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Sharp oscillation theorem for fourth-order linear delay differential equations

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Abstract

In this paper, we present a single-condition sharp criterion for the oscillation of the fourth-order linear delay differential equation

$$x^{(4)}(t) + p(t)x(\tau(t)) = 0$$

by employing a novel method of iteratively improved monotonicity properties of nonoscillatory solutions. The result obtained improves a large number of existing ones in the literature.

Keywords: Fourth-order differential equation; Delayed argument; Oscillation; Sharp criterion

1 Introduction

Consider the fourth-order linear delay differential equation

$$x^{(4)}(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0 > 0, \quad (1)$$

where $p \in C([t_0, \infty))$ is positive, and the delay function $\tau \in C([t_0, \infty))$ satisfies $\tau(t) \leq t$ and $\tau(t) \rightarrow \infty$ as $t \rightarrow \infty$.

By a solution of (1) we understand a four times differentiable real-valued function x that satisfies (1) for all t large enough. Our attention is restricted to those solutions of (1) that satisfy the condition $\sup\{|x(t)| : T \leq t < \infty\} > 0$ for any large $T \geq t_0$. We make a standing hypothesis that equation (1) possesses such solutions. A nontrivial solution of (1) is said to be oscillatory if it has infinitely many zeros and nonoscillatory otherwise. Equation (1) is called oscillatory if all its solutions are oscillatory.

The study of fourth-order differential equations originated with the vibrating rod problem in the first half of the 18th century and is generally of great practical importance. Such equations naturally arise in the modeling of physical and biological phenomena, such as, for instance, elasticity problems, deformation of structures, or oscillations of neuromuscular systems; see, e.g., [1, 2] for more detail.

With regard to their practical importance and the number of mathematical problems involved, the subject of the qualitative theory for such equations has undergone rapid de-

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velopment. In particular, oscillation theory of fourth-order differential equations involving (1) as a particular case has attracted a lot of attention over the last decades, which is evidenced by extensive research in the area, and we refer the reader to the recent related works [3–6] and the references therein. On the other hand, equation (1) can be understood as a prototype of even-order binomial differential equations, investigated in detail in the monographs of Elias [7], Kiguradze and Chanturia [8], Koplatadze [9], and Swanson [10].

The aim of this paper is to obtain an unimprovable result for (1) to be oscillatory, depending on whether the limit inferior

$$\delta_* := \liminf_{t \rightarrow \infty} \frac{t}{\tau(t)}$$

is finite or not. To start, we briefly explain where the motivation behind this research comes from. As a particular case of a more complex work for half-linear delay differential equations, Jadlovská and Džurina [11] showed, via a novel method of iteratively improved monotonicity properties of nonoscillatory solutions, that the second-order delay differential equation

$$x''(t) + p(t)x(\tau(t)) = 0 \tag{2}$$

is oscillatory if

$$\liminf_{t \rightarrow \infty} \tau(t)tp(t) > \begin{cases} 0 & \text{for } \delta_* = \infty, \\ M_1 & \text{for } \delta_* < \infty, \end{cases} \tag{3}$$

where

$$M_1 := \max \{c(1 - c)\delta_*^{-c} : 0 < c < 1\}. \tag{4}$$

We recall that the main purpose of the method is to find for any nonoscillatory, say positive, solution x of the studied binomial equation optimal values of positive constants a and b such that

$$ax(t) > x'(t)t \quad \text{and} \quad bx(t) < x'(t)t,$$

which correspond to the monotonicities

$$\left(\frac{x(t)}{t^a}\right)' < 0 \quad \text{and} \quad \left(\frac{x(t)}{t^b}\right)' > 0,$$

respectively. The oscillation criterion is just an immediate consequence of these monotonicities; see [12] for a detailed description of the method.

Very recently, Graef, Jadlovská, and Tunç [13] extended the approach from [11] and showed that any nonoscillatory solution of the third-order delay differential equation

$$x'''(t) + p(t)x(\tau(t)) = 0 \tag{5}$$

tends to zero asymptotically if

$$\liminf_{t \rightarrow \infty} \tau^2(t)tp(t) > \begin{cases} 0 & \text{for } \delta_* = \infty, \\ M_2 & \text{for } \delta_* < \infty, \end{cases} \tag{6}$$

where

$$M_2 := \max \{c(1 - c)(2 - c)\delta_*^{-c} : 0 < c < 1\}. \tag{7}$$

Three facts are important to notice about the above results. Firstly, no restriction is posed on the monotonicity or boundedness of the delay function $\tau(t)$. Secondly, the result applies also in the ordinary case $\tau(t) = t$. Thirdly, the oscillation constant M_1 (M_2) is optimal for equation (2) (equation (5)) in the sense that the strict inequality in condition (3) (condition (6)) cannot be replaced by a nonstrict one without affecting validity of the result.

A natural question that arises is whether the method of iteratively improved monotonicity properties employed in the above-mentioned works can be extended to obtain sharp results for the fourth-order delay differential equation (1). In this paper, we give a positive answer to this question. Our arguments essentially uses a classical result of Kiguradze [8, Lemma 1.1], by which the set S of all positive nonoscillatory solutions of (1) has the decomposition

$$S = S_1 \cup S_3,$$

where

$$\begin{aligned} x \in S_1 & \iff x > 0, x' > 0, x'' < 0, x''' > 0, \\ x \in S_3 & \iff x > 0, x' > 0, x'' > 0, x''' > 0. \end{aligned}$$

For each of these classes of nonoscillatory solutions, it is possible to initiate an iterative process that converges to the optimal monotonicity values a and b . As a side product of this finding, we formulate a single-condition oscillation criterion with an unimprovable oscillation constant. To the best of our knowledge, there is no qualitatively the same result for (1) in the literature for general $\tau(t)$; see Remarks 2 and 3 for more details.

The organization of the paper is as follows. In Sect. 2, we introduce the basic notations and the core of the method developed. In Sects. 3 and 4, we provide a series of lemmas, iteratively improving monotonicity properties of nonoscillatory solutions belonging to the classes S_3 and S_1 , respectively. In Sect. 5, we present our main result, an oscillation criterion for (1). As usual, the improvement made over the existing results from the literature is illustrated via Euler-type differential equations. Finally, we propose several open problems for further research.

