, we get

$$\begin{aligned}
\rho^2(u_{n+1}, u) &\leq \rho^2\big(v_n^{(0)}, u\big) \\
&\leq (1 - \alpha_n)^2 \rho^2(u_n, u) + \alpha_n^2 \rho^2(u_1, u) + 2\alpha_n(1 - \alpha_n)\langle \overrightarrow{u_n u}, \overrightarrow{u_1 u} \rangle \\
&\leq (1 - \alpha_n)\rho^2(u_n, u) + \alpha_n\big[a_n \rho^2(u_1, u) + 2(1 - \alpha_n)\langle \overrightarrow{u_n u}, \overrightarrow{u_1 u} \rangle\big] \\
&= (1 - \alpha_n)\rho^2(u_n, u) + \alpha_n \phi_n,
\end{aligned}$$

where $\phi_n = [a_n \rho^2(u_1, u) + 2(1 - \alpha_n)\langle \overrightarrow{u_n u}, \overrightarrow{u_1 u} \rangle]$. Now, by (17) and the assumption (i), we have

$$\limsup_{n\to\infty} \phi_n \leq 0.$$

Consequently, by Lemma 2.6, $\{u_n\}$ converges strongly to u. □

Corollary 3.6 *Let $T_i : D \to \mathcal{CB}(D)$, $i \in \mathbb{N}$ be a family of multi-valued quasi-nonexpansive mappings with demiclosedness-type property, $\mathcal{F} := \bigcap_{i\in\mathbb{N}} F(T_i) \neq \emptyset$ and each $T_i p = \{p\}$ for all $p \in \mathcal{F}$. Then the sequence $\{u_n\}$ generated by iterative scheme (12) with*

(i) $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*
(ii) $\{\beta_n^{(i)}\} \in [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for all* $i \in \mathbb{N}$,

strongly converges to a point in $\mathcal{F}$.

Remark 3.7 Since every hybrid mapping with a fixed point is quasi-nonexpansive, Corollary 3.6 holds for a countable family of hybrid mappings.

We have the following result from Theorem 3.5.

Corollary 3.8 *Let $T_i : D \to \mathcal{CB}(D)$, $i \in \mathbb{N}$ be a family of multi-valued nonexpansive mappings with $\mathcal{F} := \bigcap_{i\in\mathbb{N}} F(T_i) \neq \emptyset$. Then the sequence $\{u_n\}$ generated by iterative scheme (12) with*

(i) $\alpha_n \to 0$ *as* $n \to \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*
(ii) $\{\beta_n^{(i)}\} \subset [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for* $i \in \mathbb{N}$,

strongly converges to a point in $\mathcal{F}$.

Next, we consider a finite family of demicontractive mappings with fixed points. The results are analogous to the previous discussion.

Lemma 3.9 *Let $T_i : D \to \mathcal{CB}(D)$, $i = 1,2,\dots,m$ be a family of multi-valued demicontractive mappings with constants $\{k_i\} \subset (0,1)$, $\mathcal{F} := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and each $T_i p = \{p\}$ for all $p \in \mathcal{F}$. Suppose that $\{u_n\}$ is a sequence generated by*

$$\begin{cases} v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \quad u_1 \in D, \\ v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) w_n^{(i-1)}, \quad w_n^{(i-1)} \in T_i v_n^{(i-1)}, i = 1,\dots,m-1, \\ u_{n+1} = \beta_n^{(m)} v_n^{(m-1)} \oplus (1-\beta_n^{(m)}) w_n^{(m-1)}, \quad w_n^{(m-1)} \in T_n v_n^{(m-1)}, n \geq 1, \end{cases} \tag{18}$$

with $\{\alpha_n\} \subset [0,1]$, $\{\beta_n^{(i)}\} \subset [k_i,1]$ and $\alpha_n \to 0$ as $n \to \infty$. Then

(i) $\{u_n\}$ *is bounded, and*
(ii) $\limsup_{n\to\infty} (\rho(u_n,p)^2 - \rho(u_{n+1},p)^2) = 0$, *for all* $p \in \mathcal{F}$.

Proof The proof follows similar arguments as the proof of Lemma 3.3, and therefore we skip it. □

Lemma 3.10 *Let $T_i : D \to \mathcal{CB}(D)$, $i = 1,2,\dots,m$ be a family of multi-valued Lipschitzian demicontractive mappings with constants $\{k_i\} \subset (0,1)$, $\mathcal{F} := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and each $T_i p = \{p\}$ for all $p \in \mathcal{F}$. Let $\{u_n\}$ be defined by iterative process (18) with $\{\beta_n^{(i)}\} \subset [k_i,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (k_i,1)$ and $\alpha_n \to 0$. Then $\lim_{n\to\infty} \operatorname{dist}(u_n, T_i u_n) = 0$ for all $i \in \{1,2,\dots,m\}$.*

Proof The proof follows similar arguments as the proof of Lemma 3.4, and therefore we skip it. □

Theorem 3.11 *Let $T_i : D \to \mathcal{CB}(D)$, $i = 1,2,\dots,m$ be a family of multi-valued Lipschitzian demicontractive mappings satisfying demiclosedness-type property with constants $\{k_i\} \subset$*

$(0,1)$, $\mathcal{F} := \bigcap_{i}^{m} F(T_i) \neq \emptyset$ *and each* $T_i p = \{p\}$ *for all* $p \in \mathcal{F}$. *Then the sequence* $\{u_n\}$ *generated by iterative process* (18) *with*

(i) $\alpha_n \to 0$ *as* $n \to \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*
(ii) $\{\beta_n^{(i)}\} \subset [k_i, 1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (k_i, 1)$ *for* $i \in \{1, 2, \ldots, m\}$,

strongly converges to a point in $\mathcal{F}$.

Proof The proof follows similar lines as the proof of Theorem 3.5 with $i \in \{1, 2, \ldots, m\}$ only. □

We immediately have the following corollaries:

Corollary 3.12 *Let* $T_i : D \to \mathcal{CB}(D)$, $i = 1, 2, \ldots, m$ *be a family of multi-valued quasi-nonexpansive mappings with demiclosedness-type property,* $\mathcal{F} := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ *and each* $T_i p = \{p\}$ *for all* $p \in \mathcal{F}$. *Then the sequence* $\{u_n\}$ *generated by iterative scheme* (18) *with*

(i) $\alpha_n \to 0$ *as* $n \to \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*
(ii) $\{\beta_n^{(i)}\} \in [0, 1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0, 1)$ *for all* $i \in \{1, 2, \ldots, m\}$,

strongly converges to a point in $\mathcal{F}$.

Remark 3.13 Since every hybrid mapping with fixed point is quasi-nonexpansive, Corollary 3.12 holds for a finite family of hybrid mappings.

From Theorem 3.1 and Corollary 3.12, we have the following corollary:

Corollary 3.14 *Let* $T_i : D \to \mathcal{CB}(D)$, $i = 1, 2, \ldots, m$ *be a family of multi-valued nonexpansive mappings with* $\mathcal{F} := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. *Then the sequence* $\{u_n\}$ *generated by iterative scheme* (18) *with*

(i) $\alpha_n \to 0$ *as* $n \to \infty$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*
(ii) $\{\beta_n^{(i)}\} \subset [0, 1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0, 1)$ *for all* $i \in \{1, 2, \ldots, m\}$,

strongly converges to a point in $\mathcal{F}$.

4 Applications and numerical examples

In this section, we apply the proposed scheme to solve a family of minimization problems, variational inequality problems, and monotone inclusion problems. We then give numerical examples, one in non-Hilbert space and the other in Hilbert space, to show the implementation, applicability, and effectiveness of the proposed scheme.

4.1 Application to minimization problems

Let $g : D \to \mathbb{R} \cup \{+\infty\}$ be a map. The problem of finding

$$u \in H \quad \text{such that} \quad g(u) \leq g(v), \quad \forall v \in D \tag{19}$$

is known as a Minimization Problem (MP). This problem has applications in nonlinear analysis and optimization as different models can be reduced to MP.

A function $g : H \to \mathbb{R} \cup \{+\infty\}$ is called *convex* if

$$g\big(tu \oplus (1-t)v\big) \leq tg(u) + (1-t)g(v) \quad \text{for all } t \in (0,1) \text{ and } u, v \in H.$$

If the set $D(g) := \{u \in H : g(u) < +\infty\} \neq \emptyset$, then g is said to be proper. The function g is said to be *lower semi-continuous* at a point $u \in D(g)$ if $g(u) \leq \liminf_{n\to\infty} g(u_n)$ for any convergent sequence $\{u_n\}$ in $D(g)$ with limit $u \in D$. If g is lower semi-continuous at every point in $D(g)$ then it is lower semi-continuous on $D(g)$. It is known (see, e.g., [4]) that for a nonempty closed and convex subset D of a Hadamard space H, the function $\delta_D : H \to \mathbb{R}$ defined by $\delta_D(u) = 0$, if $u \in D$ and $+\infty$, elsewhere is an example of a proper, convex, lower semi-continuous function.

