Bi and Chen *Journal of Inequalities and Applications* (2022) 2022:106
https://doi.org/10.1186/s13660-022-02843-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Optimal error estimates of the local discontinuous Galerkin methods based on generalized fluxes for 1D linear fifth order partial differential equations

Hui Bi[1*] and Yixin Chen[1]

*Correspondence:
bi_hui2002@aliyun.com
[1]Department of Mathematics, Harbin University of Science and Technology, Xuefu Road, 150080 Harbin, China

**Abstract**

In this paper, we study the error estimates of local discontinuous Galerkin methods based on the generalized numerical fluxes for the one-dimensional linear fifth order partial differential equations. We use a newly developed global Gauss–Radau projection to obtain the linear type of optimal error estimates. The numerical experiments show that the scheme coupled with the third order implicit Runge–Kutta method can achieve the optimal $(k + 1)$th order of accuracy.

**Keywords:** Local discontinuous Galerkin methods; Fifth order partial differential equations; Global Gauss–Radau projection; Error estimates

## 1 Introduction

The discontinuous Galerkin (DG) method was first proposed by Reed and Hill [1] to solve neutron problems in 1973. Then, motivated by the success of Bassi and Rebay [2] for the compressible Navier–Stokes equations, Cockburn and Shu [3] designed the local discontinuous Galerkin (LDG) method for solving nonlinear equations with higher order spatial derivatives. The main idea of the LDG method is to transform higher order partial differential equations into an equivalent first order system by introducing auxiliary variables such that the Runge–Kutta discontinuous Galerkin (RKDG) method [4] can be used, so the LDG method shares many advantages of the DG schemes.

In the past twenty years, the LDG method has been widely studied in various frameworks. In [5], the LDG scheme with alternating numerical flux was applied to the linear convection-diffusion problem, and the optimal $(k+1)$th order of accuracy was obtained by virtue of local Gauss–Radau projections. In [6] and [7], Cockburn et al. studied the minimal dissipation local discontinuous Galerkin method and proved that the hp-version estimates of convection-diffusion equations can reach the optimal convergence order. Meanwhile, Cockburn et al. also analyzed the LDG method for Stokes system in [8] and then proposed a new LDG technique for incompressible stationary Navier–Stokes equations.

For partial differential equations with higher order spatial derivatives, note that the spatial discrete operator of the LDG method is usually rigid. If explicit time discretization is used, then smaller time steps must be required to ensure the stability of the scheme, which will cost more computing time. Zhang and Shu [9] improved the low precision problems of second order time discretization, proved the $L^2$-norm stability for scalar linear conservation laws, and obtained a priori error estimates of the third order TVD Runge–Kutta LDG method. Wang and Shu [10] presented the optimal error estimates for solving one-dimensional linear convection-diffusion equations in both space and time for the third order implicit-explicit Runge–Kutta time-marching coupled with LDG spatial discretization. In 2016, Wang and Shu [11, 12] further extended this result to the one-dimensional and two-dimensional nonlinear convection-diffusion equations. For more applications on implicit and explicit time-discrete method, one can refer to [13].

It is worth pointing out that when the LDG method is used, it is very important to design an appropriate numerical flux to ensure the stability of the scheme [14]. The error estimates can only reach $k + \frac{1}{2}$ order if we choose central fluxes for odd order polynomials [2]. But the numerical experiments show that the error estimates can easily reach $k + 1$ order when the alternating fluxes are used. In recent years, Meng and Shu [15] put forward more comprehensive theories of the upwind-biased fluxes and proved the optimal $k + 1$ order error estimates of the semi-discrete and fully discrete scheme for linear hyperbolic conservation equations. Li and Meng [16] analyzed the discontinuous Galerkin method based on upwind-biased numerical fluxes for one-dimensional linear hyperbolic equations with degenerate variable coefficients. For higher order partial differential equations, Meng extended the upwind-biased fluxes to the generalized alternating fluxes in [17]. Moreover, Meng, Liu, and Zhang investigated that local discontinuous Galerkin methods with generalized alternating fluxes for one-dimensional linear convection-diffusion equations can superconverge to order $2k + 1$ in [18].

The main content of this paper is as follows. In Sect. 2, we introduce the semi-discrete scheme based on the generalized numerical fluxes for the fifth order partial differential equation and obtain optimal error estimates by constructing energy equations. In Sect. 3, the theoretical results are confirmed by numerical experiments, where the strong stability preserving high order time discretization method [19] is used.

## 2 Error estimates of the LDG method

### 2.1 LDG scheme

In this paper, we consider the one-dimensional linear fifth order equation

$$u_t + u_{xxxxx} = 0, \quad (x,t) \in [0, 2\pi] \times [0, T], \tag{2.1a}$$

$$u(x,0) = u_0(x), \tag{2.1b}$$

where $u_0(x)$ is a smooth function. For convenience, we take the periodic boundary condition $u(0,t) = u(2\pi, t)$ into discussion.

#### 2.1.1 The meshes

Let us denote the computational interval $I = [0, 2\pi]$, consisting of cells $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ with $1 \le j \le N$, where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

Then we define $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$, $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, and $h = \max h_j$. Use $p^-_{j+\frac{1}{2}}$ and $p^+_{j+\frac{1}{2}}$ to denote the left and right limits of $p$ at the discontinuity point $x_{j+\frac{1}{2}}$. In what follows, we employ $[p] = p^+ - p^-$ and $\{p\} = \frac{(p^+ + p^-)}{2}$ to represent the jump and the mean value of $p$ at each element boundary point. The following piecewise polynomial space is chosen as the finite element space:

$$V_h \equiv V_h^k = \left\{ v \in L^2(I) : v|_{I_j} \in P^k(I_j), j = 1, \dots, N \right\},$$

where $P^k(I_j)$ denotes the set of polynomials of degree up to $k \geq 0$ defined on cell $I_j$.

*2.1.2 Function spaces and norms*

For any integer $l \geq 0$, the norms of the broken Sobolev spaces $W^{l,p}(I_h) = \{u \in L^2(I) : u|_{I_j} \in W^{l,p}(I_j), j = 1, \dots, N\}$ with $p = 2, \infty$ are given by

$$\|u\|_{W^{l,2}(I_h)} = \|u\|_{H^l(I_h)} = \left( \sum_{j=1}^{N} \|u\|^2_{H^l(I_j)} \right)^{\frac{1}{2}},$$

$$\|u\|_{W^{l,\infty}(I_h)} = \max_{1 \leq j \leq N} \|u\|_{W^{l,\infty}(I_j)}.$$

In the case of $l = 0$, we denote $\|u\|_{L^2(I)} = \|u\|$.

*2.1.3 The semi-discrete LDG scheme*

Next we introduce the semi-discrete LDG method of Eqs. (2.1a) and (2.1b). First, we use some variables

$$q = u_x, \qquad p = q_x, \qquad r = p_x, \qquad s = r_x$$

to transform Eq. (2.1a) into a first order linear system

$$u_t + s_x = 0, \tag{2.2a}$$

$$s = r_x, \tag{2.2b}$$

$$r = p_x, \tag{2.2c}$$

$$p = q_x, \tag{2.2d}$$

$$q = u_x. \tag{2.2e}$$

The LDG scheme is defined as follows: find $u_h, q_h, p_h, r_h, s_h \in V_h^k$ such that $\forall \rho, \xi, \phi, \psi, \varphi \in V_h^k$, there holds

$$\int_{I_j} (u_h)_t \rho \, dx + \hat{s}_h \rho^- |_{j+\frac{1}{2}} - \hat{s}_h \rho^+ |_{j-\frac{1}{2}} - \int_{I_j} s_h \rho_x \, dx = 0, \tag{2.3a}$$

$$\int_{I_j} s_h \xi \, dx - \hat{r}_h \xi^- |_{j+\frac{1}{2}} + \hat{r}_h \xi^+ |_{j-\frac{1}{2}} + \int_{I_j} r_h \xi_x \, dx = 0, \tag{2.3b}$$

$$\int_{I_j} r_h \phi \, dx - \hat{p}_h \phi^- |_{j+\frac{1}{2}} + \hat{p}_h \phi^+ |_{j-\frac{1}{2}} + \int_{I_j} p_h \phi_x \, dx = 0, \tag{2.3c}$$

$$\int_{I_j} p_h\psi\,dx - \hat{q}_h\psi^-|_{j+\frac{1}{2}} + \hat{q}_h\psi^+|_{j-\frac{1}{2}} + \int_{I_j} q_h\psi_x\,dx = 0, \tag{2.3d}$$

$$\int_{I_j} q_h\varphi\,dx - \hat{u}_h\varphi^-|_{j+\frac{1}{2}} + \hat{u}_h\varphi^+|_{j-\frac{1}{2}} + \int_{I_j} u_h\varphi_x\,dx = 0, \tag{2.3e}$$

where $\hat{s_h}$, $\hat{r_h}$, $\hat{p_h}$, $\hat{q_h}$, $\hat{u_h}$ are numerical fluxes. Here we use the generalized numerical fluxes related to parameter $\theta$. We write $\tilde{\theta} = 1 - \theta$ and choose numerical fluxes at $x_{j+\frac{1}{2}}, j = 0, \ldots, N$ as follows:

$$\begin{aligned}
\hat{u_h} &= u_h^\theta = \theta u_h^- + (1-\theta)u_h^+,\\
\hat{q_h} &= q_h^{\tilde{\theta}} = \theta q_h^+ + (1-\theta)q_h^-,\\
\hat{p_h} &= p_h^\theta = \theta p_h^- + (1-\theta)p_h^+,\\
\hat{r_h} &= r_h^\theta = \theta r_h^- + (1-\theta)r_h^+,\\
\hat{s_h} &= s_h^{\tilde{\theta}} = \theta s_h^+ + (1-\theta)s_h^-,
\end{aligned}$$

where $\theta > \frac{1}{2}$. For the initial condition, we take $u_h(0) = P_h u_0$. It holds that

$$\|u_0 - P_h u_0\|_{L^2(I)} \le Ch^{k+1}\|u_0\|_{W^{k+1,\infty}(I)}, \tag{2.4}$$

where $P_h$ is the $L^2$ projection into $V_h^k$.