2 Notation

In our proofs, we will use the constants

$$\beta_* := \liminf_{t \rightarrow \infty} \frac{\tau^3(t)tp(t)}{3!}, \quad \gamma_* := \liminf_{t \rightarrow \infty} \frac{\tau(t)t^3p(t)}{3!}, \quad \text{and} \quad \delta_* := \liminf_{t \rightarrow \infty} \frac{t}{\tau(t)}.$$

All our results require, directly or indirectly, that β_* and γ_* are positive. Obviously, for arbitrary but fixed $\beta \in (0, \beta_*)$, $\gamma \in (0, \gamma_*)$, $\delta \in (1, \delta_*)$ for $\delta_* > 1$, and $\delta = \delta_*$ for $\delta_* = 1$, there is $t_1 \geq t_0$ large enough such that

$$\frac{\tau^3(t)t^3p(t)}{3!} \geq \beta, \quad \frac{\tau(t)t^3p(t)}{3!} \geq \gamma, \quad \text{and} \quad \frac{t}{\tau(t)} \geq \delta, \quad t \geq t_1. \tag{8}$$

In view of the above, let us define (as far as they exist) the sequences $\{\beta_n\}$ and $\{\gamma_n\}$ by

$$[k]_0 := [k]_*,$$

$$[k]_n := \frac{3! [k]_0 \delta_*^{[k]_{n-1}}}{(3 - [k]_{n-1})(2 - [k]_{n-1})(1 - [k]_{n-1})}, \quad n \in \mathbb{N}, \tag{9}$$

where $[k]$ stands for either β or γ . By induction it is easy to show that if $[k]_i < 1$ for $i = 1, 2, \dots, n$, then $[k]_{n+1}$ exists, and

$$\frac{[k]_{n+1}}{[k]_n} = \ell_{[k]_n} > 1, \tag{10}$$

where

$$\ell_{[k]_0} := \frac{[k]_1}{[k]_0} = \frac{3! \delta_*^{[k]_0}}{(3 - [k]_0)(2 - [k]_0)(1 - [k]_0)} > 1,$$

$$\ell_{[k]_n} := \frac{[k]_{n+1}}{[k]_n} = \frac{\delta_*^{[k]_n} (3 - [k]_{n-1})(2 - [k]_{n-1})(1 - [k]_{n-1})}{\delta_*^{[k]_{n-1}} (3 - [k]_n)(2 - [k]_n)(1 - [k]_n)} > 1, \quad n \in \mathbb{N}.$$

For arbitrary but fixed $[k] \in (0, [k]_*)$ and $[k]_i < 1$, $i = 1, 2, \dots, n$, we also define the sequence $\{\varepsilon_{[k]_n}\}$:

$$\varepsilon_{[k]_0} = \frac{[k]}{[k]_*} < 1,$$

$$\varepsilon_{[k]_n} = \varepsilon_{[k]_0} \frac{\delta_*^{\varepsilon_{[k]_{n-1}} [k]_{n-1}} (3 - [k]_{n-1})(2 - [k]_{n-1})(1 - [k]_{n-1})}{\delta_*^{[k]_{n-1}} (3 - \varepsilon_{[k]_{n-1}} [k]_{n-1})(2 - \varepsilon_{[k]_{n-1}} [k]_{n-1})(1 - \varepsilon_{[k]_{n-1}} [k]_{n-1})}, \tag{11}$$

$$n \in \mathbb{N}.$$

Again, by induction it follows that $0 < \varepsilon_{[k]_n} < \varepsilon_{[k]_{n+1}} < 1$ for $n \in \mathbb{N}_0$ and

$$\lim_{[k] \rightarrow [k]_*} \varepsilon_{[k]_0} = 1,$$

$$\lim_{(\delta \rightarrow \delta_*) ([k] \rightarrow [k]_*)} \varepsilon_{[k]_{n+1}} = 1.$$

We will use these facts in our proofs. As usual, and without loss of generality, we can suppose from now on that nonoscillatory solutions of (1) are eventually positive.

3 Nonexistence of S_3 -type solutions

In view of (8), from (1) we see that

$$x^{(4)}(t) + \frac{3! \beta}{\tau^3(t)t} x(\tau(t)) \leq 0, \quad t \geq t_1. \tag{12}$$

We begin with a simple lemma, which gives information on the behavior of possible nonoscillatory solutions belonging to the class \mathcal{S}_3 .

Lemma 1 *Let $\beta_* > 0$ and assume that x is a solution of (1) belonging to the class \mathcal{S}_3 . Then for t sufficiently large:*

- (i) $\lim_{t \rightarrow \infty} x'''(t) = \lim_{t \rightarrow \infty} x''(t)/t = \lim_{t \rightarrow \infty} x'(t)/t^2 = \lim_{t \rightarrow \infty} x(t)/t^3 = 0$;
- (ii) $x''(t)/t$ is decreasing, and $x''(t) > tx'''(t)$;
- (iii) $x'(t)/t^2$ is decreasing, and $x'(t) > tx''(t)/2$;
- (iv) $x(t)/t^3$ is decreasing, and $x(t) > tx'(t)/3$.

Proof Let $x \in \mathcal{S}_3$ and choose $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ for $t \geq t_1$.

- (i) Since $x'''(t)$ is a nonincreasing positive function, there exists a finite limit

$$\lim_{t \rightarrow \infty} x'''(t) = \ell \geq 0.$$

If $\ell > 0$, then $x'''(t) \geq \ell > 0$, and so $x(t) \geq \ell(t - t_1)^3/3!$ eventually, say for $t \geq t_2 \geq t_1$. Using this in (12) gives

$$-x^{(4)}(t) \geq \frac{\beta \ell}{\tau^3(t)t} (\tau(t) - t_1)^3.$$

Clearly, there exists $t_3 > t_2$ such that

$$(\tau(t) - t_1)^3 > \frac{\tau^3(t)}{2}, \quad t \geq t_3,$$

which implies

$$-x^{(4)}(t) > \frac{\beta \ell}{2t}, \quad t \geq t_3.$$

Integrating the above inequality from t_3 to t gives

$$x'''(t_3) \geq x'''(t) + \frac{\beta \ell}{2} \ln \frac{t}{t_3} \geq \ell + \frac{\beta \ell}{2} \ln \frac{t}{t_3} \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

which is a contradiction. Hence $\ell = 0$.

Note that if $x \in \mathcal{S}_3$, then $x(t) \rightarrow \infty$ and $x'(t) \rightarrow \infty$ as $t \rightarrow \infty$. Also, if $x''(t) \not\rightarrow \infty$, then $\frac{x''(t)}{t} \rightarrow 0$ as $t \rightarrow \infty$ since $x''(t) > 0$ is increasing. Hence we can apply l'Hôpital's rule to see that (i) holds.

- (ii) Again using the fact that $x'''(t)$ is positive and nonincreasing, we find that

$$x''(t) = x''(t_1) + \int_{t_1}^t x'''(s) ds \geq x''(t_1) + x'''(t)(t - t_1).$$

In view of (i), there is $t_4 > t_1$ such that $x''(t_1) > t_1 x'''(t)$ for $t \geq t_4$. Therefore

$$x''(t) > tx'''(t), \quad t \geq t_4,$$

and so

$$\left(\frac{x''(t)}{t}\right)' = \frac{x'''(t)t - x''(t)}{t^2} < 0, \quad t \geq t_4,$$

which proves (ii).