For $\mu > 0$, the map J^g_μ is defined by

$$J^g_\mu(u) = \arg\min_{v\in H}\left[g(v) + \frac{1}{2\mu}\rho^2(u,v)\right].$$

Ariza-Ruiz et al. reported that, for a convex proper lower semi-continuous map g on D, the solution set of problem (19) coincides with the fixed point set $F(J^g_\mu)$ [45, Proposition 6.5]. Moreover, by Lemma 4 of [46], we have that J^g_μ is nonexpansive mapping. Since every nonexpansive mapping with a fixed point is demicontractive with demiclosedness-type property, the following results are immediate.

Theorem 4.1 *Let $g_i : H \to \mathbb{R} \cup \{+\infty\}$, $i \in \mathbb{N}$ be a family of convex, proper and lower semi-continuous functions with $\Gamma := \bigcap_{i\in\mathbb{N}} \arg\min_{u\in H} g_i(u) \neq \emptyset$. Then, for $u_1 \in D$, the sequence $\{u_n\}$ defined by*

$$\begin{cases} v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \\ v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) J^{g_i}_\mu v_n^{(i-1)}, \quad i = 1,\dots,n-1, \\ u_{n+1} = \beta_n^{(n)} v_n^{(n-1)} \oplus (1-\beta_n^{(n)}) J^{g_i}_\mu v_n^{(n-1)}, \\ \{\alpha_n\} \subset [0,1], \qquad \{\beta_n^{(i)}\} \subset [0,1], \quad i = 1,\dots,n, n \geq 1, \end{cases}$$

with

(i) $\alpha_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\beta_n^{(i)} \in [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for all* $i \in \mathbb{N}$;

strongly converges to a point in Γ.

Theorem 4.2 *Let $g_i : H \to \mathbb{R} \cup \{+\infty\}$, $i = 1,\dots,m$ be a family of convex, proper, and lower semi-continuous functions with $\Gamma := \bigcap_{i=1}^m \arg\min_{u\in H} g_i(u) \neq \emptyset$. Then, for $u_1 \in D$, the sequence $\{u_n\}$ defined by*

$$\begin{cases} v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \\ v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) J^{g_i}_\mu v_n^{(i-1)}, \quad i = 1,\dots,m-1, \\ u_{n+1} = \beta_n^{(m)} v_n^{(m-1)} \oplus (1-\beta_n^{(m)}) J^{g_i}_\mu v_n^{(m-1)}, \\ \{\alpha_n\} \subset [0,1], \qquad \{\beta_n^{(i)}\} \subset [0,1], \quad i = 1,\dots,m, n \geq 1, \end{cases}$$

with

(i) $\alpha_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\beta_n^{(i)} \in [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for all* $i \in \{1,2,\dots,m\}$;

strongly converges to a point in Γ.

Furthermore, some combined problems involving MP can be obtained from our result. For example, the main result in [29] can be obtained using scheme (18) as follows:

Theorem 4.3 *Let $g : D \to \mathbb{R} \cup \{+\infty\}$ be a convex, proper, and lower semi-continuous function, and let $T_i : D \to D$, $i = 1, \dots, m$ be a family of single-valued nonexpansive mappings with $\Gamma := \bigcap_{i=1}^m F(T_i) \cap \arg\min_{u\in H} g(u) \neq \emptyset$. Then, for $u_1 \in D$, the sequence $\{u_n\}$ defined by*

$$\begin{cases} v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \\ v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) T_i v_n^{(i-1)}, \quad i = 1, \dots, m, \\ u_{n+1} = \beta_n^{(m+1)} v_n^{(m)} \oplus (1-\beta_n^{(m+1)}) J_\mu^g v_n^{(m)}, \\ \{\alpha_n\} \subset [0,1], \qquad \{\beta_n^{(i)}\} \subset [0,1], \quad i = 1, \dots, m+1, n \geq 1, \end{cases}$$

with

(i) $\alpha_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $\beta_n^{(i)} \in [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for all* $i \in \{1, 2, \dots, m+1\}$;

strongly converges to a point in Γ.

4.2 Application to variational inequality problems

Let $f : D \to H$ be a single-valued map. The problem of finding

$$u \in D \quad \text{such that} \quad \langle \overrightarrow{fuu}, \overrightarrow{uw} \rangle \geq 0, \quad \text{for every } w \in D \tag{20}$$

is known as a Variational Inequality Problem (VIP), and it has several applications in optimizations (see, e.g., [47]). We denote the set of solution of VIP by $\mathrm{VI}(D,f)$. It is known that for any $u \in H$, there exists a unique point $P_D u$ in D such that $\rho(u, P_D u) = \min_{w\in H} \rho(u, w)$ [22, Proposition 2.4]. Furthermore, as in [48, Theorem 2.2], $y = P_D u$ if and only if $\langle \overrightarrow{uy}, \overrightarrow{yw} \rangle \geq 0$ for every $w \in D$. Thus, the fixed point set $F(P_D f)$ coincides with the solution of (20).

Recall that a single-valued map $T : D \to H$ is said to be *α-inverse strongly monotone* if there exist $\alpha > 0$ such that for every $u, w \in D$,

$$\rho^2(u,w) - \langle \overrightarrow{fufw}, \overrightarrow{uw} \rangle \leq \alpha\rho^2(u,w) + \alpha\rho^2(fu,fw) - 2\alpha \langle \overrightarrow{fufw}, \overrightarrow{uw} \rangle. \tag{21}$$

It is known (see, e.g., [49, Lemma 2]) that if f is α-inverse strongly monotone then the map $f^{(\lambda)}$ defined by $f^{(\lambda)} u = (1-\lambda)u \oplus \lambda fu$ with $\lambda \in (0, 2\alpha)$ is nonexpansive and $F(f^{(\lambda)}) = F(f)$. Thus, $F(P_D f^{(\lambda)}) = \mathrm{VI}(D,f)$, and we have the following results.

Theorem 4.4 *Let $f_i : D \to H$, $i \in \mathbb{N}$ be a family of α_i-inverse strongly monotone mappings with $\Gamma := \bigcap_{i\in\mathbb{N}} \mathrm{VI}(D,f_i) \neq \emptyset$. Then, for $u_1 \in D$, the sequence $\{u_n\}$ defined by*

$$\begin{cases} v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \\ v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) P_D f_i^{(\lambda_i)} v_n^{(i-1)}, \quad i = 1, \dots, n-1, \\ u_{n+1} = \beta_n^{(n)} v_n^{(n-1)} \oplus (1-\beta_n^{(n)}) P_D f_i^{(\lambda_i)} v_n^{(n-1)}, \\ \{\alpha_n\} \subset [0,1], \qquad \{\beta_n^{(i)}\} \subset [0,1], \qquad \lambda_i \in (0, 2\alpha_i) \cap (0,1], \quad n \geq 1, \end{cases}$$

with

(i) $\alpha_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\beta_n^{(i)} \in [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for all* $i \in \mathbb{N}$;
strongly converges to a point in Γ.

Theorem 4.5 *Let* $f_i : D \to H$, $i = 1,\dots,m$ *be a family of inverse strongly monotone mappings with* $\Gamma := \bigcap_{i=1}^{m} \mathrm{VI}(D,f_i) \neq \emptyset$. *Then, for* $u_1 \in D$, *the sequence* $\{u_n\}$ *defined by*

$$\begin{cases} v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \\ v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) P_D f_i^{(\lambda_i)} v_n^{(i-1)}, \quad i = 1,\dots,m-1, \\ u_{n+1} = \beta_n^{(m)} v_n^{(m-1)} \oplus (1-\beta_n^{(m)}) P_D f_i^{(\lambda_i)} v_n^{(m-1)}, \\ \{\alpha_n\} \subset [0,1], \qquad \{\beta_n^{(i)}\} \subset [0,1], \qquad \lambda_i \in (0,2\alpha_i) \cap (0,1], \quad n \geq 1, \end{cases}$$

with
(i) $\alpha_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\beta_n^{(i)} \in [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for all* $i \in \{1,2,\dots,m\}$;
strongly converges to a point in Γ.