We define $\langle z, p\rangle = \int_I z \cdot p dx$. For simplicity, we would like to introduce the DG discrete operator $H(z, p, \hat{z})$. That is,

$$H(z,p,\hat{z}) = \sum_{j=1}^{N} H_j(z,p,\hat{z}),$$

where for each cell $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$,

$$H_j(z,p,\hat{z}) = \int_{I_j} \hat{z}p_x\,dx - \hat{z}p^-|_{j+\frac{1}{2}} + \hat{z}p^+|_{j-\frac{1}{2}}. \tag{2.5}$$

By simple calculations, we obtain the following lemma for a DG discrete operator.

**Lemma 1**

$$H(z,p,z^\theta) + H(p,z,p^{\tilde{\theta}}) = 0, \tag{2.6a}$$

$$H(z,p,z^\theta) + H(p,z,p^\theta) = (1-2\theta)\sum_{j=1}^{N}\big([z]\cdot[p]\big)_{j+\frac{1}{2}}, \tag{2.6b}$$

$$H(z,p,z^{\tilde{\theta}}) + H(p,z,p^{\tilde{\theta}}) = (2\theta-1)\sum_{j=1}^{N}\big([z]\cdot[p]\big)_{j+\frac{1}{2}}, \tag{2.6c}$$

$$H(z,z,z^\theta) = \left(\frac{1}{2}-\theta\right)\sum_{j=1}^{N}\big([z]^2\big)_{j+\frac{1}{2}}, \tag{2.6d}$$

$$H\left(z,z,z^{\tilde{\theta}}\right)=\left(\theta-\frac{1}{2}\right)\sum_{j=1}^{N}\left([z]^2\right)_{j+\frac{1}{2}}. \tag{2.6e}$$

### 2.1.4 The numerical initial condition

In this subsection, to derive optimal error estimates for the fifth order equation, we need to obtain the optimal initial error estimates for all variables first [20]. We consider the corresponding steady-state problem

$$u+u_{xxxxx}=g(x)$$

satisfying periodic conditions and a source term $g(x)=u_0(x)+u_0(x)^{(5)}$ so that its exact solution is identically the initial condition of (2.1a)–(2.1b), $u_0(x)$. That is, find $u_h,s_h,r_h,p_h,q_h\in V_h^k$ such that

$$\int_{I_j}u_h\rho\,dx-H_j\left(s_h,\rho,s_h^{\tilde{\theta}}\right)=\int_{I_j}g\rho\,dx, \tag{2.7a}$$

$$\int_{I_j}s_h\xi\,dx+H_j\left(r_h,\xi,r_h^{\theta}\right)=0, \tag{2.7b}$$

$$\int_{I_j}r_h\phi\,dx+H_j\left(p_h,\phi,p_h^{\theta}\right)=0, \tag{2.7c}$$

$$\int_{I_j}p_h\psi\,dx+H_j\left(q_h,\psi,q_h^{\tilde{\theta}}\right)=0, \tag{2.7d}$$

$$\int_{I_j}q_h\varphi\,dx+H_j\left(u_h,\varphi,u_h^{\theta}\right)=0, \tag{2.7e}$$

hold for any $\rho,\xi,\phi,\psi,\varphi\in V_h^k$ and $j=1,\ldots,N$.

**Lemma 2** *The numerical initial condition (2.7a)–(2.7e) is well defined. That is, LDG solutions $u_h$, $s_h$, $r_h$, $p_h$, $q_h$ of (2.7a)–(2.7e) uniquely exist.*

*Proof* Since (2.7a) is a linear system with a known right-hand side and $s_h$, $r_h$, $p_h$, $q_h$ can be expressed by $u_h$, we can first prove the uniqueness of $(u_h,s_h,r_h,p_h,q_h)$, then, obviously, the existence will follow. Assuming that $(u_h^1,s_h^1,r_h^1,p_h^1,q_h^1)$ and $(u_h^2,s_h^2,r_h^2,p_h^2,q_h^2)$ are two different solutions of (2.7a)–(2.7e) and denoting $g_u=u_h^1-u_h^2$, $g_s=s_h^1-s_h^2$, $g_r=r_h^1-r_h^2$, $g_p=p_h^1-p_h^2$, $g_q=q_h^1-q_h^2$, we have

$$\int_{I_j}g_u\rho\,dx-H_j\left(g_s,\rho,g_s^{\tilde{\theta}}\right)=0, \tag{2.8a}$$

$$\int_{I_j}g_s\xi\,dx+H_j\left(g_r,\xi,g_r^{\theta}\right)=0, \tag{2.8b}$$

$$\int_{I_j}g_r\phi\,dx+H_j\left(g_p,\phi,g_p^{\theta}\right)=0, \tag{2.8c}$$

$$\int_{I_j}g_p\psi\,dx+H_j\left(g_q,\psi,g_q^{\tilde{\theta}}\right)=0, \tag{2.8d}$$

$$\int_{I_j}g_q\varphi\,dx+H_j\left(g_u,\varphi,g_u^{\theta}\right)=0. \tag{2.8e}$$

We take $(\rho, \xi, \phi, \psi, \varphi) = (g_u, g_q, -g_p, g_r, -g_s)$ in (2.8a)–(2.8e). By Lemma 1 and direct calculation, we have

$$\|g_u\|^2 - \left(\frac{1}{2} - \theta\right) \sum_{j=1}^{N} \left([z]^2\right)_{j+\frac{1}{2}} = 0. \tag{2.9}$$

Thus $g_u = 0$ since $\theta > \frac{1}{2}$. Then, substituting $g_u = 0$ into (2.8e) and letting $\varphi = g_q$, we have $g_q = 0$. Similarly, we have $g_p = g_r = g_s = 0$, which implies that $u_h$, $s_h$, $r_h$, $p_h$, $q_h$ are unique. This ends the proof of Lemma 2. □

*2.1.5 The global Gauss–Radau projections*

For the LDG scheme, using the generalized numerical fluxes, we need to construct a globally defined projection $P_h^*$. For $u \in H^1(I)$, the projection $P_h^* u$ is defined as the element of $V_h^k$ that satisfies

$$\int_{I_j} P_h^* u(x) v_h \, dx = \int_{I_j} u(x) v_h \, dx, \quad \forall v_h \in P^{k-1}(I_j), \tag{2.10a}$$

$$\widehat{P_h^* u} = \hat{u}, \quad at x_{j+\frac{1}{2}}, j = 1, \ldots, N, \tag{2.10b}$$

with $\theta > \frac{1}{2}$.

**Lemma 3** *Assume that $u$ is sufficiently smooth and periodic. Then there exists unique $P_h^*$ satisfying conditions* (2.10a) *and* (2.10b). *Moreover, there holds the following property*:

$$\left\| u - P_h^* u \right\|_{L^2(I)} \le C h^{k+1} \|u\|_{W^{k+1,\infty}(I)}. \tag{2.11}$$

Using the initial condition and the triangle inequality, we have

$$\left\| \left(P_h^* u - u_h\right)(\cdot, 0) \right\|_{L^2(I)} \le C h^{k+1} \|u\|_{W^{k+1,\infty}(I)}, \tag{2.12}$$

where $C = C(\theta)$ is independent of the mesh size $h$. The lemma has been proved in [15].