(iii) Since $x''(t)/t$ is a decreasing function tending to zero (see (i) and (ii)), we have

$$\begin{aligned} x'(t) &= x'(t_4) + \int_{t_3}^t x''(s) \, ds \geq x'(t_4) + \frac{x''(t)}{t} \left(\frac{t^2}{2} - \frac{t_4^2}{2}\right) \\ &= \frac{x''(t)t}{2} + x'(t_4) - \frac{x''(t)t_4^2}{2t} > \frac{tx''(t)}{2}, \quad t \geq t_5, \end{aligned}$$

for some $t_5 > t_4$. Therefore

$$\left(\frac{x'(t)}{t^2}\right)' = \frac{x''(t)t - 2x'(t)}{t^3} < 0, \quad t \geq t_5,$$

which proves (iii).

(iv) Similarly, since $x'(t)/t^2$ is a decreasing function tending to zero (see (i) and (iii)), we get

$$x(t) = x(t_5) + \int_{t_5}^t x'(s) \, ds \geq x(t_5) + \frac{x'(t)}{t^2} \left(\frac{t^3}{3} - \frac{t_5^3}{3}\right) > \frac{x'(t)t}{3}, \quad t \geq t_6,$$

for some $t_6 > t_5$. Thus

$$\left(\frac{x(t)}{t^3}\right)' = \frac{x'(t)t - 3x(t)}{t^4} < 0, \quad t \geq t_6,$$

which proves (iv) and completes the proof of the lemma. □

Remark 1 When investigating the asymptotic properties of solutions of higher-order differential equations, authors often refer to the famous monograph of Kiguradze and Chanturia [8] and use the following result: for the function h satisfying $h^{(i)}(t) > 0, i = 0, 1, \dots, m$, and $h^{(m+1)}(t) \leq 0$ eventually, $h(t)/h'(t) \geq t/m$. However, as remarked in [14], only

$$\frac{h(t)}{h'(t)} \geq \ell \frac{t}{m} \tag{13}$$

holds eventually for every $\ell \in (0, 1)$. The necessity of the constant $\ell \in (0, 1)$ in inequality (13) has been shown by means of counterexamples; see [14] for details. Obviously, the application of (13) to the solution $x(t) \in \mathcal{S}_3$ would lead to

$$x^{(i)}(t) \geq \ell \frac{x^{(i+1)}(t)}{i+1}, \quad i = 0, 1, 2,$$

which is a weaker result than Lemma 1 provides. The omission of the constant ℓ was made possible by the requirement of having β_* positive.

The next lemma provides some additional properties of solutions in the class \mathcal{S}_3 .

Lemma 2 Let $\beta_* > 0$ and assume that x is a solution of (1) belonging to the class \mathcal{S}_3 . Then for any $\beta \in (0, \beta_*)$ and t sufficiently large:

- (v) $x''(t)/t^{1-\beta}$ is decreasing, and $(1 - \beta)x''(t) > tx'''(t)$;
- (vi) $\beta < 1$;
- (vii) $\lim_{t \rightarrow \infty} x''(t)/t^{1-\beta} = \lim_{t \rightarrow \infty} x'(t)/t^{2-\beta} = \lim_{t \rightarrow \infty} x(t)/t^{3-\beta} = 0$;
- (viii) $x'(t)/t^{2-\beta}$ is decreasing, and $x'(t) > tx''(t)/(2 - \beta)$;
- (ix) $x(t)/t^{3-\beta}$ is decreasing, and $x(t) > tx'(t)/(3 - \beta)$.

Proof Let $x \in \mathcal{S}_3$ and choose $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ and parts (i)–(iv) of Lemma 1 hold for $t \geq t_1$.

(v) Define the function

$$z(t) := x''(t) - tx'''(t), \tag{14}$$

which is positive in view of (ii). Differentiating z and using (12) and the monotonicity of $x(t)/t^3$ (see (iv)), we obtain

$$z'(t) = -tx^{(4)}(t) \geq 3! \beta \frac{x(\tau(t))}{\tau^3(t)} \geq 3! \beta \frac{x(t)}{t^3}. \tag{15}$$

Using the estimates from (iv) and (iii), respectively, in the above inequality, we find

$$z'(t) > 2\beta \frac{x'(t)}{t^2} > \beta \frac{x''(t)}{t}. \tag{16}$$

Integrating (16) from t_1 to t and using the fact that $x''(t)/t$ is decreasing and tends to zero (see (i) and (ii)), we obtain

$$z(t) > z(t_1) + \beta \int_{t_1}^t \frac{x''(s)}{s} ds \geq z(t_1) + \beta \frac{x''(t)}{t}(t - t_1) > \beta x''(t), \quad t \geq t_2,$$

for some $t_2 > t_1$, that is,

$$(1 - \beta)x''(t) > tx'''(t), \quad t \geq t_2,$$

and so

$$\left(\frac{x''(t)}{t^{1-\beta}} \right)' = \frac{x'''(t)t - (1 - \beta)x''(t)}{t^{2-\beta}} < 0, \quad t \geq t_2. \tag{17}$$

Hence part (v) holds.

(vi) This clearly follows from (v) and the fact that $x''(t)$ is increasing.

(vii) To show

$$\lim_{t \rightarrow \infty} \frac{x''(t)}{t^{1-\beta}} = 0, \tag{18}$$

it suffices to prove that there is $\varepsilon > 1$ such that, for sufficiently large t ,

$$\left(\frac{x''(t)}{t^{1-\varepsilon\beta}} \right)' < 0. \tag{19}$$

Indeed, if

$$\frac{x''(t)}{t^{1-\beta}} \geq c > 0,$$

then

$$\frac{x''(t)}{t^{1-\beta-\beta(\varepsilon-1)}} \geq ct^{\beta(\varepsilon-1)} \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

which is a contradiction. Using (17), we see that for any $k \in ((2 - \beta)/2, 1)$, there is $t_3 \geq t_2$ sufficiently large such that

$$\begin{aligned} x'(t) &= x'(t_2) + \int_{t_2}^t \frac{x''(s)}{s^{1-\beta}} s^{1-\beta} ds \\ &\geq x'(t_2) + \frac{x''(t)}{t^{1-\beta}} \int_{t_2}^t s^{1-\beta} ds \\ &= x'(t_2) + \frac{x''(t)}{t^{1-\beta}} \frac{(t^{2-\beta} - t_2^{2-\beta})}{2-\beta} > \frac{k}{2-\beta} x''(t)t, \quad t \geq t_3. \end{aligned} \tag{20}$$

Using this in (16), we have

$$z'(t) > 2\beta \frac{x'(t)}{t^2} > \frac{2k\beta}{(2-\beta)} \frac{x''(t)}{t}.$$

Integrating from t_3 to t and using (i), this becomes

$$z(t) \geq z(t_3) + \frac{2k\beta}{(2-\beta)} \frac{x''(t)}{t} (t - t_3) > \frac{2k\beta}{(2-\beta)} x''(t), \quad t \geq t_4,$$

for some $t_4 > t_3$. In view of the definition of z (see (14)), the above inequality implies

$$\left(1 - \frac{2k\beta}{2-\beta}\right) x''(t) > x'''(t).$$

Then it is easy to see that (19) holds with

$$\varepsilon = \frac{2k}{2-\beta} > 1.$$

The remaining limits in (vii) follow from (18) and an application of l'Hôpital's rule.