4.3 Application to monotone inclusion problems

A multi-valued mapping $A : D(A) \to 2^{H^*}$ is called *monotone* if

$$\langle u^* - v^*, \overrightarrow{uv} \rangle \geq 0 \quad \forall u, v \in D(A), u^* \in Au, v^* \in Av,$$

where $D(A) := \{u \in H : Au \neq \emptyset\}$. The problem of finding

$$u \in D(A) \quad \text{such that} \quad 0 \in Au, \tag{22}$$

is known as a Monotone Inclusion Problem (MIP). This problem composes many other problems and has significant applications in nonlinear analysis and optimizations.

The *resolvent of* A *of order* $\mu > 0$ is the multi-valued mapping $J_\mu^A : H \to 2^H$ defined by

$$J_\mu^A(w) := \left\{ v \in H : \left[\frac{1}{\mu} \overrightarrow{vw} \right] \in Av \right\},$$

where $[t\overrightarrow{vw}] := \{s\overrightarrow{xy} : t\langle \overrightarrow{vw}, \overrightarrow{uz} \rangle = s\langle \overrightarrow{xy}, \overrightarrow{uz} \rangle, \forall u, z \in H\}$. It is shown in [50, Theorem 3.9] that for any monotone mapping A satisfying the range condition and $\mu > 0$, the resolvent operator J_μ^A is firmly nonexpansive, and the fixed point set $F(J_\mu^A)$ coincides with the solution set of (22). Consequently, we have the following results.

Theorem 4.6 *Let* $A_i : D(A) \to 2^{H^*}$, $i \in \mathbb{N}$ *be a family of monotone mappings that satisfy the range condition with* $\Gamma := \bigcap_{i\in\mathbb{N}} A_i^{-1}(0) \neq \emptyset$. *Then, for* $u_1 \in D$, *the sequence* $\{u_n\}$ *defined by*

$$\begin{cases} v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \\ v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) w_n^{(i-1)}, \quad w_n^{(i-1)} \in J_\mu^{A_i} v_n^{(i-1)}, i = 1,\dots,n-1, \\ u_{n+1} = \beta_n^{(n)} v_n^{(n-1)} \oplus (1-\beta_n^{(n)}) w_n^{(n-1)}, \quad w_n^{(n-1)} \in J_\mu^{A_i} v_n^{(n-1)}, \\ \{\alpha_n\} \subset [0,1], \qquad \{\beta_n^{(i)}\} \subset [0,1], \qquad \mu > 0, \quad i = 1,\dots,n, n \geq 1, \end{cases}$$

with

(i) $\alpha_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n^{(i)} \in [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for all* $i \in \mathbb{N}$;

strongly converges to a point in Γ.

Theorem 4.7 *Let* $A_i : D(A) \to 2^{H^*}$, $i = 1 \cdots, m$ *be a family of monotone mappings that satisfy the range condition with* $\Gamma := \bigcap_{i=1}^{m} A_i^{-1}(0) \neq \emptyset$. *Then, for* $u_1 \in D$, *the sequence* $\{u_n\}$ *defined by*

$$\begin{cases} v_n^{(0)} = (1-\alpha_n)u_n \oplus \alpha_n u_1, \\ v_n^{(i)} = \beta_n^{(i)} v_n^{(i-1)} \oplus (1-\beta_n^{(i)}) w_n^{(i-1)}, \quad w_n^{(i-1)} \in J_\mu^{A_i} v_n^{(i-1)}, i = 1, \dots, m-1, \\ u_{n+1} = \beta_n^{(m)} v_n^{(m-1)} \oplus (1-\beta_n^{(m)}) w_n^{(m-1)}, \quad w_n^{(m-1)} \in J_\mu^{A_i} v_n^{(m-1)}, \\ \{\alpha_n\} \subset [0,1], \qquad \{\beta_n^{(i)}\} \subset [0,1], \qquad \mu > 0, \quad i = 1, \dots, m, n \geq 1, \end{cases}$$

with

(i) $\alpha_n \to 0$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n^{(i)} \in [0,1]$, $\liminf_{n\to\infty} \beta_n^{(i)} \in (0,1)$ *for all* $i \in \{1, \dots, m\}$;

strongly converges to a point in Γ.

4.4 Numerical examples

In this part, we give two examples, one from non-Hilbert space and the other from Hilbert space. Moreover, we set the control parameters $\{\alpha_n\} = \{\frac{1}{\sqrt{n}}\}$, $\{\beta_n^{(i)}\} = \{\frac{n+i}{5n+i}\}$ and consider different initial points in showing the convergence result. All codes are written and executed in Matlab (8.3.0.532) and run on HP Compaq (Presario Cq56) AMD Dual-core laptop.

Example 4.1 Let $H = \mathbb{R}^2$ and $\rho : H \times H \to [0, \infty)$ be a radial metric defined by

$$\rho(u,v) = \begin{cases} \|u-v\|_2, & \text{if } u = \gamma v \text{ for some } \gamma \in \mathbb{R}, \\ \|u\|_2 + \|v\|_2, & \text{otherwise.} \end{cases}$$

Then, (H, ρ) is a CAT(0) space with the geodesic path connecting u, v defined by $\tau_u^v(t) = (1-t)u + tv$ if $u = \gamma v$ for some γ in $\mathbb{R}$, otherwise

$$\tau_u^v(t) := \begin{cases} (1 - t\frac{\rho(u,v)}{\rho(0,u)})u, & \text{if } 0 \leq t \leq \frac{\rho(0,u)}{\rho(u,v)}, \\ \frac{\rho(0,u)}{\rho(u,v)}(t\frac{\rho(u,v)}{\rho(0,u)} - 1)v, & \text{if } \frac{\rho(0,u)}{\rho(u,v)} \leq t \leq 1. \end{cases}$$

Let $\ell > 0$. Take $D := [0,\ell] \times [0,\ell]$; then for $i \in \mathbb{N}$, consider the map $T_i : D \to \mathcal{CB}(D)$ defined by $T_i u := \prod_{j=1}^{2} [0, \frac{\ell u_j}{\ell + i}]$, for every $u = (u_1, u_2) \in D$. Then all the assumptions of Theorem 3.5 are satisfied. Moreover, $\mathcal{F} = \{0\}$ and $T_i 0 = \{0\}$ for every $i \in \mathbb{N}$. Now, setting $w_n^{(i-1)} = \frac{\ell}{\ell+i} v_n^{(i-1)}$ and taking the starting points as Case 1: $u_1 = (672, 218)^T$, Case 2: $u_1 = (305, 471)^T$, Case 3: $u_1 = (182, 391)^T$, Case 4: $u_1 = (728, 639)^T$, we have Figs. 1–4 due to Theorem 3.5.