**Lemma 4** *Assuming that $u_0 \in W^{k+1,\infty}(I_h)$ and periodic, we have the following error estimates*:

$$\|u_0 - u_h\| + \|s_0 - s_h\| + \|r_0 - r_h\| + \|p_0 - p_h\| + \|q_0 - q_h\| \le C h^{k+1}, \tag{2.13}$$

*where $q_0 = u_0'$, $p_0 = u_0''$, $r_0 = u_0^{(3)}$, $s_0 = u_0^{(4)}$, and $C$ is independent of $h$.*

*Proof* In this part, we denote

$$\begin{aligned}
e_u &= u_0 - u_h = \left(u_0 - P_h^* u\right) + \left(P_h^* u - u_h\right) = \eta_u + \bar{e_u}, \\
e_q &= q_0 - q_h = \left(q_0 - P_h^* q\right) + \left(P_h^* q - q_h\right) = \eta_q + \bar{e_q}, \\
e_p &= p_0 - p_h = \left(p_0 - P_h^* p\right) + \left(P_h^* p - p_h\right) = \eta_p + \bar{e_p}, \\
e_r &= r_0 - r_h = \left(r_0 - P_h^* r\right) + \left(P_h^* r - r_h\right) = \eta_r + \bar{e_r},
\end{aligned}$$

$$e_s = s_0 - s_h = \left(s_0 - P_h^* s\right) + \left(P_h^* s - s_h\right) = \eta_s + \bar{e}_s.$$

Considering scheme (2.8a)–(2.8e) and summing over all $j$, we have the following equations:

$$\int_I e_u \rho \, dx - H\left(e_s, \rho, e_s^{\tilde{\theta}}\right) = 0,$$

$$\int_I e_s \xi \, dx + H\left(e_r, \xi, e_r^{\theta}\right) = 0,$$

$$\int_I e_r \phi \, dx + H\left(e_p, \phi, e_p^{\theta}\right) = 0,$$

$$\int_I e_p \psi \, dx + H\left(e_q, \psi, e_q^{\tilde{\theta}}\right) = 0,$$

$$\int_I e_q \varphi \, dx + H\left(e_u, \varphi, e_u^{\theta}\right) = 0.$$

By the orthogonality, we have

$$\int_I \bar{e}_u \rho \, dx - H\left(\bar{e}_s, \rho, \bar{e}_s^{\tilde{\theta}}\right) = -\int_I \eta_u \rho \, dx, \tag{2.14a}$$

$$\int_I \bar{e}_s \xi \, dx + H\left(\bar{e}_r, \xi, \bar{e}_r^{\theta}\right) = -\int_I \eta_s \xi \, dx, \tag{2.14b}$$

$$\int_I \bar{e}_r \phi \, dx + H\left(\bar{e}_p, \phi, \bar{e}_p^{\theta}\right) = -\int_I \eta_r \phi \, dx, \tag{2.14c}$$

$$\int_I \bar{e}_p \psi \, dx + H\left(\bar{e}_q, \psi, \bar{e}_q^{\tilde{\theta}}\right) = -\int_I \eta_p \psi \, dx, \tag{2.14d}$$

$$\int_I \bar{e}_q \varphi \, dx + H\left(\bar{e}_u, \varphi, \bar{e}_u^{\theta}\right) = -\int_I \eta_q \varphi \, dx, \tag{2.14e}$$

that hold for any $\rho, \xi, \phi, \psi, \varphi \in V_h^k$. In what follows, we will prove the optimal initial error estimates. Taking $(\rho, \xi) = (-\bar{e}_r, \bar{e}_s)$ in (2.14a)–(2.14b), summing the corresponding equations up, and using Lemma 1, we obtain the following equation:

$$\int_I \bar{e}_s \bar{e}_s \, dx - \int_I \bar{e}_u \bar{e}_r \, dx = -\int_I \eta_s \bar{e}_s \, dx + \int_I \eta_u \bar{e}_r \, dx. \tag{2.15a}$$

Taking $(\psi, \phi) = (\bar{e}_p, \bar{e}_q)$, $(\rho, \varphi) = (\bar{e}_u, -\bar{e}_s)$, $(\varphi, \psi) = (\bar{e}_q, \bar{e}_u)$, $(\phi, \xi) = (\bar{e}_r, \bar{e}_p)$ and $(\xi, \phi) = (-\bar{e}_r, -\bar{e}_p)$, by the same way, we have

$$\int_I \bar{e}_p \bar{e}_p \, dx + \int_I \bar{e}_r \bar{e}_q \, dx = -\int_I \eta_p \bar{e}_p \, dx - \int_I \eta_r \bar{e}_q \, dx, \tag{2.15b}$$

$$\int_I \bar{e}_u \bar{e}_u \, dx - \int_I \bar{e}_q \bar{e}_s \, dx = -\int_I \eta_u \bar{e}_u \, dx + \int_I \eta_q \bar{e}_s \, dx, \tag{2.15c}$$

$$\int_I \bar{e}_q \bar{e}_q \, dx + \int_I \bar{e}_p \bar{e}_u \, dx = -\int_I \eta_q \bar{e}_q \, dx - \int_I \eta_p \bar{e}_u \, dx, \tag{2.15d}$$

$$\begin{aligned} &\int_I \bar{e}_r \bar{e}_r \, dx + H\left(\bar{e}_p, \bar{e}_r, \bar{e}_p^{\theta}\right) + \int_I \bar{e}_s \bar{e}_p \, dx + H\left(\bar{e}_r, \bar{e}_p, \bar{e}_r^{\theta}\right) \\ &\quad = -\int_I \eta_r \bar{e}_r \, dx - \int_I \eta_s \bar{e}_p \, dx, \end{aligned} \tag{2.15e}$$

$$-\int_I \bar{e_s}\bar{e_r}\,dx - H(\bar{e_r},\bar{e_r},\bar{e_r}^{\theta}) - \int_I \bar{e_r}\bar{e_p}\,dx - H(\bar{e_p},\bar{e_p},\bar{e_p}^{\theta})$$
$$= \int_I \eta_s\bar{e_r}\,dx + \int_I \eta_r\bar{e_p}\,dx. \tag{2.15f}$$

Adding (2.15e) and (2.15f), we get

$$\int_I \bar{e_r}\bar{e_r}\,dx + \int_I \bar{e_s}\bar{e_p}\,dx - \int_I \bar{e_s}\bar{e_r}\,dx - \int_I \bar{e_r}\bar{e_p}\,dx + \Omega$$
$$= -\int_I \eta_r\bar{e_r}\,dx - \int_I \eta_s\bar{e_p}\,dx \int_I \eta_s\bar{e_r}\,dx + \int_I \eta_r\bar{e_p}\,dx, \tag{2.16}$$

where

$$\Omega = H(\bar{e_p},\bar{e_r},\bar{e_p}^{\theta}) + H(\bar{e_r},\bar{e_p},\bar{e_r}^{\theta}) - H(\bar{e_r},\bar{e_r},\bar{e_r}^{\theta}) - H(\bar{e_p},\bar{e_p},\bar{e_p}^{\theta}).$$

It is easy to find that $\Omega \geq 0$ by Lemma 1. Then, using Lemmas 1 and 3, the Cauchy–Schwarz and Young's inequalities to (2.15a)–(2.15d) and (2.16), we have

$$\|\bar{e_s}\|^2 \leq C_{u_1}\|\bar{e_u}\|^2 + C_{r_1}\|\bar{e_r}\|^2 + C_{s_1}\|\bar{e_s}\|^2 + Ch^{2k+2}, \tag{2.17a}$$
$$\|\bar{e_p}\|^2 \leq C_{p_2}\|\bar{e_p}\|^2 + C_{r_2}\|\bar{e_r}\|^2 + C_{q_2}\|\bar{e_q}\|^2 + Ch^{2k+2}, \tag{2.17b}$$
$$\|\bar{e_q}\|^2 \leq C_{u_3}\|\bar{e_u}\|^2 + C_{q_3}\|\bar{e_q}\|^2 + C_{p_3}\|\bar{e_p}\|^2 + Ch^{2k+2}, \tag{2.17c}$$
$$\|\bar{e_u}\|^2 \leq C_{u_4}\|\bar{e_u}\|^2 + C_{q_4}\|\bar{e_4}\|^2 + C_{s_4}\|\bar{e_s}\|^2 + Ch^{2k+2}, \tag{2.17d}$$
$$\|\bar{e_r}\|^2 \leq C_{p_5}\|\bar{e_p}\|^2 + C_{r_5}\|\bar{e_r}\|^2 + C_{s_5}\|\bar{e_s}\|^2 + Ch^{2k+2}. \tag{2.17e}$$

Furthermore, by adjusting the coefficients in (2.17a)–(2.17e) with Young's inequality, we have the estimate

$$\|s_0 - s_h\|^2 + \|r_0 - r_h\|^2 + \|p_0 - p_h\|^2 + \|q_0 - q_h\|^2 \leq \|u_0 - u_h\|^2 + Ch^{2k+2}.$$

Hence we arrive at

$$\|s_0 - s_h\| + \|r_0 - r_h\| + \|p_0 - p_h\| + \|q_0 - q_h\| \leq \|u_0 - u_h\| + Ch^{k+1}. \tag{2.17f}$$