(viii) Using (18) in (20), we have

$$x'(t) \geq \frac{x''(t)t}{2-\beta} + x'(t_2) - \frac{x''(t)t_2^{2-\beta}}{t^{1-\beta}(2-\beta)} > \frac{x''(t)t}{2-\beta}, \quad t \geq t_5,$$

for some $t_5 > t_4$. Thus

$$\left(\frac{x'(t)}{t^{2-\beta}}\right)' = \frac{x''(t)t - (2-\beta)x'(t)}{t^{3-\beta}} < 0, \quad t \geq t_5.$$

(ix) Finally, since $x'(t)/t^{2-\beta}$ is a decreasing function tending to zero (see (vii) and (viii)), we have

$$\begin{aligned} x(t) &= x(t_5) + \int_{t_5}^t \frac{x'(s)}{s^{2-\beta}} s^{2-\beta} ds \\ &\geq x(t_5) + \frac{x'(t)}{t^{2-\beta}} \frac{(t^{3-\beta} - t_5^{3-\beta})}{3-\beta} \\ &= \frac{x'(t)t}{3-\beta} + x(t_5) - \frac{x'(t)t_5^{3-\beta}}{t^{2-\beta}(3-\beta)} > \frac{x'(t)t}{3-\beta}, \quad t \geq t_6, \end{aligned}$$

for some $t_6 > t_5$. Therefore

$$\left(\frac{x(t)}{t^{3-\beta}}\right)' = \frac{x'(t)t - (3-\beta)x(t)}{t^{4-\beta}} < 0, \quad t \geq t_6.$$

The proof is complete. □

The next result is a simple consequence of (ix).

Lemma 3 *Assume that $\beta_* > 0$ and $\delta_* = \infty$. Then $S_3 = \emptyset$.*

Proof Suppose to the contrary that $x \in S_3$ and let $t_1 \geq t_0$ be such that $x(\tau(t)) > 0$ for $t \geq t_1$. Using (ix) and (8) in (15), we find

$$z'(t) = -tx^{(4)}(t) \geq 3!\beta \frac{x(\tau(t))}{\tau^{3-\beta}(t)\tau^\beta(t)} \geq 3!\beta \frac{x(t)}{t^3} \left(\frac{t}{\tau(t)}\right)^\beta \geq 3!\beta\delta^\beta \frac{x(t)}{t^3}.$$

Also, from (ix) and (viii) it follows, respectively, that

$$z'(t) > \frac{3!\beta\delta^\beta}{(3-\beta)} \frac{x'(t)}{t^2} > \frac{3!\beta\delta^\beta}{(3-\beta)(2-\beta)} \frac{x''(t)}{t}$$

eventually, say for $t \geq t_2$ for some $t_2 \geq t_1$. Proceeding as in the proof of (v), we arrive at

$$\left(1 - \frac{3!\beta\delta^\beta}{(3-\beta)(2-\beta)}\right)x''(t) > tx'''(t).$$

Since δ can be arbitrarily large, we choose it so that

$$\delta^\beta > \frac{(3-\beta)(2-\beta)}{3!\beta}.$$

This implies that $-x''(t) > tx'''(t)$, which is a contradiction. This proves the lemma. □

In view of Lemma 2(vi) and Lemma 3, it is reasonable to assume that $\delta_* < \infty$, so that $S_3 \neq \emptyset$. The following lemma can be seen as an iterative version of Lemma 2.

Lemma 4 *Let $\beta_* > 0$ and assume that x is a solution of (1) belonging to the class S_3 . Then for any $\varepsilon_{\beta_n} \in (0, 1)$ and sufficiently large t ,*

- (I)_n $x''(t)/t^{1-\tilde{\beta}_n}$ is decreasing, and $(1 - \tilde{\beta}_n)x''(t) > tx'''(t)$;
- (II)_n $\tilde{\beta}_n < 1$;
- (III)_n $\lim_{t \rightarrow \infty} x''(t)/t^{1-\tilde{\beta}_n} = \lim_{t \rightarrow \infty} x'(t)/t^{2-\tilde{\beta}_n} = \lim_{t \rightarrow \infty} x(t)/t^{3-\tilde{\beta}_n} = 0$;
- (IV)_n $x'(t)/t^{2-\tilde{\beta}_n}$ is decreasing, and $x'(t) > tx''(t)/(2 - \tilde{\beta}_n)$;
- (V)_n $x(t)/t^{3-\tilde{\beta}_n}$ is decreasing, and $x(t) > tx'(t)/(3 - \tilde{\beta}_n)$;

where $\tilde{\beta}_n = \varepsilon_{\beta_n} \beta_n$.

Proof Let $x \in \mathcal{S}_3$ with $x(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. We will proceed by induction on n . For $n = 0$, the conclusion clearly follows from Lemma 2 with $\beta = \tilde{\beta}_0$. Next, assume that (I)_n–(IV)_n hold for $n \geq 1$ and $t \geq t_n \geq t_1$. We need to show that they all hold for $n + 1$.