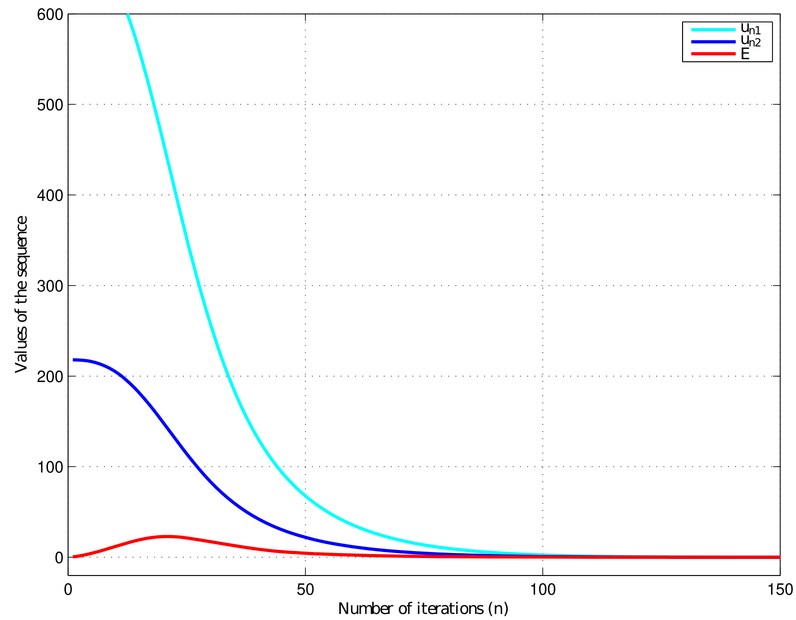

Figure 1 Convergence result of Theorem 3.5 with starting point of Case 1

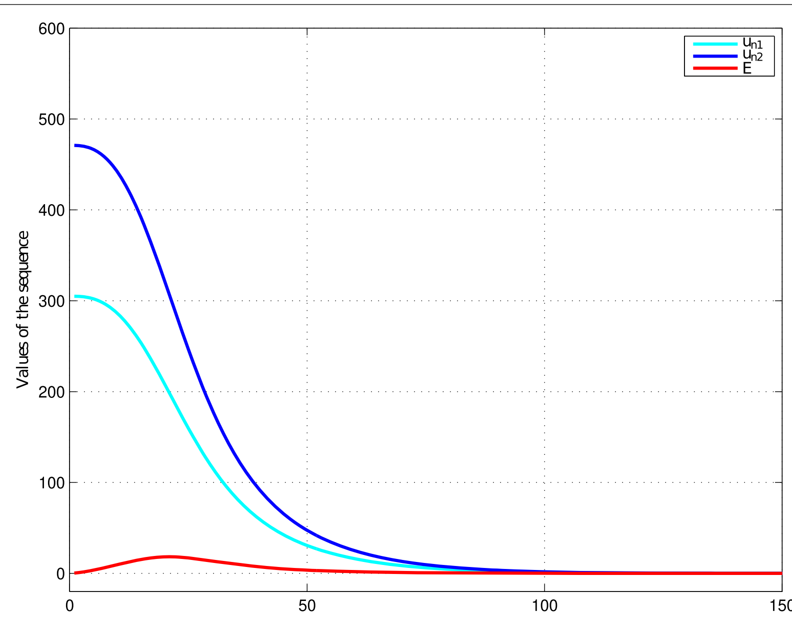

Figure 2 Convergence result of Theorem 3.5 with starting point of Case 2

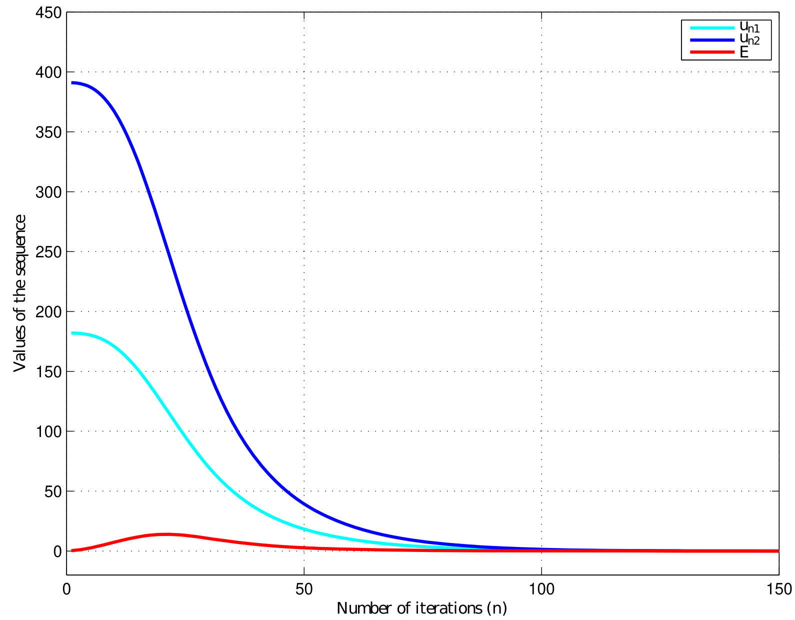

Figure 3 Convergence result of Theorem 3.5 with starting point of Case 3

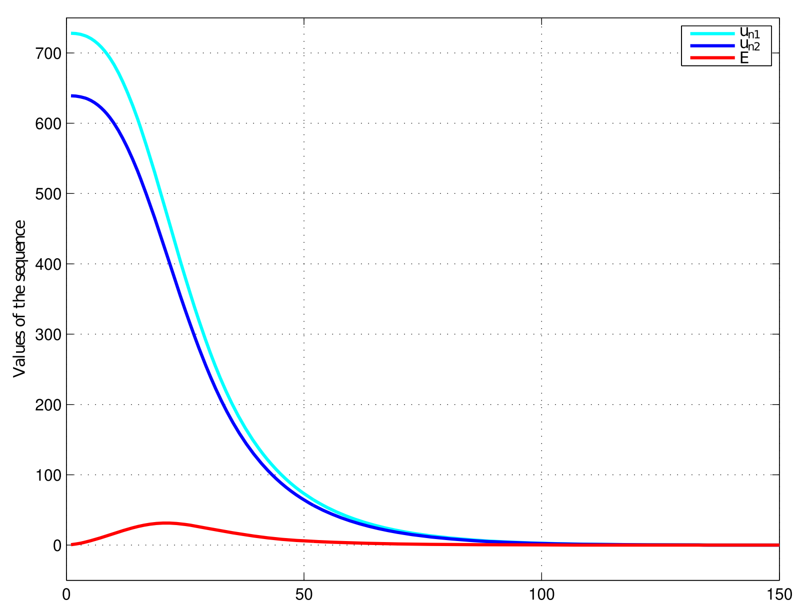

Figure 4 Convergence result of Theorem 3.5 with starting point of Case 4

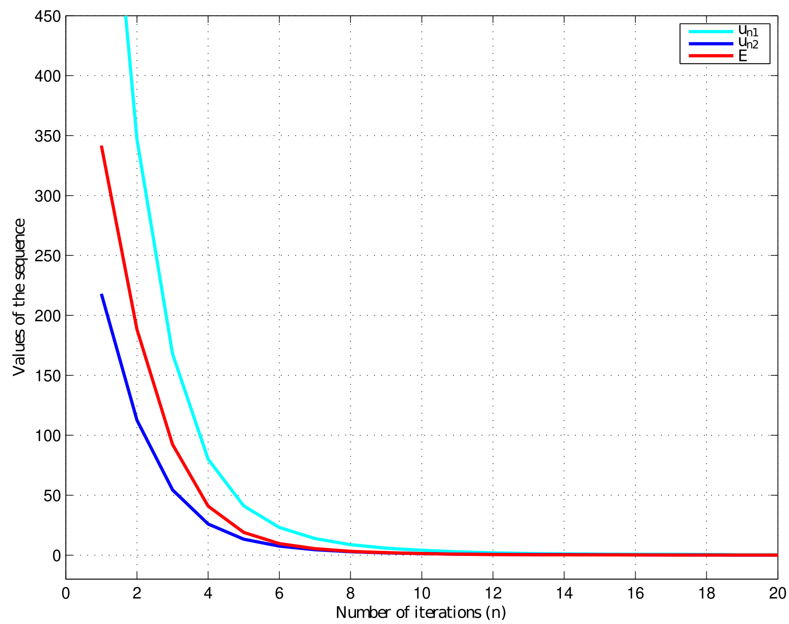

Figure 5 Convergence result of Theorem 3.11 with starting point of Case 1

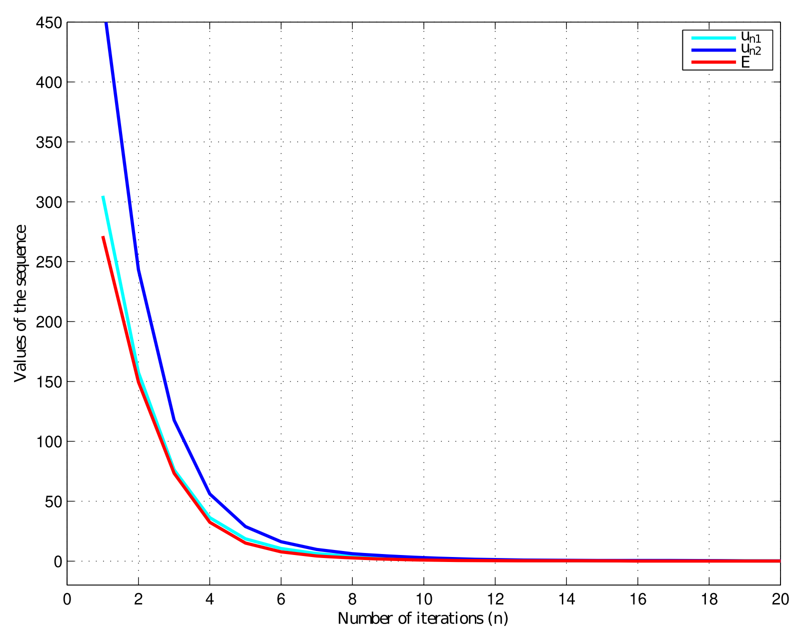

Figure 6 Convergence result of Theorem 3.11 with starting point of Case 2

Now, if we consider the same example but for a finite family of the mappings taking only first 200 of them, i.e., $\{T_i : i = 1, 2, \dots, 200\}$, then, it is easy to see that the assumptions of Theorem 3.11 are satisfied. Moreover, using the same initial points given in Cases 1–4, we have Figs. 5–8 due to Theorem 3.11. Also, Table 1 provides the approximate values of few terms of the sequence $\{u_n\}$ in Example 4.1 using the starting points of Case 1 and Case 2 for both infinite and finite family of the mappings.