We take $(\rho,\xi,\phi,\psi,\varphi) = (\bar{e_u},\bar{e_q},-\bar{e_p},\bar{e_r},-\bar{e_s})$ in (2.14a)–(2.14e). Through direct calculation, from Lemma 1 we have

$$\|\bar{e_u}\|^2 - \left(\frac{1}{2}-\theta\right)\sum_{j=1}^{N}\left([\bar{e_p}]^2\right)_{j+\frac{1}{2}}$$
$$= \int_I \eta_p\bar{e_p}\,dx - \int_I \eta_u\bar{e_u}\,dx - \int_I \eta_q\bar{e_q}\,dx + \int_I \eta_s\bar{e_s}\,dx - \int_I \eta_r\bar{e_r}\,dx. \tag{2.17g}$$

Substituting (2.17f) into (2.17g) and using Lemma 3, we finally get $\|\bar{e_u}\| \leq Ch^{k+1}$. This completes the proof of Lemma 4. □

### 2.2 Error estimates

In this subsection, we state the error estimate of the LDG method using the generalized numerical fluxes. First, we define

$$\begin{aligned}
e_u &= u - u_h = \left(u - P_h^* u\right) + \left(P_h^* u - u_h\right) = \eta_u + \bar{e_u},\\
e_q &= q - q_h = \left(q - P_h^* q\right) + \left(P_h^* q - q_h\right) = \eta_q + \bar{e_q},\\
e_p &= p - p_h = \left(p - P_h^* p\right) + \left(P_h^* p - p_h\right) = \eta_p + \bar{e_p},\\
e_r &= r - r_h = \left(r - P_h^* r\right) + \left(P_h^* r - r_h\right) = \eta_r + \bar{e_r},\\
e_s &= s - s_h = \left(s - P_h^* s\right) + \left(P_h^* s - s_h\right) = \eta_s + \bar{e_s}.
\end{aligned}$$

Then we have the following theorem.

**Theorem 1** *Assume that $u, q, p, r, s$ are the exact solutions of system (2.2a)–(2.2e), and for $t \in [0,T]$, $\|u\|_{k+5}$, $\|u_t\|_{k+5}$, $\|u_{tt}\|_{k+5}$ are bounded uniformly. We take the generalized numerical fluxes and the finite element space $V_h^k$, there holds the following $L^2$-norm error estimates*:

$$\|e_u\| + \|e_q\| + \|e_p\| + \|e_r\| + \|e_s\| + \left\|(e_u)_t\right\| \le Ch^{k+1}(t+3), \tag{2.18}$$

*where $C$ depends on $\theta$, $\|u\|_{k+5}$, $\|u_t\|_{k+5}$, and $\|u_{tt}\|_{k+5}$, but is independent of $h$.*

*Proof* Using the DG discrete operator, the LDG scheme can be written as

$$\int_{I_j} (u_h)_t \rho \, dx - H_j\left(s_h, \rho, s_h^{\tilde{\theta}}\right) = 0, \tag{2.19a}$$

$$\int_{I_j} s_h \xi \, dx + H_j\left(r_h, \xi, r_h^{\theta}\right) = 0, \tag{2.19b}$$

$$\int_{I_j} r_h \phi \, dx + H_j\left(p_h, \phi, p_h^{\theta}\right) = 0, \tag{2.19c}$$

$$\int_{I_j} p_h \psi \, dx + H_j\left(q_h, \psi, q_h^{\tilde{\theta}}\right) = 0, \tag{2.19d}$$

$$\int_{I_j} q_h \varphi \, dx + H_j\left(u_h, \varphi, u_h^{\theta}\right) = 0. \tag{2.19e}$$

Summing (2.19a)–(2.19e) over all $j = 1, \dots, N$, we obtain

$$\int_{I} (u_h)_t \rho \, dx - H\left(s_h, \rho, s_h^{\tilde{\theta}}\right) = 0, \tag{2.20a}$$

$$\int_{I} s_h \xi \, dx + H\left(r_h, \xi, r_h^{\theta}\right) = 0, \tag{2.20b}$$

$$\int_{I} r_h \phi \, dx + H\left(p_h, \phi, p_h^{\theta}\right) = 0, \tag{2.20c}$$

$$\int_{I} p_h \psi \, dx + H\left(q_h, \psi, q_h^{\tilde{\theta}}\right) = 0, \tag{2.20d}$$

$$\int_I q_h\varphi\,dx + H\big(u_h,\varphi,u_h^{\theta}\big) = 0. \tag{2.20e}$$

Thus, we get the following error equations:

$$\int_I (e_u)_t\rho\,dx - H\big(e_s,\rho,e_s^{\tilde{\theta}}\big) = 0, \tag{2.21a}$$

$$\int_I e_s\xi\,dx + H\big(e_r,\xi,e_r^{\theta}\big) = 0, \tag{2.21b}$$

$$\int_I e_r\phi\,dx + H\big(e_p,\phi,e_p^{\theta}\big) = 0, \tag{2.21c}$$

$$\int_I e_p\psi\,dx + H\big(e_q,\psi,e_q^{\tilde{\theta}}\big) = 0, \tag{2.21d}$$

$$\int_I e_q\varphi\,dx + H\big(e_u,\varphi,e_u^{\theta}\big) = 0. \tag{2.21e}$$

To prove the theorem, we need to establish six equations by repeatedly taking different $\rho$, $\xi$, $\phi$, $\psi$, $\varphi$ in Eqs. (2.21a)–(2.21e). The specific method is as follows.

**The first equation:**

Taking $(\rho,\xi,\phi,\psi,\varphi) = (\bar{e_u},\bar{e_q},-\bar{e_p},\bar{e_r},-\bar{e_s})$ in Eqs. (2.21a)–(2.21e) and summing over all equations, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\bar{e_u}\|^2 + \big\langle(\eta_u)_t,\bar{e_u}\big\rangle + \langle e_s,\bar{e_q}\rangle - \langle e_r,\bar{e_p}\rangle \\
&\quad + \langle e_p,\bar{e_r}\rangle - \langle e_q,\bar{e_s}\rangle - H\big(\bar{e_s},\bar{e_u},\bar{e_s}^{\tilde{\theta}}\big) - H\big(\bar{e_u},\bar{e_s},\bar{e_u}^{\theta}\big) \\
&\quad + H\big(\bar{e_r},\bar{e_q},\bar{e_r}^{\theta}\big) + H\big(\bar{e_q},\bar{e_r},\bar{e_q}^{\tilde{\theta}}\big) - H\big(\bar{e_p},\bar{e_p},\bar{e_p}^{\theta}\big) = 0.
\end{aligned} \tag{2.22}$$

Then, by Lemma 1, Eq. (2.22) is finally turned into

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\bar{e_u}\|^2 + \left(\theta - \frac{1}{2}\right)\sum_{j=1}^{N}[\bar{e_p}]^2 \\
&\quad = -\big\langle(\eta_u)_t,\bar{e_u}\big\rangle - \langle\eta_s,\bar{e_q}\rangle + \langle\eta_r,\bar{e_p}\rangle - \langle\eta_p,\bar{e_r}\rangle + \langle\eta_q,\bar{e_s}\rangle.
\end{aligned} \tag{2.23}$$

**The second equation:**

Taking the derivatives on both sides of (2.21a)–(2.21e) with respect to $t$ and taking $(\rho,\xi,\phi,\psi,\varphi) = ((\bar{e_u})_t,(\bar{e_q})_t,-(\bar{e_p})_t,(\bar{e_r})_t,-(\bar{e_s})_t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|(\bar{e_u})_t\big\|^2 + \left(\theta - \frac{1}{2}\right)\sum_{j=1}^{N}\big[(\bar{e_p})_t\big]^2 \\
&\quad = -\big\langle(\eta_u)_{tt},(\bar{e_u})_t\big\rangle - \big\langle(\eta_s)_t,(\bar{e_q})_t\big\rangle + \big\langle(\eta_r)_t,(\bar{e_p})_t\big\rangle - \big\langle(\eta_p)_t,(\bar{e_r})_t\big\rangle + \big\langle(\eta_q)_t,(\bar{e_s})_t\big\rangle.
\end{aligned} \tag{2.24}$$

**The third equation:**

Substituting $(\psi,\varphi) = (\bar{e_u},\bar{e_q})$ into (2.21d) and (2.21e), we have

$$\begin{aligned}
&\langle\eta_p,\bar{e_u}\rangle + \langle\bar{e_p},\bar{e_u}\rangle + \langle\eta_q,\bar{e_q}\rangle + \langle\bar{e_q},\bar{e_q}\rangle \\
&\quad + H\big(\bar{e_q},\bar{e_u},e_q^{\tilde{\theta}}\big) + H\big(\bar{e_u},\bar{e_q},e_u^{\theta}\big) = 0.
\end{aligned}$$

Using Lemma 1, we have

$$\|\bar{e_q}\|^2 = -\langle \bar{e_p}, \bar{e_u}\rangle - \langle \eta_q, \bar{e_q}\rangle - \langle \eta_p, \bar{e_u}\rangle. \tag{2.25}$$