(I)_{n+1} Using (viii) and (8) in (15), we have

$$\begin{aligned} z'(t) &= (x''(t) - tx'''(t))' \geq 3!\tilde{\beta}_0 \frac{x(\tau(t))}{\tau^3(t)} \\ &= 3!\tilde{\beta}_0 \frac{x(\tau(t))}{\tau^{3-\tilde{\beta}_n}(t)\tau^{\tilde{\beta}_n}(t)} \\ &\geq 3!\tilde{\beta}_0 \frac{x(t)}{t^3} \left(\frac{t}{\tau(t)}\right)^{\tilde{\beta}_n} \\ &\geq 3!\tilde{\beta}_0 \delta^{\tilde{\beta}_n} \frac{x(t)}{t^3}. \end{aligned}$$

Then by (V)_n and (IV)_n we get

$$z'(t) > \frac{3!\tilde{\beta}_0 \delta^{\tilde{\beta}_n} x'(t)}{(3 - \tilde{\beta}_n) t^2} > \frac{3!\tilde{\beta}_0 \delta^{\tilde{\beta}_n} x''(t)}{(3 - \tilde{\beta}_n)(2 - \tilde{\beta}_n) t}. \tag{21}$$

Integrating (21) from t_n to t and using (I)_n and (III)_n in the resulting inequality, we see that there exists $t'_n > t_n$ such that

$$\begin{aligned} z(t) &\geq z(t_n) + \frac{3!\tilde{\beta}_0 \delta^{\tilde{\beta}_n}}{(3 - \tilde{\beta}_n)(2 - \tilde{\beta}_n)} \int_{t_n}^t \frac{x''(s)}{s^{1-\tilde{\beta}_n} s^{\tilde{\beta}_n}} ds \\ &\geq z(t_n) + \frac{3!\tilde{\beta}_0 \delta^{\tilde{\beta}_n}}{(3 - \tilde{\beta}_n)(2 - \tilde{\beta}_n)(1 - \tilde{\beta}_n)} \frac{x''(t)}{t^{1-\tilde{\beta}_n}} (t^{1-\tilde{\beta}_n} - t_n^{1-\tilde{\beta}_n}) \\ &> \frac{3!\tilde{\beta}_0 \delta^{\tilde{\beta}_n}}{(3 - \tilde{\beta}_n)(2 - \tilde{\beta}_n)(1 - \tilde{\beta}_n)} x''(t) \\ &= \tilde{\beta}_{n+1} x''(t), \quad t \geq t'_n, \end{aligned}$$

i.e.,

$$(1 - \tilde{\beta}_{n+1})x''(t) > tx'''(t) \tag{22}$$

and

$$\left(\frac{x''(t)}{t^{1-\tilde{\beta}_{n+1}}}\right)' < 0, \tag{23}$$

which proves (I)_{n+1}.

(II)_{n+1} This follows immediately from (I)_{n+1} and the fact that $x''(t)$ is increasing.

(III)_{n+1} As for the case $n = 0$, it suffices to show that there is $\varepsilon > 1$ such that

$$\left(\frac{x''(t)}{t^{1-\tilde{\beta}_{n+1}}}\right)' < 0. \tag{24}$$

Using (23), we see that for any $k \in (0, 1)$, there is $t''_n \geq t'_n$ sufficiently large such that

$$\begin{aligned} x'(t) &= x'(t'_n) + \int_{t'_n}^t \frac{x''(s)}{s^{1-\tilde{\beta}_{n+1}}} s^{1-\tilde{\beta}_{n+1}} ds \\ &\geq x'(t'_n) + \frac{x''(t)}{t^{1-\tilde{\beta}_{n+1}}} \int_{t'_n}^t s^{1-\tilde{\beta}_{n+1}} ds \\ &= x'(t'_n) + \frac{x''(t)}{t^{1-\tilde{\beta}_{n+1}}} \frac{(t^{2-\tilde{\beta}_{n+1}} - (t'_n)^{2-\tilde{\beta}_{n+1}})}{2 - \tilde{\beta}_{n+1}} \\ &> \frac{k}{2 - \tilde{\beta}_{n+1}} x''(t)t, \quad t \geq t''_n. \end{aligned} \tag{25}$$

Combining this with (21), we obtain

$$z'(t) > \frac{3! \tilde{\beta}_0 \delta^{\tilde{\beta}_n}}{(3 - \tilde{\beta}_n)} \frac{x'(t)}{t^2} > \frac{3! k \tilde{\beta}_0 \delta^{\tilde{\beta}_n}}{(3 - \tilde{\beta}_n)(2 - \tilde{\beta}_{n+1})} \frac{x''(t)}{t},$$

which after integrating from t''_n to t and using (III)_n yields

$$\begin{aligned} z(t) &> z(t''_n) + \frac{3! k \tilde{\beta}_0 \delta^{\tilde{\beta}_n}}{(3 - \tilde{\beta}_n)(2 - \tilde{\beta}_{n+1})} \int_{t''_n}^t \frac{x''(s)}{s^{1-\tilde{\beta}_n} s^{\tilde{\beta}_n}} ds \\ &> \frac{3! k \tilde{\beta}_0 \delta^{\tilde{\beta}_n}}{(3 - \tilde{\beta}_n)(2 - \tilde{\beta}_{n+1})(1 - \tilde{\beta}_n)} x''(t) \\ &= \frac{k(2 - \tilde{\beta}_n)}{(2 - \tilde{\beta}_{n+1})} \tilde{\beta}_{n+1} x''(t) \\ &= \varepsilon \tilde{\beta}_{n+1} x''(t), \quad t \geq t''_n > t'_n. \end{aligned}$$

Since $\tilde{\beta}_n < \tilde{\beta}_{n+1}$, we can choose k such that $\varepsilon > 1$, and hence (19) holds. The rest of proof is the same as that for $n = 0$.

(IV)_{n+1} Using (III)_{n+1} in (25) gives

$$(2 - \tilde{\beta}_{n+1})x'(t) > x''(t)t,$$

and so (IV)_{n+1} holds.

(V)_{n+1} As in the case $n = 0$, using the fact that $x'/t^{2-\tilde{\beta}_{n+1}}$ is decreasing and tends to zero, we get

$$x(t) = x(t''_n) + \int_{t''_n}^t \frac{x'(s)}{s^{2-\tilde{\beta}_{n+1}}} s^{2-\tilde{\beta}_{n+1}} ds > \frac{x'(t)t}{3 - \tilde{\beta}_{n+1}}, \tag{26}$$

which implies (V)_{n+1} and completes the proof of the lemma. □

From the above arguments we can immediately obtain the following lemma.

Lemma 5 *Assume that $\delta_* < \infty$ and*

$$\liminf_{t \rightarrow \infty} \tau^3(t)tp(t) > M, \tag{27}$$

where

$$M := \max\{c(1 - c)(2 - c)(3 - c)\delta_*^{-c} : 0 < c < 1\}. \tag{28}$$

Then $\mathcal{S}_3 = \emptyset$.