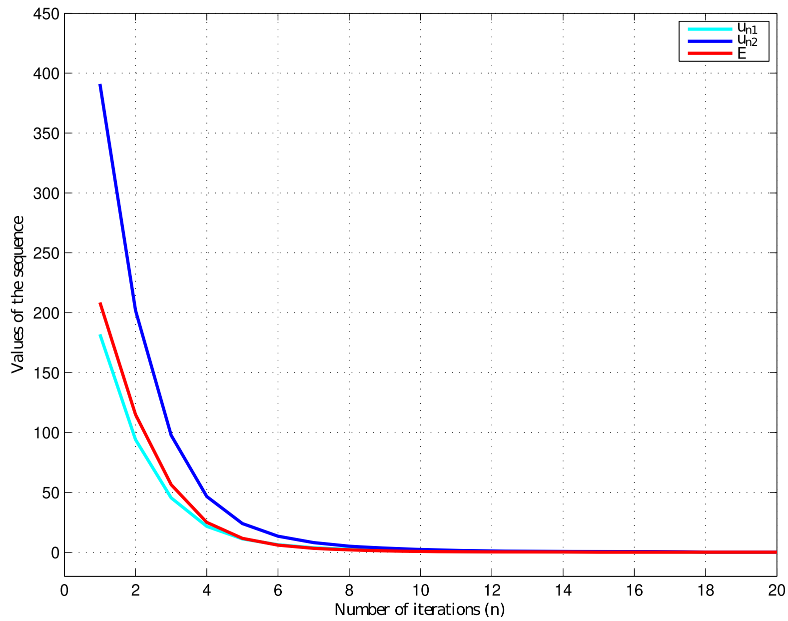

Figure 7 Convergence result of Theorem 3.11 with starting point of Case 3

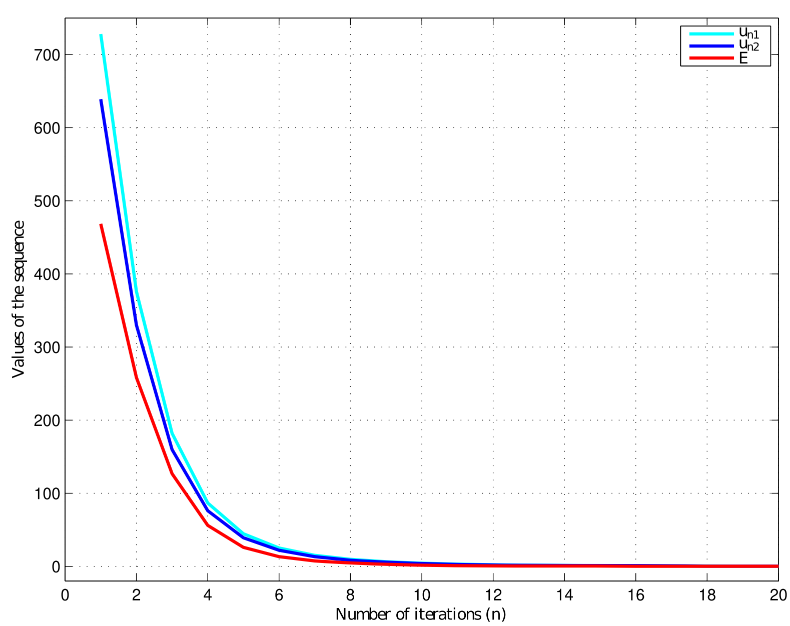

Figure 8 Convergence result of Theorem 3.11 with starting point of Case 4

Table 1 Some values of $\{u_n\}$ from Example 4.1 with starting points of Case 1 and 2

n	$u_1 = (672, 218)$, infinite	$u_1 = (305, 471)$, infinite	$u_1 = (672, 218)$, $m = 200$	$u_1 = (305, 471)$, $m = 200$
	u_n	u_n	u_n	u_n
1	(672, 218)	(305, 471)	(672, 218)	(305, 471)
2	(671.5524, 217.8548)	(304.7969, 470.6863)	(346.9873, 112.5643)	(157.4868, 243.2009)
3	(670.4874, 217.5093)	(304.3135, 469.9398)	(167.8756, 54.4597)	(76.1936, 117.6628)
4	(668.5767, 216.8895)	(303.4462, 468.6006)	(79.9849, 25.9475)	(36.3027, 56.0609)
5	(665.6416, 215.9373)	(302.1141, 466.5434)	(41.1475, 13.3485)	(18.6756, 28.84)
6	(661.5287, 214.6031)	(300.2474, 463.6608)	(23.0989, 7.4934)	(10.4839, 16.1898)
7	(656.1094, 212.845)	(297.7878, 459.8624)	(13.8878, 4.5053)	(6.3032, 9.7338)
8	(649.2807, 210.6298)	(294.6884, 455.0762)	(8.7931, 2.8525)	(3.9909, 6.163)
9	(640.9668, 207.9327)	(290.915, 449.2491)	(5.7981, 1.8809)	(2.6316, 4.0639)
10	(631.1207, 204.7386)	(286.4461, 442.348)	(3.9521, 1.2821)	(1.7938, 2.77)
11	(619.7253, 201.0418)	(281.2741, 434.3611)	(2.7701, 0.89864)	(1.2573, 1.9416)
12	(606.7948, 196.8471)	(275.4054, 425.2981)	(1.9888, 0.64517)	(0.90265, 1.3939)
13	(592.3742, 192.169)	(268.8603, 415.1908)	(1.4581, 0.47301)	(0.66178, 1.022)
14	(576.5391, 187.032)	(261.6733, 404.0921)	(1.089, 0.35329)	(0.49428, 0.76329)
15	(559.394, 181.4701)	(253.8916, 392.0753)	(0.82698, 0.26828)	(0.37534, 0.57963)
16	(541.0697, 175.5256)	(245.5748, 379.2319)	(0.63745, 0.20679)	(0.28932, 0.44678)
17	(521.7193, 169.2482)	(236.7922, 365.6693)	(0.49805, 0.16157)	(0.22605, 0.34908)
18	(501.5146, 162.6937)	(227.6219, 351.508)	(0.39397, 0.12781)	(0.17881, 0.27613)
19	(480.6403, 155.922)	(218.1478, 336.8774)	(0.31519, 0.10225)	(0.14305, 0.22091)
20	(459.2891, 148.9956)	(208.4571, 321.9125)	(0.2548, 0.082659)	(0.11565, 0.17859)
⋮	⋮	⋮	⋮	⋮
100	(2.4355, 0.79007)	(1.1054, 1.707)	(0.00055682, 0.00018063)	(0.00025272, 0.00039027)
101	(2.2624, 0.73395)	(1.0269, 1.5857)	(0.0005406, 0.00017537)	(0.00024536, 0.0003789)
102	(2.1008, 0.6815)	(0.95347, 1.4724)	(0.00052508, 0.00017034)	(0.00023832, 0.00036803)
103	(1.9498, 0.63251)	(0.88493, 1.3666)	(0.00051022, 0.00016552)	(0.00023157, 0.00035761)
104	(1.8088, 0.58677)	(0.82094, 1.2678)	(0.00049598, 0.0001609)	(0.00022511, 0.00034763)
105	(1.6772, 0.54409)	(0.76122, 1.1755)	(0.00048233, 0.00015647)	(0.00021892, 0.00033806)
106	(1.5544, 0.50427)	(0.70551, 1.0895)	(0.00046924, 0.00015222)	(0.00021297, 0.00032889)
107	(1.44, 0.46714)	(0.65357, 1.0093)	(0.00045668, 0.00014815)	(0.00020727, 0.00032008)
108	(1.3333, 0.43254)	(0.60516, 0.93452)	(0.00044461, 0.00014423)	(0.0002018, 0.00031163)
109	(1.234, 0.4003)	(0.56006, 0.86487)	(0.00043303, 0.00014048)	(0.00019654, 0.00030351)
110	(1.1414, 0.37029)	(0.51806, 0.80003)	(0.00042189, 0.00013686)	(0.00019148, 0.0002957)
⋮	⋮	⋮	⋮	⋮
190	(0.00038278, 0.00012417)	(0.00017373, 0.00026828)	(0.0001086, 3.5231e–05)	(4.9292e–05, 7.6119e–05)
191	(0.00033863, 0.00010985)	(0.00015369, 0.00023734)	(0.00010739, 3.4839e–05)	(4.8743e–05, 7.5272e–05)
192	(0.00029941, 9.7131e–05)	(0.00013589, 0.00020986)	(0.00010621, 3.4454e–05)	(4.8205e–05, 7.444e–05)
193	(0.00026459, 8.5835e–05)	(0.00012009, 0.00018545)	(0.00010504, 3.4077e–05)	(4.7676e–05, 7.3625e–05)
194	(0.00023369, 7.581e–05)	(0.00010606, 0.00016379)	(0.0001039, 3.3707e–05)	(4.7158e–05, 7.2825e–05)
195	(0.00020629, 6.692e–05)	(9.3627e–05, 0.00014458)	(0.00010278, 3.3343e–05)	(4.665e–05, 7.204e–05)
196	(0.000182, 5.904e–05)	(8.2602e–05, 0.00012756)	(0.00010168, 3.2987e–05)	(4.6151e–05, 7.1269e–05)
197	(0.00016048, 5.206e–05)	(7.2836e–05, 0.00011248)	(0.0001006, 3.2637e–05)	(4.5661e–05, 7.0513e–05)
198	(0.00014143, 4.5879e–05)	(6.4189e–05, 9.9124e–05)	(9.9546e–05, 3.2293e–05)	(4.5181e–05, 6.9771e–05)
199	(0.00012457, 4.041e–05)	(5.6537e–05, 8.7309e–05)	(9.8506e–05, 3.1956e–05)	(4.4709e–05, 6.9042e–05)
200	(0.00010966, 3.5574e–05)	(4.9771e–05, 7.686e–05)	(9.7485e–05, 3.1625e–05)	(4.4246e–05, 6.8327e–05)