**The fourth equation:**

Similar to the third equation, taking $(\phi, \psi) = (\bar{e_q}, \bar{e_p})$ in (2.21c) and (2.21d), we get

$$\|\bar{e_p}\|^2 = -\langle \bar{e_r}, \bar{e_q}\rangle - \langle \eta_p, \bar{e_p}\rangle - \langle \eta_r, \bar{e_q}\rangle. \tag{2.26}$$

**The fifth equation:**

Substituting $(\xi, \phi) = (\bar{e_p}, \bar{e_r})$ into (2.21b) and (2.21c) yields that

$$\langle \bar{e_s}, \bar{e_p}\rangle + \langle \eta_s, \bar{e_p}\rangle + \langle \bar{e_r}, \bar{e_r}\rangle + \langle \eta_r, \bar{e_r}\rangle + H\big(\bar{e_r}, \bar{e_p}, \bar{e_r}^{\theta}\big) + H\big(\bar{e_p}, \bar{e_r}, \bar{e_p}^{\theta}\big) = 0.$$

Using Lemma 1, we obtain

$$\|\bar{e_r}\|^2 = (2\theta - 1)\sum_{j=1}^{N}\big([\bar{e_p}][\bar{e_r}]\big)_{j+\frac{1}{2}} - \langle \bar{e_s}, \bar{e_p}\rangle - \langle \eta_s, \bar{e_p}\rangle - \langle \eta_r, \bar{e_r}\rangle. \tag{2.27}$$

**The sixth equation:**

Finally, substituting $(\rho, \xi) = (\bar{e_r}, -\bar{e_s})$ into (2.21a) and (2.21b), it is easy to see by Lemma 1 that

$$\|\bar{e_s}\|^2 = -\langle \eta_s, \bar{e_s}\rangle + \big\langle (\eta_u)_t, \bar{e_r}\big\rangle + \big\langle (\bar{e_u})_t, \bar{e_r}\big\rangle. \tag{2.28}$$

Now we have six equations. In what follows we need to find some appropriate coefficients, and according to (2.25), (2.26), (2.27), and (2.28) we can infer the relationship among $\|\bar{e_u}\|$, $\|\bar{e_q}\|$, $\|\bar{e_p}\|$, $\|\bar{e_r}\|$, $\|\bar{e_s}\|$, $\|(\bar{e_u})_t\|$.

By multiplying some constants 19 × (2.25) + 3 × (2.26) + (2.27) + (2.28), we obtain

$$\begin{aligned}
&\|\bar{e_s}\|^2 + \|\bar{e_r}\|^2 + 3\|\bar{e_p}\|^2 + 19\|\bar{e_q}\|^2 \\
&\quad = (2\theta - 1)\sum_{j=1}^{N}\big([\bar{e_p}][\bar{e_r}]\big)_{j+\frac{1}{2}} \\
&\qquad - 19\langle \bar{e_p}, \bar{e_u}\rangle - 3\langle \bar{e_r}, \bar{e_q}\rangle - \langle \bar{e_s}, \bar{e_p}\rangle + \big\langle (\bar{e_u})_t, , \bar{e_r}\big\rangle \\
&\qquad - 19\langle \eta_q, \bar{e_q}\rangle - 19\langle \eta_p, \bar{e_u}\rangle - 3\langle \eta_p, \bar{e_p}\rangle - 3\langle \eta_r, \bar{e_q}\rangle \\
&\qquad - \langle \eta_r, \bar{e_r}\rangle - \langle \eta_s, \bar{e_p}\rangle - \langle \eta_s, \bar{e_s}\rangle + \big\langle (\eta_u)_t, \bar{e_r}\big\rangle.
\end{aligned} \tag{2.29}$$

Furthermore, by Young's inequality and Lemma 1, we have

$$\begin{aligned}
(2\theta - 1)\sum_{j=1}^{N}\big([\bar{e_p}][\bar{e_r}]\big)_{j+\frac{1}{2}} &\le \frac{1}{2}\left\{(2\theta - 1)\sum_{j=1}^{N}[\bar{e_p}]^2_{j+\frac{1}{2}} + (2\theta - 1)\sum_{j=1}^{N}[\bar{e_r}]^2_{j+\frac{1}{2}}\right\} \\
&= \langle e_r, \bar{e_p}\rangle + \langle e_s, \bar{e_r}\rangle \\
&= \langle \bar{e_r}, \bar{e_p}\rangle + \langle \bar{e_s}, \bar{e_r}\rangle + \langle \eta_r, \bar{e_p}\rangle + \langle \eta_s, \bar{e_r}\rangle.
\end{aligned} \tag{2.30}$$

Then, substituting (2.30) into (2.29), we obtain

$$\begin{aligned}
&\|\bar{e_s}\|^2 + \|\bar{e_r}\|^2 + 3\|\bar{e_p}\|^2 + 19\|\bar{e_q}\|^2 \\
&\quad \le \left|\langle \bar{e_r}, \bar{e_p}\rangle\right| + \left|\langle \bar{e_s}, \bar{e_r}\rangle\right| + 19\left|\langle \bar{e_p}, \bar{e_u}\rangle\right| + 3\left|\langle \bar{e_r}, \bar{e_q}\rangle\right| + \left|\langle \bar{e_s}, \bar{e_p}\rangle\right| \\
&\qquad + \left|\langle (\bar{e_u})_t, \bar{e_r}\rangle\right| + 19\left|\langle \eta_q, \bar{e_q}\rangle\right| + 19\left|\langle \eta_p, \bar{e_u}\rangle\right| + 3\left|\langle \eta_p, \bar{e_p}\rangle\right| \\
&\qquad + 3\left|\langle \eta_r, \bar{e_q}\rangle\right| + \left|\langle \eta_r, \bar{e_r}\rangle\right| + \left|\langle \eta_s, \bar{e_p}\rangle\right| + \left|\langle \eta_s, \bar{e_s}\rangle\right| \\
&\qquad + \left|\langle (\eta_u)_t, \bar{e_r}\rangle\right| + \left|\langle \eta_r, \bar{e_p}\rangle\right| + \left|\langle \eta_s, \bar{e_r}\rangle\right|. \qquad (2.31)
\end{aligned}$$

It follows from the Cauchy–Schwarz inequality and the properties of the projections that

$$\begin{aligned}
&\|\bar{e_s}\|^2 + \|\bar{e_r}\|^2 + 3\|\bar{e_p}\|^2 + 19\|\bar{e_q}\|^2 \\
&\quad \le \|\bar{e_r}\|\|\bar{e_p}\| + \|\bar{e_s}\|\|\bar{e_r}\| + 19\|\bar{e_p}\|\|\bar{e_u}\| + 3\|\bar{e_r}\|\|\bar{e_q}\| + \|\bar{e_s}\|\|\bar{e_p}\| \\
&\qquad + \left\|(\bar{e_u})_t\right\|\|\bar{e_r}\| + Ch^{k+1}\left(\|\bar{e_s}\| + \|\bar{e_r}\| + \|\bar{e_p}\| + \|\bar{e_q}\| + \|\bar{e_u}\|\right). \qquad (2.32)
\end{aligned}$$

Next, using Young's inequality, we have

$$\begin{aligned}
&\|\bar{e_s}\|^2 + \|\bar{e_r}\|^2 + 3\|\bar{e_p}\|^2 + 19\|\bar{e_q}\|^2 \\
&\quad \le \left(\frac{1}{4}\|\bar{e_r}\|^2 + \|\bar{e_p}\|^2\right) + \left(\frac{1}{2}\|\bar{e_s}\|^2 + \frac{1}{2}\|\bar{e_r}\|^2\right) \\
&\qquad + \left(\frac{1}{8}\|\bar{e_r}\|^2 + 18\|\bar{e_q}\|^2\right) + \left(\frac{1}{4}\|\bar{e_s}\|^2 + \|\bar{e_p}\|^2\right) \\
&\qquad + \left(\frac{1}{16}\|\bar{e_r}\|^2 + 4\left\|(\bar{e_u})_t\right\|^2\right) + \left(\frac{1}{4}\|\bar{e_p}\|^2 + 361\|\bar{e_u}\|^2\right) \\
&\qquad + Ch^{2k+2} + \left(\frac{1}{8}\|\bar{e_s}\|^2 + \frac{1}{32}\|\bar{e_r}\|^2 + \frac{1}{4}\|\bar{e_p}\|^2 + \frac{1}{2}\|\bar{e_q}\|^2 + \|\bar{e_u}\|^2\right). \qquad (2.33)
\end{aligned}$$

After a very simple arrangement, we have

$$\begin{aligned}
&\frac{1}{8}\|\bar{e_s}\|^2 + \frac{1}{32}\|\bar{e_r}\|^2 + \frac{1}{2}\|\bar{e_p}\|^2 + \frac{1}{2}\|\bar{e_q}\|^2 \\
&\quad \le Ch^{2k+2} + C\left(\|\bar{e_u}\|^2 + \left\|(\bar{e_u})_t\right\|^2\right).
\end{aligned}$$