Proof Suppose to the contrary that $x \in \mathcal{S}_3$ and let $t_1 \geq t_0$ be such that $x(\tau(t)) > 0$ for $t \geq t_1$. We claim that

$$\beta_{n-1} < 1, \quad n \in \mathbb{N}. \tag{29}$$

By (II)_n, $\tilde{\beta}_n < 1$. Since $\varepsilon_{\beta_n} \in (0, 1)$ can be chosen arbitrarily, set $\varepsilon_{\beta_n} > 1/\ell_{\beta_n}$, where ℓ_{β_n} is defined by (10). Then

$$1 > \tilde{\beta}_n = \varepsilon_{\beta_n} \ell_{\beta_n} \beta_{n-1} > \beta_{n-1},$$

which proves the claim. In view of (29), we conclude that the sequence $\{\beta_n\}_{n=0}^\infty$ defined by (9) is increasing and bounded from above, that is, there exists a finite limit

$$\lim_{n \rightarrow \infty} \beta_n = c,$$

where $c \in (0, 1)$ is a root of the equation

$$c(3 - c)(2 - c)(1 - c)\delta_*^{-c} = 3!\beta_*. \tag{30}$$

However, condition (27) implies that (30) does not possess positive solutions. Hence $\mathcal{S}_3 = \emptyset$, and the proof is complete. □

4 Nonexistence of \mathcal{S}_1 -type solutions

In this section, we prove similar results to those in Sect. 3 for solutions in the class \mathcal{S}_1 . In view of (8), equation (1) becomes

$$x^{(4)}(t) + \frac{3!\gamma}{\tau(t)t^3}x(\tau(t)) \leq 0, \quad t \geq t_1. \tag{31}$$

Lemma 6 *Let $\gamma_* > 0$ and assume that x is a solution of (1) belonging to the class \mathcal{S}_1 . Then for t sufficiently large:*

- (i) $\lim_{t \rightarrow \infty} x'(t) = \lim_{t \rightarrow \infty} x(t)/t = 0$;
- (ii) $x(t)/t$ is decreasing.

Proof Let $x \in \mathcal{S}_1$ and choose $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ for $t \geq t_1$.

(i) Since $x'(t)$ is a decreasing positive function, there exists a finite limit

$$\lim_{t \rightarrow \infty} x'(t) = \ell \geq 0.$$

If $\ell > 0$, then $x'(t) \geq \ell > 0$, and so $x(t) \geq \ell(t - t_1) > \ell t/3$ for $t \geq t_2$ for some $t_2 \geq t_1$. Using this in (31) gives

$$-x^{(4)}(t) \geq \frac{2\ell\gamma}{t^3}.$$

Integrating twice from t to ∞ , we have

$$-x''(t) \geq \frac{\ell\gamma}{t},$$

and after integrating from t_2 to t ,

$$x'(t_1) \geq x(t) + \ell\gamma \ln \frac{t}{t_2} \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

which is a contradiction. Hence $\ell = 0$. Applying l'Hôpital's rule, we see that (i) holds.

(ii) Again using the monotonicity of x' and (i), we see that

$$x(t) = x(t_1) + \int_{t_1}^t x'(s) \, ds \geq x(t_1) + x'(t)(t - t_1) > x'(t)t$$

for $t \geq t_3 > t_1$, where t_3 is sufficiently large such that $x(t_1) - x'(t)t_1 > 0$ for $t \geq t_3$. Thus

$$\left(\frac{x(t)}{t}\right)' = \frac{x'(t)t - x(t)}{t^2} < 0, \quad t \geq t_3,$$

and the proof is complete. □

Lemma 7 *Let $\gamma_* > 0$ and assume that x is a solution of (1) belonging to the class \mathcal{S}_1 . Then for any $\gamma \in (0, \gamma_*)$ and t sufficiently large:*

- (iii) $x(t)/t^{1-\gamma}$ is decreasing;
- (iv) $\gamma < 1$;
- (v) $\lim_{t \rightarrow \infty} x(t)/t^{1-\gamma} = 0$;
- (vi) $x(t)/t^\gamma$ is nondecreasing.

Proof Let $x \in \mathcal{S}_1$ and choose $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ for $t \geq t_1$.

(iii) Since $x(t)/t$ is decreasing (see (ii) in (31)), we obtain

$$-x^{(4)}(t) \geq \frac{3!\gamma}{t^3} \frac{x(\tau(t))}{\tau(t)} \geq 3!\gamma \frac{x(t)}{t^4}.$$

Integrating this inequality twice from t to ∞ and using at each step that x is increasing yields

$$-x''(t) \geq \gamma \frac{x(t)}{t^2}. \tag{32}$$

Define the function

$$w(t) = x(t) - tx'(t),$$

which is clearly positive in view of (ii). Differentiating w and using (32), we see that

$$w'(t) := -tx''(t) \geq \gamma \frac{x(t)}{t}. \tag{33}$$

Integrating from t_1 to t and using again that $x(t)/t$ is decreasing and tends to zero, we have

$$w(t) \geq w(t_1) + \gamma \int_{t_1}^t \frac{x(s)}{s} ds \geq w(t_1) + \gamma \frac{x(t)}{t}(t - t_1) > \gamma x(t), \quad t \geq t_2, \tag{34}$$

where $t_2 > t_1$ is sufficiently large such that $w(t_1) - t_1x(t)/t > 0$ for $t \geq t_2$. Hence

$$(1 - \gamma)x(t) > tx'(t)$$

and

$$\left(\frac{x(t)}{t^{1-\gamma}}\right)' = \frac{x'(t)t - (1 - \gamma)x(t)}{t^{2-\gamma}} < 0, \quad t \geq t_2, \tag{35}$$

so (iii) holds.

(iv) This clearly follows from (iii) and the fact that x is increasing.

(v) To prove this, similarly to the proof for the class \mathcal{S}_3 , it suffices to show that

$$\left(\frac{x(t)}{t^{1-\varepsilon\gamma}}\right)' < 0 \tag{36}$$

for some $\varepsilon > 1$. Using (35) in (34), we see that for any $k \in (1 - \gamma, 1)$, there is $t_3 \geq t_2$ such that

$$\begin{aligned} w(t) &\geq w(t_2) + \gamma \int_{t_2}^t \frac{x(s)}{s^{1-\gamma}s^\gamma} ds \\ &\geq w(t_2) + \frac{\gamma}{1-\gamma} \frac{x(t)}{t^{1-\gamma}}(t^{1-\gamma} - t_2^{1-\gamma}) > \frac{k\gamma}{1-\gamma}x(t), \quad t \geq t_3, \end{aligned} \tag{37}$$

from which it follows that

$$\left(1 - \frac{k\gamma}{1-\gamma}\right)x(t) > tx'(t).$$

Now it is clear that (36) holds with $\varepsilon = k/(1 - \gamma) > 1$.