Example 4.2 Let $(H,\rho) = (\mathbb{R}^4, \|\cdot\|_2)$. Then (H,ρ) is a Hadamard space with the geodesic path connecting u, v defined by $\tau_u^v(t) = (1-t)u + tv$ for every $t \in [0,1]$. Take $D := \prod_{i=1}^4 [-30, 30]$ and consider the map $T_i : D \to \mathcal{CB}(D)$ defined by

$$T_i u := \left\{ \left(\frac{u_1 + 8i - 2}{4i}, \frac{2u_2 + 6i - 3}{4i}, \frac{2u_3 + 7i - 7}{2i}, \frac{u_4 + 3i - 3}{i} \right) \right\},$$

for every $u = (u_1, u_2, u_3, u_4) \in D$ and $i = 1, 2, 3$. Then all the assumptions of Theorem 3.11 are satisfied. Moreover, $p = (2, \frac{3}{2}, \frac{7}{2}, 3)$ is the only element of $\mathcal{F}$ and $T_i p = \{p\}$ for ev-

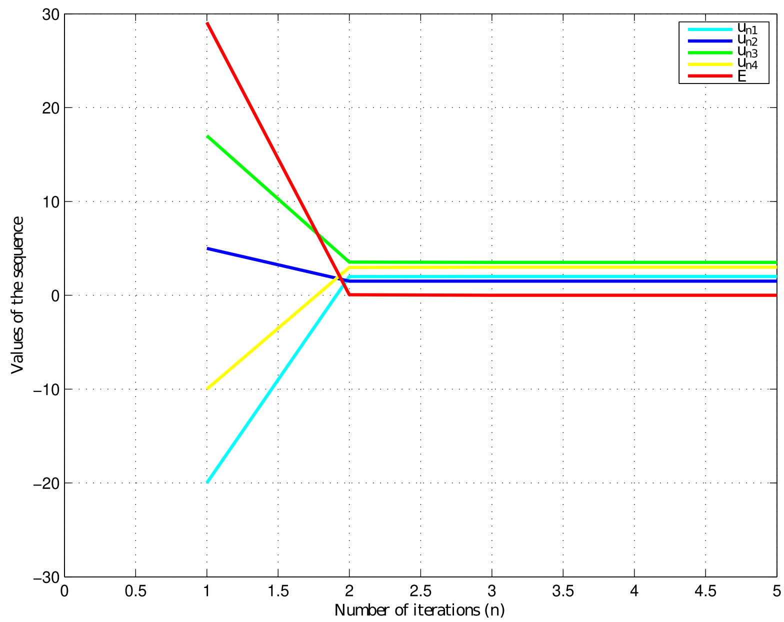

Figure 9 Convergence result of Theorem 3.11 with starting point of Case 5

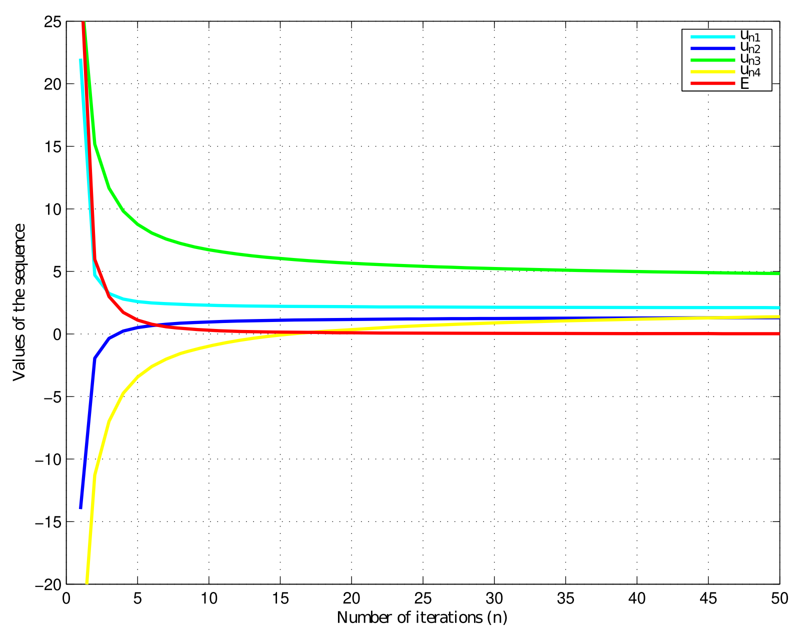

Figure 10 Convergence result of Theorem 3.11 with starting point of Case 6

ery i. Now, using Case 5: $u_1 = (-20, 5, 17, -10)$, Case 6: $u_2 = (22, -20, -19, 30)$, Case 7: $u_1 = (23, -30, -28, 24)$, and case 8: $u_1 = (27, -20, -19, 30)$, we have Figs. 9–12 due to Theorem 3.11. Also, Table 2 displays the approximate values of few terms of the sequence $\{u_n\}$ in Example 4.2 using the starting points of Case 5 and Case 6.