Using Young's inequality for further simplification, we obtain

$$\begin{aligned}
&\frac{1}{128}\left(\|\bar{e_s}\| + \|\bar{e_r}\| + \|\bar{e_p}\| + \|\bar{e_q}\|\right)^2 \\
&\quad \le Ch^{2k+2} + C\left(\|\bar{e_u}\|^2 + \left\|(\bar{e_u})_t\right\|^2\right). \qquad (2.34)
\end{aligned}$$

That is,

$$\|\bar{e_s}\| + \|\bar{e_r}\| + \|\bar{e_p}\| + \|\bar{e_q}\| \le Ch^{k+1} + C\left(\|\bar{e_u}\| + \left\|(\bar{e_u})_t\right\|\right), \qquad (2.35)$$

where the constant $C$ depends on $\theta$, $\|u\|_{k+5}$, and $\|u_t\|_{k+5}$, but is independent of $h$. Next, adding (2.23) and (2.24) to estimate $\|\bar{e_u}\| + \|(\bar{e_u})_t\|$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\bar{e_u}\|^2 + \frac{1}{2}\frac{d}{dt}\big\|(\bar{e_u})_t\big\|^2 + \left(\theta - \frac{1}{2}\right)\sum_{j=1}^{N}[\bar{e_p}]^2_{j+\frac{1}{2}} + \left(\theta - \frac{1}{2}\right)\sum_{j=1}^{N}\big[(\bar{e_p})_t\big]^2_{j+\frac{1}{2}}\\ &\quad = A + B.\end{aligned} \tag{2.36}$$

Here

$$A = \langle \eta_q, \bar{e_s}\rangle - \langle \eta_s, \bar{e_q}\rangle - \langle \eta_p, \bar{e_r}\rangle + \langle \eta_r, \bar{e_p}\rangle - \big\langle (\eta_u)_t, \bar{e_u}\big\rangle - \big\langle (\eta_u)_{tt}, (\bar{e_u})_t\big\rangle \tag{2.37}$$

and

$$B = \big\langle (\eta_q)_t, (\bar{e_s})_t\big\rangle - \big\langle (\eta_s)_t, (\bar{e_q})_t\big\rangle - \big\langle (\eta_p)_t, (\bar{e_r})_t\big\rangle + \big\langle (\eta_r)_t, (\bar{e_p})_t\big\rangle \tag{2.38}$$

denote the right-hand sides of (2.23) and (2.24), respectively. For $A$, by using the Cauchy–Schwarz and Young's inequalities, we obtain

$$A \le Ch^{k+1}\big(\|\bar{e_r}\| + \|\bar{e_p}\| + \|\bar{e_s}\| + \|\bar{e_q}\| + \|\bar{e_u}\| + \big\|(\bar{e_u})_t\big\|\big).$$

Then, combining the above inequality with (2.35), we arrive at

$$A \le Ch^{2k+2} + Ch^{k+1}\big(\|\bar{e_u}\| + \big\|(\bar{e_u})_t\big\|\big), \tag{2.39}$$

where $C$ depends on $\theta$, $\|u\|_{k+5}$, $\|u_t\|_{k+5}$, and $\|u_{tt}\|_{k+5}$, but is independent of $h$.

In order to estimate $B$, we need to handle the four integrations in $B$, respectively. In fact, the processing technique for each integration is similar, so we take the first term $\langle (\eta_q)_t, (\bar{e_s})_t\rangle$ as an example. First, we integrate $\langle (\eta_q)_t, (\bar{e_s})_t\rangle$ with respect to time between 0 and $t$, then exchange integral sequence and get the integration by parts

$$\begin{aligned}\int_0^t \int_I (\eta_q)_t(\bar{e_s})_t\,dx\,dt &= \int_I \int_0^t (\eta_q)_t(\bar{e_s})_t\,dt\,dx\\ &= \int_I \big[(\eta_q)_t\bar{e_s} - \big((\eta_q)_t\bar{e_s}\big)(\cdot,0)\big]\,dx - \int_0^t \int_I (\eta_q)_{tt}\bar{e_s}\,dx\,dt.\end{aligned} \tag{2.40}$$

Similarly, doing the same calculations for each integration of $B$ and integrating $B$ with respect to $t$, we obtain

$$\begin{aligned}\int_0^t B\,dt = {}& \int_I \big[(\eta_q)_t\bar{e_s} - \big((\eta_q)_t\bar{e_s}\big)(\cdot,0)\big]\,dx - \int_0^t \int_I (\eta_q)_{tt}\bar{e_s}\,dx\,dt\\ &- \int_I \big[(\eta_s)_t\bar{e_q} - \big((\eta_s)_t\bar{e_q}\big)(\cdot,0)\big]\,dx - \int_0^t \int_I (\eta_s)_{tt}\bar{e_q}\,dx\,dt\\ &- \int_I \big[(\eta_p)_t\bar{e_r} - \big((\eta_p)_t\bar{e_r}\big)(\cdot,0)\big]\,dx - \int_0^t \int_I (\eta_p)_{tt}\bar{e_r}\,dx\,dt\\ &+ \int_I \big[(\eta_r)_t\bar{e_p} - \big((\eta_r)_t\bar{e_p}\big)(\cdot,0)\big]\,dx - \int_0^t \int_I (\eta_r)_{tt}\bar{e_p}\,dx\,dt.\end{aligned} \tag{2.41}$$

By using the Cauchy–Schwarz inequality, Young's inequality, and the properties of projections from Lemma 4, we have

$$\int_0^t B\,dt \le Ch^{k+1}\big(\|\bar{e_s}\| + \|\bar{e_q}\| + \|\bar{e_r}\| + \|\bar{e_p}\|\big) + Ch^{2k+2} + Ch^{k+1}\int_0^t \big(\|\bar{e_s}\| + \|\bar{e_q}\| + \|\bar{e_r}\| + \|\bar{e_p}\|\big)\,dt. \tag{2.42}$$

Furthermore, it follows from (2.35) and Young's inequality that

$$\int_0^t B\,dt \le Ch^{2k+2} + Ch^{k+1}\int_0^t \big(\|\bar{e_u}\| + \big\|(\bar{e_u})_t\big\|\big)\,dt + \frac{1}{4}\big(\|\bar{e_u}\|^2 + \big\|(\bar{e_u})_t\big\|^2\big). \tag{2.43}$$

Now, integrating both sides of equality (2.36) with respect to $t$, and combining inequalities (2.39) and (2.43), we obtain

$$\begin{aligned}
&\frac{1}{2}\|\bar{e_u}\|^2 + \frac{1}{2}\big\|(\bar{e_u})_t\big\|^2 + \int_0^t \left(\theta - \frac{1}{2}\right)\sum_{j=1}^{N}[\bar{e_p}]^2_{j+\frac{1}{2}}\,dt \\
&\quad + \int_0^t \left(\theta - \frac{1}{2}\right)\sum_{j=1}^{N}\big[(\bar{e_p})_t\big]^2_{j+\frac{1}{2}}\,dt \\
&\le Ch^{2k+2} + \frac{1}{4}\big(\|\bar{e_u}\|^2 + \big\|(\bar{e_u})_t\big\|^2\big) \\
&\quad + Ch^{k+1}\int_0^t \big(\|\bar{e_u}\| + \big\|(\bar{e_u})_t\big\|\big)\,dt.
\end{aligned} \tag{2.44}$$

Thus, we have

$$\big(\|\bar{e_u}\| + \big\|(\bar{e_u})_t\big\|\big)^2 \le Ch^{2k+2} + Ch^{k+1}\int_0^t \big(\|\bar{e_u}\| + \big\|(\bar{e_u})_t\big\|\big)\,dt. \tag{2.45}$$

We denote

$$\begin{aligned}
&A(t) = \|\bar{e_u}\| + \big\|(\bar{e_u})_t\big\|, \\
&E(t) = Ch^{2k+2} + Ch^{k+1}\int_0^t \big(\|\bar{e_u}\| + \big\|(\bar{e_u})_t\big\|\big)\,dt.
\end{aligned}$$

By (2.37) we have

$$A(t) \le \sqrt{E(t)}. \tag{2.46}$$

Note that

$$\frac{d}{dt}E(t) = Ch^{k+1}A(t) \le Ch^{k+1}\sqrt{E(t)} \tag{2.47}$$

and

$$\frac{d}{dt}E(t) = 2\sqrt{E(t)}\frac{d}{dt}\sqrt{E(t)}. \tag{2.48}$$

Combining (2.39) with (2.48), we have

$$\frac{d}{dt}\sqrt{E(t)} \le Ch^{k+1}. \tag{2.49}$$

Integrating inequality (2.41) with respect to $t$, we have

$$\sqrt{E(t)} \le \sqrt{E(0)} + Ch^{k+1}t \le Ch^{k+1}(t+1). \tag{2.50}$$

Combining (2.50) with (2.46), we get

$$\|\bar{e_u}\| + \left\|(\bar{e_u})_t\right\| \le Ch^{k+1}(t+1). \tag{2.51}$$

Therefore, from (2.35) and (2.51) we have

$$\|\bar{e_s}\| + \|\bar{e_r}\| + \|\bar{e_p}\| + \|\bar{e_q}\| + \|\bar{e_u}\| + \left\|(\bar{e_u})_t\right\| \le Ch^{k+1}(t+2), \tag{2.52}$$

where $C$ depends on $\theta$, $\|u\|_{k+5}$, $\|u_t\|_{k+5}$, and $\|u_{tt}\|_{k+5}$, but is independent of $h$. Finally, combining inequality (2.44), Lemma 2, and the triangle inequality, we conclude that

$$\|e_u\| + \|e_q\| + \|e_p\| + \|e_r\| + \|e_s\| + \left\|(e_u)_t\right\| \le Ch^{k+1}(t+3), \tag{2.53}$$

and we prove the theorem. □

## 3 Numerical experiments

In this subsection, we present some numerical experiments to validate the error estimates of the LDG method based on generalized numerical fluxes. We adopt $P^k$ elements on the nonuniform mesh, which is 10% random perturbation coordinates of the uniform mesh, with $N = 10, 20, 40$.