(vi) Integrating (32) from t to ∞ and using the monotonicity of x , we get

$$x'(t) \geq \gamma \int_t^\infty \frac{x(s)}{s^2} ds \geq \gamma \frac{x(t)}{t},$$

and so

$$\left(\frac{x(t)}{t^\gamma}\right)' \geq 0.$$

The proof of the lemma is now complete. □

Lemma 8 *Assume that $\gamma_* > 0$ and $\delta_* = \infty$. Then $S_1 = \emptyset$.*

Proof Suppose to the contrary that $x \in S_1$ and let $t_1 \geq t_0$ be such that $x(\tau(t)) > 0$ for $t \geq t_1$. Using (iii) and (8) in (31), we see that

$$-x^{(4)}(t) \geq \frac{3!\gamma}{t^3} \frac{x(\tau(t))}{\tau^{1-\gamma}(t)\tau^\gamma(t)} \geq 3!\gamma \frac{x(t)}{t^4} \left(\frac{t}{\tau(t)}\right)^\gamma \geq 3!\gamma \delta^\gamma \frac{x(t)}{t^4}.$$

Integrating twice from t to ∞ and using repeatedly that x is increasing, we obtain

$$-x''(t) \geq \gamma \delta^\gamma \frac{x(t)}{t^2}. \tag{38}$$

Then using (38) in (33) yields

$$w'(t) \geq \gamma \delta^\gamma \frac{x(t)}{t}.$$

Integrating as in (34) with γ replaced by $\gamma \delta^\gamma$ leads to

$$(1 - \gamma \delta^\gamma)x(t) > tx'(t).$$

Since δ can be arbitrarily large, we can choose it so that

$$\delta^\gamma > \frac{1}{\gamma},$$

which implies that $-x(t) > tx'(t)$, a contradiction. This proves the lemma. □

Next, we obtain an iterative form of Lemma 7.

Lemma 9 *Let $\gamma_* > 0$ and assume that x is a solution of (1) belonging to the class S_1 . Then for any $\varepsilon_{\gamma_n} \in (0, 1)$ and sufficiently large t :*

- (I)_n $x(t)/t^{1-\tilde{\gamma}_n}$ is decreasing;
- (II)_n $\tilde{\gamma}_n < 1$;
- (III)_n $\lim_{t \rightarrow \infty} x(t)/t^{1-\tilde{\gamma}_n} = 0$;
- (IV)_n $x(t)/t^{\tilde{\gamma}_n}$ is nondecreasing;

where $\tilde{\gamma}_n = \varepsilon_{\gamma_n} \gamma_n$.

Proof Let $x \in S_1$ with $x(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. We will proceed by induction on n . For $n = 0$, the conclusion follows from Lemma 7. Next, assume that (I)_n–(IV)_n hold for $n \geq 1$ and $t \geq t_n \geq t_1$ and let us show that (I)_{n+1} holds. Using (I)_n in (31) gives

$$-x^{(4)}(t) \geq \frac{3!\tilde{\gamma}_0}{t^3} \frac{x(\tau(t))}{\tau^{1-\tilde{\gamma}_n}(t)\tau^{\tilde{\gamma}_n}(t)} \geq 3!\gamma \frac{x(t)}{t^4} \left(\frac{t}{\tau(t)}\right)^{\tilde{\gamma}_n} \geq 3!\tilde{\gamma}_0 \delta^{\tilde{\gamma}_n} \frac{x(t)}{t^4}.$$

Integrating the above inequality from t to ∞ and using the fact that $x(t)/t^{\tilde{\gamma}_n}$ is increasing (see (IV)_n), we obtain

$$\begin{aligned} x'''(t) &\geq 3!\tilde{\gamma}_0\delta^{\tilde{\gamma}_n} \int_t^\infty \frac{x(s)}{s^{\tilde{\gamma}_n}s^{4-\tilde{\gamma}_n}} ds \\ &\geq 3!\tilde{\gamma}_0\delta^{\tilde{\gamma}_n} \frac{x(t)}{t^{\tilde{\gamma}_n}} \int_t^\infty \frac{1}{s^{4-\tilde{\gamma}_n}} ds \\ &= \frac{3!\gamma\delta^{\tilde{\gamma}_n}}{3-\tilde{\gamma}_n} \frac{x(t)}{t^3}. \end{aligned} \tag{39}$$

Repeating this step, we get

$$-x''(t) \geq \frac{3!\gamma\delta^{\tilde{\gamma}_n}}{(3-\tilde{\gamma}_n)(2-\tilde{\gamma}_n)} \frac{x(t)}{t^2},$$

and using this in (33), we have

$$w'(t) := -tx''(t) \geq \frac{3!\gamma\delta^{\tilde{\gamma}_n}}{(3-\tilde{\gamma}_n)(2-\tilde{\gamma}_n)} \frac{x(t)}{t}.$$

Integrating from t_n to t and using (I)_n and (III)_n, we obtain

$$\begin{aligned} w(t) &\geq w(t_n) + \frac{3!\tilde{\gamma}_0\delta^{\tilde{\gamma}_n}}{(3-\tilde{\gamma}_n)(2-\tilde{\gamma}_n)} \int_{t_n}^t \frac{x(s)}{s^{1-\tilde{\gamma}_n}s^{\tilde{\gamma}_n}} ds \\ &\geq w(t_n) + \frac{3!\tilde{\gamma}_0\delta^{\tilde{\gamma}_n}}{(3-\tilde{\gamma}_n)(2-\tilde{\gamma}_n)(1-\tilde{\gamma}_n)} \frac{x(t)}{t^{1-\tilde{\gamma}_n}} (t^{1-\tilde{\gamma}_n} - t_n^{1-\tilde{\gamma}_n}) \\ &> \tilde{\gamma}_{n+1}x(t), \quad t \geq t'_n, \end{aligned} \tag{40}$$

where $t'_n > t_n$ is sufficiently large such that $w(t_n) - t_n^{1-\tilde{\gamma}_n}x(t_n)/t_n^{1-\tilde{\gamma}_n} > 0$ for $t \geq t'_n$. By the definition of w we see that

$$(1 - \tilde{\gamma}_{n+1})x(t) > tx'(t)$$

and

$$\left(\frac{x(t)}{t^{1-\tilde{\gamma}_{n+1}}}\right)' = \frac{x'(t)t - (1 - \tilde{\gamma}_{n+1})x(t)}{t^{2-\tilde{\gamma}_{n+1}}} < 0, \quad t \geq t'_n, \tag{41}$$

which proves (I)_{n+1}. Since the proofs of the other parts are similar to those in the case $n = 0$, we omit the details. □

Lemma 10 *Assume that $\delta_* < \infty$ and*

$$\liminf_{t \rightarrow \infty} \tau(t)t^3 p(t) > M, \tag{42}$$

where M is defined by (28). Then $\mathcal{S}_1 = \emptyset$.

Proof The proof is similar to that of Lemma 5 and hence is omitted. □

5 Main result and discussion

Combining the results from the previous two sections, we now present the main result in this paper.