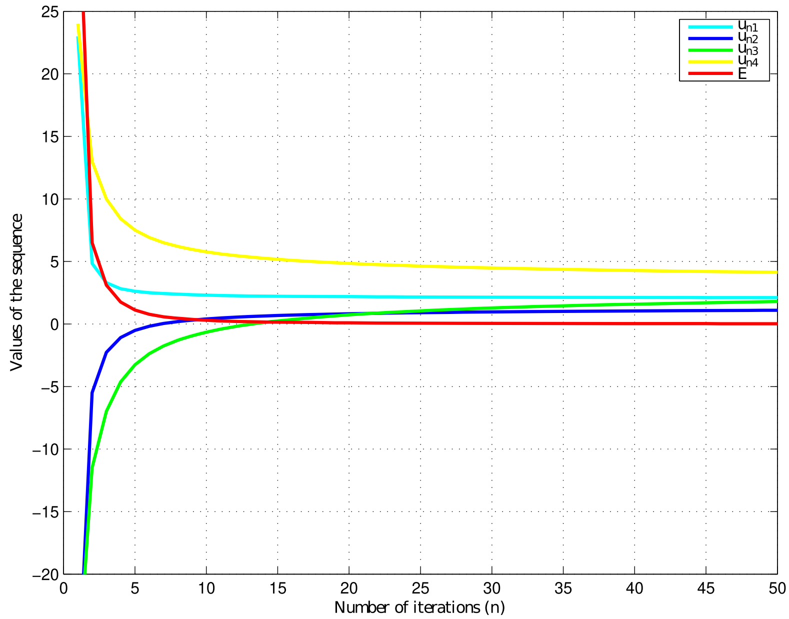

Figure 11 Convergence result of Theorem 3.11 with starting point of Case 7

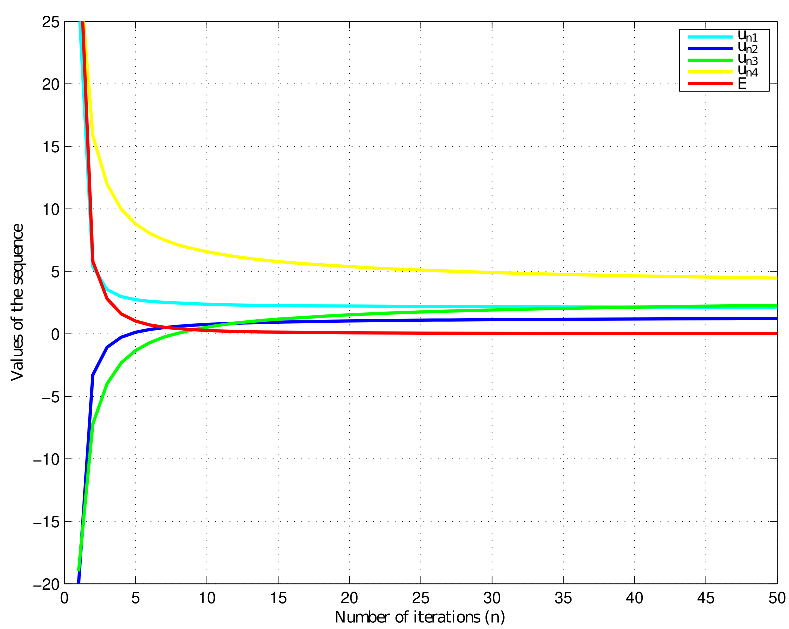

Figure 12 Convergence result of Theorem 3.11 with starting point of Case 8

Table 2 Some values of $\{u_n\}$ from Example 4.2 with starting points of Case 1 and Case 2

n	$u_1 = (-20, 5, 17, -10)$	$u_1 = (22, -20, -19, 30)$
	u_n	u_n
1	(−20, 5, 17, −10)	(22, −20, −19, 30)
2	(−0.97917, 2.2778, 9.9286, −3.1905)	(4.7083, −3.2778, −7.2143, 15.8571)
3	(0.64369, 1.9189, 7.9934, −1.327)	(3.233, −1.0735, −3.989, 11.9868)
4	(1.1364, 1.7873, 6.9845, −0.35548)	(2.7851, −0.2648, −2.3076, 9.9691)
5	(1.3545, 1.7239, 6.3979, 0.20944)	(2.5868, 0.12431, −1.3298, 8.7958)
6	(1.476, 1.687, 6.0195, 0.57383)	(2.4763, 0.35138, −0.69914, 8.039)
7	(1.5535, 1.6626, 5.7545, 0.82898)	(2.4059, 0.50103, −0.25753, 7.509)
8	(1.6074, 1.6452, 5.5573, 1.0189)	(2.3569, 0.60777, 0.071223, 7.1145)
9	(1.6472, 1.6322, 5.4036, 1.1669)	(2.3208, 0.68821, 0.32728, 6.8073)
10	(1.6778, 1.6219, 5.2798, 1.2861)	(2.2929, 0.75132, 0.53363, 6.5596)
11	(1.7022, 1.6136, 5.1774, 1.3847)	(2.2707, 0.80236, 0.70433, 6.3548)
12	(1.7222, 1.6067, 5.0909, 1.468)	(2.2525, 0.84464, 0.8485, 6.1818)
13	(1.7389, 1.6009, 5.0166, 1.5396)	(2.2374, 0.88036, 0.97231, 6.0332)
14	(1.7531, 1.5959, 4.9519, 1.6019)	(2.2245, 0.911, 1.0801, 5.9038)
15	(1.7653, 1.5915, 4.8949, 1.6567)	(2.2134, 0.93765, 1.1751, 5.7899)
16	(1.7759, 1.5877, 4.8442, 1.7055)	(2.2037, 0.96108, 1.2596, 5.6885)
17	(1.7853, 1.5843, 4.7988, 1.7493)	(2.1951, 0.98188, 1.3354, 5.5975)
18	(1.7937, 1.5813, 4.7577, 1.7889)	(2.1875, 1.0005, 1.4039, 5.5154)
19	(1.8012, 1.5786, 4.7203, 1.8249)	(2.1807, 1.0173, 1.4662, 5.4406)
20	(1.8079, 1.5761, 4.6861, 1.8578)	(2.1746, 1.0325, 1.5231, 5.3722)
⋮	⋮	⋮
300	(1.9576, 1.5176, 3.7984, 2.7127)	(2.0385, 1.3917, 3.0027, 3.5968)
⋮	⋮	⋮
500	(1.9673, 1.5136, 3.7317, 2.7769)	(2.0297, 1.4163, 3.1139, 3.4634)
501	(1.9673, 1.5136, 3.7315, 2.7771)	(2.0297, 1.4164, 3.1142, 3.4629)
502	(1.9674, 1.5136, 3.7312, 2.7773)	(2.0297, 1.4164, 3.1146, 3.4625)
503	(1.9674, 1.5136, 3.731, 2.7776)	(2.0296, 1.4165, 3.115, 3.462)
504	(1.9674, 1.5136, 3.7308, 2.7778)	(2.0296, 1.4166, 3.1154, 3.4615)
505	(1.9675, 1.5136, 3.7305, 2.778)	(2.0296, 1.4167, 3.1158, 3.4611)
506	(1.9675, 1.5135, 3.7303, 2.7782)	(2.0296, 1.4168, 3.1161, 3.4606)
507	(1.9675, 1.5135, 3.7301, 2.7784)	(2.0295, 1.4169, 3.1165, 3.4602)
508	(1.9676, 1.5135, 3.7299, 2.7786)	(2.0295, 1.4169, 3.1169, 3.4597)
509	(1.9676, 1.5135, 3.7296, 2.7789)	(2.0295, 1.417, 3.1173, 3.4593)
510	(1.9676, 1.5135, 3.7294, 2.7791)	(2.0294, 1.4171, 3.1176, 3.4589)
⋮	⋮	⋮
1000	(1.9769, 1.5096, 3.6643, 2.8418)	(2.021, 1.4409, 3.2261, 3.3287)
⋮	⋮	⋮
2000	(1.9837, 1.5068, 3.6165, 2.8878)	(2.0148, 1.4582, 3.3058, 3.233)
⋮	⋮	⋮
10,000	(1.9927, 1.503, 3.5523, 2.9496)	(2.0066, 1.4813, 3.4128, 3.1046)

5 Conclusions

A new iterative scheme is proposed for a countable family of multi-valued mappings in Hadamard spaces. The proposed scheme does not involve the *CQ* hybrid projection technique to show that it strongly converges to a common fixed point of a family of demicontractive mappings. Moreover, some modified schemes, derived from the proposed iterative scheme, were given as applications for solving the family of minimization, variational inequality, and monotone inclusion problems. Furthermore, the results obtained here also hold in all complete CAT($k < 0$), Hadamard manifolds, Hilbert spaces, and so on. Our results generalized some recent results, for instance, result of [11] follows from Corollary 3.12, and result of [29] follows from Theorem 4.3, and so on.

We have proved strong and Δ convergences for Lipschitzian multivalued demicontractive mappings with the demiclosedness-type property. For future work, one might ask wether the Lipschitzian condition or even the demiclosedness-type property can be dropped and/or wether the result obtained here could be generalized to wider classes of mappings, such as a class of hemicontractive mappings, as well as applying the fixed point results in handling risk assessment model [51].

Acknowledgements
The authors would like to thank Parin Chaipunya and Jamilu Abubakar for comments and suggestions on this paper and acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research project was supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2022 under "High Potential Researchers Incubation Projec" (project number FRB650048/0164).

Funding
This research was funded by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, KMUTT. Sani Salisu is supported by the Petchra Pra Jom Klao Ph.D. Research Scholarship from King Mongkut's University of Technology Thonburi (Contract No. 67/2563). Moreover, this research project is supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2022 under "High Potential Researchers Incubation Project" (project number FRB650048/0164).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contribution
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]Center of Excellence in Theoretical and Computational Science (TaCS-CoE) & KMUTT Fixed Point Research Laboratory, Room SCL 802, Fixed Point Laboratory, Science Laboratory Building, Departments of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand. [2]Department of Mathematics, Faculty of Natural and Applied Sciences, Sule Lamido University Kafin Hausa, P.M.B. o48, Jigawa, Nigeria. [3]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan.