*Example* 3.1 In this example, consider the fifth order equation

$$\begin{cases} u_t + u_{xxxxx} = 0, \\ u(x,0) = u_0(x) = \sin x. \end{cases}$$

The exact solution of the equation is $u(x,t) = \sin(x-t)$. The $L^2$ error estimates and the corresponding convergence rates are listed in the following tables. To reduce time errors, we use the third order implicit Runge–Kutta method, taking $\triangle t = 0.1h^2$ at $k = 0, 1$ and $\triangle t = 0.01h^2$ at $k = 2$, computing until $T = 1$ and $T = 2$.

From Tables 1 and 2 we observe that the errors achieve the desired $(k+1)$ order accuracy for different $\theta$ when $T = 1$ and $T = 2$, which demonstrates the sharpness of error estimates in Theorem 1. Moreover, since the iterative matrix of fifth order equation has strong rigidity, which will cost too much computing time, so we will focus on studying a more appropriate implicit Runge–Kutta LDG method in the future work.

**Table 1** The $L^2$-norm error estimates at T = 1

| k = 0 | $\theta = 0.8$ | | $\theta = 1$ | | $\theta = 1.2$ | | $\theta = 1.5$ | |
|---|---|---|---|---|---|---|---|---|
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 1.63E–01 | – | 1.54E–01 | – | 1.50E–01 | – | 1.89E–01 | – |
| $N = 20$ | 7.27E–02 | 1.16 | 7.20E–02 | 1.10 | 7.19E–02 | 1.06 | 7.7E–02 | 1.29 |
| $N = 40$ | 3.38E–02 | 1.10 | 3.45E–02 | 1.06 | 3.57E–02 | 1.01 | 3.86E–02 | 1.00 |
| $N = 80$ | 1.62E–02 | 1.06 | 1.69E–02 | 1.03 | 1.78E–02 | 1.00 | 1.97E–02 | 0.97 |
| k = 1 | $\theta = 0.8$ | | $\theta = 1$ | | $\theta = 1.2$ | | $\theta = 1.5$ | |
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 6.42E–02 | – | 7.28E–02 | – | 4.76E–02 | – | 9.59E–02 | – |
| $N = 20$ | 1.65E–02 | 1.96 | 1.69E–02 | 2.11 | 1.38E–02 | 1.80 | 2.40E–02 | 2.00 |
| $N = 40$ | 4.08E–03 | 2.02 | 4.19E–03 | 2.01 | 3.29E–03 | 2.07 | 6.30E–03 | 1.93 |
| $N = 80$ | 1.05E–03 | 1.96 | 1.20E–03 | 1.80 | 9.1E–04 | 1.85 | 1.89E–03 | 1.74 |
| k = 2 | $\theta = 0.8$ | | $\theta = 1$ | | $\theta = 1.2$ | | $\theta = 1.5$ | |
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 1.73E–03 | – | 2.37E–03 | – | 2.84E–03 | – | 6.49E–03 | – |
| $N = 20$ | 2.58E–04 | 2.75 | 2.91E–04 | 3.03 | 3.82E–04 | 2.89 | 8.09E–04 | 3.00 |
| $N = 40$ | 3.47E–05 | 2.89 | 3.51E–05 | 3.05 | 5.16E–05 | 2.88 | 9.09E–05 | 3.15 |

**Table 2** The $L^2$-norm error estimates at T = 2

| k = 0 | $\theta = 0.8$ | | $\theta = 1$ | | $\theta = 1.2$ | | $\theta = 1.5$ | |
|---|---|---|---|---|---|---|---|---|
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 2.09E–01 | – | 1.98E–01 | – | 2.08E–01 | – | 3.29E–01 | – |
| $N = 20$ | 2.88E–02 | 1.23 | 9.16E–02 | 1.06 | 9.94E–02 | 1.07 | 1.28E–01 | 1.36 |
| $N = 40$ | 3.95E–02 | 1.17 | 4.35E–02 | 1.07 | 4.92E–02 | 1.01 | 6.11E–02 | 1.07 |
| $N = 80$ | 1.84E–02 | 1.10 | 2.11E–02 | 1.04 | 2.45E–02 | 1.01 | 3.08E–02 | 0.99 |
| k = 1 | $\theta = 0.8$ | | $\theta = 1$ | | $\theta = 1.2$ | | $\theta = 1.5$ | |
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 1.62E–02 | – | 5.41E–02 | – | 8.83E–02 | – | 4.58E–02 | – |
| $N = 20$ | 4.33E–03 | 1.90 | 1.21E–02 | 2.16 | 2.51E–02 | 1.81 | 1.34E–02 | 1.77 |
| $N = 40$ | 1.18E–03 | 1.88 | 2.81E–03 | 2.11 | 7.32E–03 | 1.78 | 4.17E–03 | 1.70 |
| $N = 80$ | 3.03E–04 | 1.96 | 7.03E–04 | 2.00 | 1.93E–03 | 1.92 | 1.36E–03 | 1.61 |
| k = 2 | $\theta = 0.8$ | | $\theta = 1$ | | $\theta = 1.2$ | | $\theta = 1.5$ | |
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 3.57E–03 | – | 4.90E–03 | – | 5.25E–03 | – | 6.70E–03 | – |
| $N = 20$ | 4.76E–04 | 2.91 | 6.12E–04 | 3.00 | 6.97E–04 | 2.91 | 8.34E–04 | 3.01 |
| $N = 40$ | 6.57E–05 | 2.86 | 7.96E–05 | 3.01 | 9.30E–05 | 2.90 | 8.96E–05 | 3.22 |

*Example* 3.2 In this example, consider the fifth order nonlinear equation

$$\begin{cases} u_t + (u^2)_{xx} + u_{xxxxx} = 0, \\ u(x,0) = u_0(x) = \sin x. \end{cases}$$

For this nonlinear equation, we choose the generalized local Lax–Friedrichs (GLLF) flux

$$\hat{f}(a,b) = \left(\frac{1}{2} + \theta\right) f(a) + \left(\frac{1}{2} - \theta\right) f(b) - \lambda\alpha(b-a), \alpha = \max_{u\in[a,b]} \left|f'(u)\right|. \tag{3.1}$$

The time step is taken as $\triangle t = 5 \times 10^{-7} h^4$. Table 3 lists the $L^\infty$ errors and orders for Example 3.2, from which we observe that the error estimates achieve the expected $(k+1)$th

**Table 3** The $L^\infty$ error estimates at T = 0.001

| k = 0 | λ = 0.25, θ = 0 | | λ = 0.5, θ = 0.25 | | λ = 0.5, θ = −0.25 | | λ = 0.5, θ = 0 | |
|---|---|---|---|---|---|---|---|---|
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 6.03E−04 | – | 1.30E−03 | – | 1.02E−03 | – | 9.64E−04 | – |
| $N = 20$ | 2.60E−04 | 1.21 | 6.98E−04 | 0.90 | 5.37E−04 | 0.93 | 4.78E−04 | 1.01 |
| $N = 40$ | 1.21E−04 | 1.10 | 3.52E−04 | 0.99 | 2.73E−04 | 0.98 | 2.36E−04 | 1.02 |
| k = 1 | λ = 0.25, θ = 0 | | λ = 0.5, θ = 0.25 | | λ = 0.5, θ = −0.25 | | λ = 0.5, θ = 0 | |
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 1.67E−02 | – | 1.68E−02 | – | 1.68E−02 | – | 1.68E−02 | – |
| $N = 20$ | 4.15E−03 | 2.01 | 4.15E−03 | 2.11 | 4.15E−03 | 2.01 | 4.15E−03 | 2.01 |
| $N = 40$ | 1.03E−03 | 2.01 | 1.03E−03 | 2.01 | 1.03E−03 | 2.01 | 1.03E−03 | 2.01 |
| k = 2 | λ = 0.25, θ = 0 | | λ = 0.5, θ = 0.25 | | λ = 0.5, θ = −0.25 | | λ = 0.5, θ = 0 | |
| | error | order | error | order | error | order | error | order |
| $N = 10$ | 9.72E−04 | – | 9.65E−04 | – | 9.72E−04 | – | 9.68E−04 | – |
| $N = 20$ | 1.27E−04 | 2.93 | 1.27E−04 | 2.93 | 1.27E−04 | 2.93 | 1.27E−04 | 2.93 |
| $N = 40$ | 1.61E−05 | 2.98 | 1.61E−05 | 2.98 | 1.61E−05 | 2.98 | 1.61E−05 | 2.98 |