Theorem 1 *Assume that*

$$\liminf_{t \rightarrow \infty} \tau^3(t)tp(t) > \begin{cases} 0 & \text{for } \delta_* = \infty, \\ M & \text{for } \delta_* < \infty, \end{cases} \tag{43}$$

where

$$M := \max\{c(1 - c)(2 - c)(3 - c)\delta_*^{-c} : 0 < c < 1\}.$$

Then equation (1) is oscillatory.

Proof Notice that condition (43) implies $\beta_* > 0$, and since

$$\liminf_{t \rightarrow \infty} \tau^3(t)tp(t) \leq \liminf_{t \rightarrow \infty} \tau(t)t^3p(t),$$

we see that $\gamma_* > 0$. Now if $\delta_* = \infty$, then Lemmas 3 and 8 imply that $S_1 = S_3 = \emptyset$. For $\delta_* < \infty$, the same conclusion follows from Lemmas 5 and 10. This proves the theorem. \square

Corollary 1 *Let $\tau(t) = \alpha t$ with $0 < \alpha \leq 1$. If*

$$\liminf_{t \rightarrow \infty} t^4p(t) > \max\{c(1 - c)(2 - c)(3 - c)\alpha^{c-3} : 0 < c < 1\}, \tag{44}$$

then (1) is oscillatory.

Remark 2 As an important related result, we recall a particular case of integral-type criterion due to Koplatadze [9, Corollary 6.4] obtained by a different technique for even-order differential equations with deviating arguments. He proved that (1) is oscillatory if $\tau(t) \geq \alpha t, 0 < \alpha \leq 1$, and

$$\liminf_{t \rightarrow \infty} t \int_t^\infty s^2p(s) ds > \max\{c(1 - c)(2 - c)(3 - c)\alpha^{c-3} : 0 < c < 1\}. \tag{45}$$

Clearly, if $\tau(t) = \alpha t$, then (44) and (45) are qualitatively the same for (1). Hence, for $n = 4$, Theorem 1 can be regarded as a generalization of [9, Corollary 6.4], removing the restrictive condition $\tau(t) \geq \alpha t$. For similar related comparison results, we refer the reader to [15].

We demonstrate the sharpness of the newly obtained oscillation criterion on Euler-type differential equations.

Example 1 Consider the fourth-order Euler delay differential equation

$$x^{(4)}(t) + \frac{p_0\delta_*^3}{t^4}x\left(\frac{1}{\delta_*}t\right) = 0, \quad p_0 > 0, \delta_* \geq 1, t > 1. \tag{46}$$

The associated characteristic equation is

$$f(\alpha) := \alpha(1 - \alpha)(2 - \alpha)(3 - \alpha)\delta_*^{\alpha-3} = p_0, \tag{47}$$

which is obtained by setting

$$x(t) = t^\alpha. \tag{48}$$

Denoting the local maxima of $f(\alpha)$ by

$$m_1(\delta_*) = \max\{f(\alpha) : 0 < \alpha < 1\},$$

$$m_3(\delta_*) = \max\{f(\alpha) : 2 < \alpha < 3\} = \max\{f(3 - c) : 0 < c < 1\} = M,$$

and noting that for any δ_* ,

$$m_1(\delta_*) \leq m_3(\delta_*) = M,$$

it is easy to verify that (46) has the nonoscillatory solution (48) if

$$p_0 \leq M.$$

By Theorem 1, equation (46) is oscillatory if

$$p_0 > M, \tag{49}$$

which shows that our oscillation constant cannot be improved. As a result, we see that this paper provides an optimal method for the study of oscillatory properties of fourth-order delay differential equations.

In the particular case $\delta_* = 2$, we see that equation (46) is oscillatory if

$$p_0 > m_3(2) \simeq 0.78483. \tag{50}$$

Remark 3 For completeness, we review a few known oscillation constants for equation (46) with $\delta_* = 2$ given in the previous works.

1. Grace and Lalli [16, 17]:

$$p_0 > 1728;$$

2. Zafer [18]:

$$p_0 > \frac{192}{e \ln 2} \simeq 101.902;$$

3. Grace [19]:

$$p_0 > \frac{6}{1 + \ln 2} \simeq 3.545;$$

4. Karpuz et al. [20], Zhang and Yan [21]:

$$p_0 > \frac{6}{e \ln 2} \simeq 3.184;$$

5. Koplatadze [22]:

$$p_0 > \frac{3!}{3 + \ln 2} \simeq 1.625;$$

6. Baculíková and Džurina [23]:

$$p_0 > 1.$$

It is worth noting that cases (2)–(5) above require $\tau(t) < t$ and $\tau'(t) \geq 0$, which is not needed in Theorem 1. Hence Theorem 1 significantly improves many existing results in the literature, even without the usual restrictive assumptions on the deviating argument.

Remark 4 The results presented in this paper open many fruitful problems for further research, and we state at least the most obvious ones.

The first problem consists in extending the sharp results from this paper to the more general fourth-order equation

$$(r_3(t)(r_2(t)(r_1(t)x'(t)))')' + p(t)x(\tau(t)) = 0 \tag{51}$$

with $r_i \in \mathcal{C}([t_0, \infty), (0, \infty))$ in the so-called canonical form, i.e., where

$$\int_{t_0}^{\infty} \frac{1}{r_i(s)} ds = \infty, \quad i = 1, 2, 3. \tag{52}$$

For a similar extension in the case of third-order equations, we refer the reader to the recent paper [24].

It is also an open question how to obtain a sharp single-condition oscillation criterion for (51) in the noncanonical case where

$$\int_{t_0}^{\infty} \frac{1}{r_i(s)} ds < \infty, \quad i = 1, 2, 3. \tag{53}$$

Note that the nonexistence of eight possible classes of nonoscillatory solutions must be shown for (51) to be oscillatory (see [2, 25] for more detail).

It would be also interesting to establish the corresponding results for equation (51) with an advanced argument $\tau(t) \geq t$ in either the canonical or noncanonical case. We refer the reader to [26] for a similar sharp oscillation criteria for second-order advanced differential equations. It is also worth mentioning that the method of iteratively improved monotonicities of nonoscillatory solutions has not as yet been applied to equations with damping. In view of the increasing interest in the study of third- and fourth-order difference equations with deviating arguments, another possible direction for future research is to extend the approach used in this paper to the discrete case, similar to what was done for second-order difference equations with deviating arguments in [27, 28].

Finally, another possibility is developing a unified approach by investigating the oscillatory and asymptotic properties of solutions of fourth-order dynamic equations with deviating arguments on time scales via the method of iteratively improved monotonicities.

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Author contribution

IJ made the major analysis and the original draft preparation. JD, JG, and SG analyzed all the results, gave many comments, and made necessary improvements. All authors read and approved the final manuscript.

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