Publisher's Note

Received: 20 May 2021 Accepted: 4 October 2022 Published online: 15 November 2022

References

1. Bruck, R.E.: Asymptotic behavior of nonexpansive mappings. Contemp. Math. **18**, 1–47 (1983)
2. Chidume, C.: Some Geometric Properties of Banach Spaces. Springer, Berlin (2009)
3. Oden, J.T., Kikuchi, N.: Theory of variational inequalities with applications to problems of flow through porous media. Int. J. Eng. Sci. **18**(10), 1173–1284 (1980)
4. Dehghan, H., Izuchukwu, C., Mewomo, O., Taba, D., Ugwunnadi, G.: Iterative algorithm for a family of monotone inclusion problems in CAT(0) spaces. Quaest. Math. **43**(7), 975–998 (2020)
5. Banach, S.: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**(1), 133–181 (1922)
6. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**(3), 506–510 (1953)
7. Ishikawa, S.: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**(1), 147–150 (1974)
8. Noor, M.A.: New approximation schemes for general variational inequalities. J. Math. Anal. Appl. **251**(1), 217–229 (2000)
9. Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**(1), 240–256 (2002)
10. Dhompongsa, S., Kaewkhao, A., Panyanak, B.: On Kirk's strong convergence theorem for multivalued nonexpansive mappings on CAT(0) spaces. Nonlinear Anal., Theory Methods Appl. **75**(2), 459–468 (2012)
11. Chaolamjiak, W., Yambangwai, D., Hammad, H.A.: Modified hybrid projection methods with SP iterations for quasi-nonexpansive multivalued mappings in Hilbert spaces. Bull. Iran. Math. Soc. **47**, 1399–1422 (2021)
12. Nadler, S.B.: Some results on multi-valued contraction mappings. In: Set-Valued Mappings, Selections and Topological Properties of 2^x, pp. 64–69. Springer, Berlin (1970)

13. Abbas, M., Khan, S.H., Khan, A.R., Agarwal, R.P.: Common fixed points of two multivalued nonexpansive mappings by one-step iterative scheme. Appl. Math. Lett. **24**(2), 97–102 (2011)
14. Chidume, C., Chidume, C., Djitte, N., Minjibir, M.: Convergence theorems for fixed points of multivalued strictly pseudocontractive mappings in Hilbert spaces. Abstr. Appl. Anal. **2013**, Article ID 629468 (2013)
15. Chidume, C., Minjibir, M.: Krasnoselskii algorithm for fixed points of multivalued quasi-nonexpansive mappings in certain Banach spaces. Fixed Point Theory **17**(2), 301–311 (2016)
16. Minjibir, M.S., Strong, S.S.: Δ-Convergence theorems for a countable family of multi-valued demicontractive maps in Hadamard spaces. Nonlinear Funct. Anal. Appl. **27**(1), 45–58 (2022)
17. Tang, J., Zhu, J.: A new modified proximal point algorithm for a finite family of minimization problem and fixed point for a finite family of demicontractive mappings in Hadamard spaces. Nonlinear Funct. Anal. Appl. **25**(3), 563–577 (2020)
18. Kirk, W., Shahzad, N.: Fixed Point Theory in Distance Spaces. Springer, Berlin (2014)
19. Eskandani, G.Z., Raeisi, M.: On the zero point problem of monotone operators in Hadamard spaces. Numer. Algorithms **80**(4), 1155–1179 (2019)
20. Izuchukwu, C., Aremu, K.O., Oyewole, O.K., Mewomo, O.T., Khan, S.H.: On mixed equilibrium problems in Hadamard spaces. J. Math. **2019**, Article ID 3210649 (2019)
21. Bacak, M.: Old and new challenges in Hadamard spaces. arXiv preprint (2018). arXiv:1807.01355
22. Bridson, M.R., Haefliger, A.: Metric Spaces of Non-positive Curvature, vol. 319. Springer, Berlin (2013)
23. Kirk, W.: Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees. Fixed Point Theory Appl. **2004**(4), 309–316 (2004)
24. Bačák, M.: The proximal point algorithm in metric spaces. Isr. J. Math. **194**(2), 689–701 (2013)
25. Zamani Eskandani, G., Kim, J.K., Moharami, R., Lim, W.H.: An extragradient method for equilibrium problems and fixed point problems of an asymptotically nonexpansive mapping in CAT(0) spaces. J. Nonlinear Convex Anal. **23**(7), 1509–1523 (2022)
26. Sokhuma, K., Sokhuma, K.: Convergence theorems for two nonlinear mappings in CAT(0) spaces. Nonlinear Funct. Anal. Appl. **27**(3), 499–512 (2022)
27. Kirk, W.A.: Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), vol. 64, pp. 195–225 (2003)
28. Dhompongsa, S., Panyanak, B.: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**(10), 2572–2579 (2008)
29. Cuntavepanit, A., Phuengrattana, W.: On solving the minimization problem and the fixed-point problem for a finite family of non-expansive mappings in CAT(0) spaces. Optim. Methods Softw. **33**(2), 311–321 (2018)
30. Aremu, K., Jolaoso, L., Izuchukwu, C., Mewomo, O.: Approximation of common solution of finite family of monotone inclusion and fixed point problems for demicontractive multivalued mappings in CAT(0) spaces. Ric. Mat. **69**(1), 13–34 (2020)
31. Chaipunya, P., Kumam, P.: On the proximal point method in Hadamard spaces. Optimization **66**(10), 1647–1665 (2017)
32. Kumam, P., Chaipunya, P.: Equilibrium problems and proximal algorithms in Hadamard spaces. arXiv preprint (2018). arXiv:1807.10900
33. Kumam, W., Pakkaranang, N., Kumam, P., Cholamjiak, P.: Convergence analysis of modified Picard-S hybrid iterative algorithms for total asymptotically nonexpansive mappings in Hadamard spaces. Int. J. Comput. Math. **97**(1–2), 175–188 (2020)
34. Phuengrattana, W., Suantai, S.: On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval. J. Comput. Appl. Math. **235**(9), 3006–3014 (2011)
35. Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. Publ. Math. Inst. Hautes Études Sci., **41**, 5–251 (1972)
36. Goebel, K., Simeon, R.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Dekker, New York (1984)
37. Reich, S., Shafrir, I.: Nonexpansive iterations in hyperbolic spaces. Nonlinear Anal., Theory Methods Appl. **15**(6), 537–558 (1990)
38. Brown, K.S.: Buildings. In: Buildings, pp. 76–98. Springer, Berlin (1989)
39. Dhompongsa, S., Kirk, W.A., Sims, B.: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal., Theory Methods Appl. **65**(4), 762–772 (2006)
40. Dhompongsa, S., Kirk, W., Panyanak, B.: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**(1), 35 (2007)
41. Kirk, W., Panyanak, B.: A concept of convergence in geodesic spaces. Nonlinear Anal., Theory Methods Appl. **68**(12), 3689–3696 (2008)
42. Kakavandi, B.: Weak topologies in complete CAT(0) metric spaces. Proc. Am. Math. Soc. **141**(3), 1029–1039 (2013)
43. Maingé, P.E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**(7–8), 899–912 (2008)
44. Dehghan, H., Rooin, J.: Metric projection and convergence theorems for nonexpansive mappings in Hadamard spaces. arXiv preprint (2014). arXiv:1410.1137
45. Ariza-Ruiz, D., Leuştean, L., López-Acedo, G.: Firmly nonexpansive mappings in classes of geodesic spaces. Trans. Am. Math. Soc. **366**(8), 4299–4322 (2014)
46. Jost, J.: Convex functionals and generalized harmonic maps into spaces of non positive curvature. Comment. Math. Helv. **70**(1), 659–673 (1995)
47. Khatibzadeh, H., Ranjbar, S.: A variational inequality in complete CAT(0) spaces. J. Fixed Point Theory Appl. **17**(3), 557–574 (2015)
48. Dehghan, H., Rooin, J.: A characterization of metric projection in CAT(0) spaces. arXiv preprint (2013). arXiv:1311.4174
49. Aremu, K.O., Izuchukwu, C., Abass, H.A., Mewomo, O.T.: On a viscosity iterative method for solving variational inequality problems in Hadamard spaces. Axioms **9**(4), 143 (2020)
50. Khatibzadeh, H., Ranjbar, S.: Monotone operators and the proximal point algorithm in complete CAT(0) metric spaces. J. Aust. Math. Soc. **103**(1), 70–90 (2017)
51. Rezaei, M., Borjalilu, N.: A dynamic risk assessment modeling based on fuzzy ANP for safety management systems. Aviation **22**(4), 143–155 (2018)