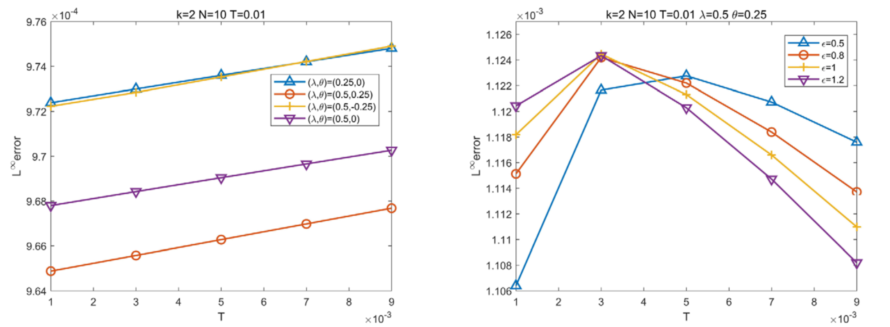
**Table 4** The $L^\infty$ error estimates when T = 0.001 and $\varepsilon = 1$

| k = 0 | λ = 0.5, θ = 0 | | λ = 0.5, θ = 0.25 | |
|---|---|---|---|---|
| | error | order | error | order |
| $N = 10$ | 1.66E−03 | – | 1.94E−03 | – |
| $N = 20$ | 6.00E−04 | 1.46 | 9.50E−04 | 1.03 |
| $N = 40$ | 2.20E−04 | 1.44 | 4.71E−04 | 1.01 |
| k = 1 | λ = 0.5, θ = 0 | | λ = 0.5, θ = 0.25 | |
| | error | order | error | order |
| $N = 10$ | 1.68E−02 | – | 1.69E−02 | – |
| $N = 20$ | 4.17E−03 | 2.01 | 4.20E−03 | 2.01 |
| $N = 40$ | 1.03E−03 | 2.01 | 1.04E−03 | 2.02 |
| k = 2 | λ = 0.5, θ = 0 | | λ = 0.5, θ = 0.25 | |
| | error | order | error | order |
| $N = 10$ | 1.12E−03 | – | 1.12E−03 | – |
| $N = 20$ | 1.37E−04 | 3.03 | 1.37E−04 | 3.03 |
| $N = 40$ | 1.69E−05 | 3.02 | 1.69E−05 | 3.02 |

order of accuracy for different $\theta$ and $\lambda$ when $T = 0.001$. In addition, $(k + \frac{1}{2})$ convergence orders are also observed for the $L^2$ norm, which are omitted to save space. For different parameters $\theta$ and $\lambda$, Fig. 1(a) shows the time growth of $L^2$-norm errors when $N = 10$.

*Example* 3.3 In this example, consider the fifth order nonlinear equation

$$\begin{cases} u_t + \varepsilon (u_{xx})^2_{xxx} = 0, \\ u(x,0) = u_0(x) = \sin x. \end{cases}$$




(a) Time history of the $L^\infty$ error of the numerical solution in Example 3.2

(b) Time history of the $L^\infty$ error of the numerical solution in Example 3.3

**Figure 1** Time history of the $L^\infty$ error of the numerical solution

For this nonlinear equation, we still use the generalized local Lax–Friedrichs (GLLF) flux in (3.1). The time step is also taken as $\triangle t = 5 \times 10^{-7} h^4$. Table 4 lists the $L^\infty$ errors and orders for Example 3.3, from which we again observe the $(k + 1)$th order of accuracy for different $\theta$ and $\lambda$ when $T = 0.001$ and $\varepsilon = 1$. In addition, we also observe that the $L^2$ errors can reach $(k + \frac{1}{2})$ order for the nonlinear problem, which are omitted to save space. Figure 1(b) shows the time growth with different $\varepsilon$, when $N = 10$, $\lambda = 0.5$, and $\theta = 0.25$.

**Acknowledgements**
Not applicable.

**Funding**
This work was supported by the Fundamental Research Foundation for Universities of Heilongjiang Province of China under Grant LGYC2018JC001.

**Availability of data and materials**
Not applicable.

## Declarations

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
HB framed the problems. HB and YC carried out the results and wrote the manuscripts. Both the authors contributed equally to the writing of this paper. YC completed the numerical experiments. All authors read and approved the final manuscript.

## Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 September 2020 Accepted: 22 July 2022 Published online: 09 August 2022

**References**
1. Reed, W.-H., Hill, T.-R.: Triangular mesh methods for the neutron transport equation. Los Alamos Report LA–UR, 73–479 (1973)
2. Bassi, F., Rebay, S.: A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier–Stokes equations. J. Comput. Phys. **131**(2), 267–279 (1997)
3. Cockburn, B., Shu, C.-W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM J. Numer. Anal. **35**(6), 2440–2463 (1998)
4. Cockburn, B., Lin, S.-Y., Shu, C.-W.: TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws: one- dimensional systems. J. Comput. Phys. **84**(1), 90–113 (1989)
5. Castillo, P.: An optimal error estimate for the local discontinuous Galerkin method. J. Comput. Math., 285–290 (2000)
6. Castillo, P., Cockburn, B., Schwab, C., Schotzau, D.: Optimal prior error estimates of the hp-version of the local discontinuous Galerkin method for the convention-diffusion problems. Math. Comput. **71**, 455–478 (2001)

7. Cockburn, B., Dong, B.: An analysis of the minimal dissipation local discontinuous Galerkin method for convection-diffusion problems. J. Sci. Comput. **32**(2), 233–262 (2007)
8. Cockburn, B., Kanschat, G., Schotzau, D., Schwab, C.: Local discontinuous Galerkin method for the Stokes systems. SIAM J. Numer. Anal. **40**, 319–343 (2002)
9. Zhang, Q., Shu, C.-W.: Stability analysis and a priori error estimates of the third order explicit Runge-Kutta discontinuous Galerkin method for scalar conservation laws. SIAM J. Numer. Anal. **48**(3), 1038–1063 (2010)
10. Shu, C.-W., Wang, H., Zhang, Q.: Stability and error estimates of local discontinuous Galerkin methods with implict-explict time-marching for advection-diffusion problems. SIAM J. Numer. Anal. **53**(1), 206–227 (2015)
11. Wang, H., Wang, S., Zhang, Q., Shu, C.-W.: Local discontinuous Galerkin methods with implicit-explicit time-marching for multi-dimensional convection-diffusion problems. ESAIM: Math. Model. Numer. Anal. **50**, 1083–1105 (2016)
12. Wang, H., Shu, C.-W., Zhang, Q.: Stability analysis and error estimates of local discontinuous Galerkin methods with implicit–explicit time-marching for nonlinear convection–diffusion problems. Appl. Math. Comput. **272**, 237–258 (2016)
13. Bi, H., Qian, C.-G., Sun, Y.: The optimal error estimates and superconvergence of the local discontinuous Galerkin methods for one-dimensional linear fifth order time dependent equations. Comput. Math. Appl. **72**(3), 687–703 (2016)
14. Xu, Y., Shu, C.-W.: Local discontinuous Galerkin methods for high-order time-dependent partial differential equations. Commun. Comput. Phys. **7**(1), 1–46 (2010)
15. Meng, X., Shu, C.-W., Wu, B.: Optimal error estimates for discontinuous Galerkin methods based on upwind-biased fluxes for linear hyperbolic equations. Math. Comput. **85**, 1225–1261 (2016)
16. Li, J., Zhang, D., Meng, X.: Analysis of discontinuous Galerkin methods with upwind-biased fluxes for one dimensional linear hyperbolic equations with degenerate variable coefficients. J. Sci. Comput. **78**(3), 1305–1328 (2019)
17. Cheng, Y., Meng, X., Zhang, Q.: Application of generalized Gauss- Radau projections for the local discontinuous Galerkin method for linear convection-diffusion equations. Math. Comput. **86**(305), 1233–1267 (2017)
18. Liu, X.-B., Zhang, D.-Z., Meng, X.: Superconvergence of local discontinuous Galerkin methods with generalized alternating fluxes for 1D linear convection diffusion equations. Sci. China Math. (2020)
19. Gottlieb, S., Shu, C.-W., Tadmor, E.: Strong stability-preserving high-order time discretization methods. SIAM Rev. **43**(1), 89–112 (2001)
20. Li, J., Zhang, D., Meng, X., Wu, B.: Analysis of local discontinuous Galerkin methods with generalized numerical fluxes for linearized KdV equations. Math. Comput. **89**(325), 2085–2111 (2020)