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# Novel results of $\alpha_*$-$\psi$-$\Lambda$-contraction multivalued mappings in $F$-metric spaces with an application

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**Abstract**

The objective of this paper is to introduce a new motif of $\alpha_*$-$\psi$-$\Lambda$-contraction multivalued mappings, some novel fixed-point and coincidence-point results for this contraction will be investigated in the scope of $F$-metric spaces, and some examples are given to illustrate our main results and we derive the existence and uniqueness of a solution of a functional equation to support our main result.

**MSC:** 47H09; 47H10; 54H25

**Keywords:** Fixed and coincidence points; $F$-metric spaces; $\alpha_*$-$\psi$-$\Lambda$-contraction; Functional equations

## 1 Introduction and preliminaries

The conception of $F$-metric space ($F$-MS) was given by Jleli and Samet [21] in 2018 as a generalization of metric space (MS) [16], that has gained importance due to the development of the metric fixed-point theory; they proved that every metric space is an $F$-MS, but the converse is not true, confirming that $F$-MS is more general than the metric space with the help of concrete examples, and compared this concept with existing generalizations from the literature. They defined a natural topology $\tau_F$ on these spaces and studied their topological properties. Moreover, a new fixed-point theorem of the Banach Contraction Principle (BCP) was established in the scope of $F$-MS. This article is arranged into four sections. The first section contains a short history of the literature, providing motivation for this article and some basic definitions that will help readers understand our results. In Sect. 2, new fixed-point theorems for $\alpha_*$-$\psi$-$\Lambda$-contraction multivalued mappings in the scope of $F$-MS and the given example will be discussed. In Sect. 3, the coincidence-point results for said contraction mappings in $F$-MS are investigated as consequences. Section 4 is concerned with an application of the said results to the functional equations in dynamic programming with its example.

**Definition 1.1** ([16]) A mapping $d : \Upsilon \times \Upsilon \to [0,\infty)$ on a nonempty set $\Upsilon$, satisfying the following conditions for all $\gamma, \delta, \kappa \in \Upsilon$,

($d_1$) $d(\gamma,\delta) = 0 \Longleftrightarrow \gamma = \delta$;
($d_2$) $d(\gamma,\delta) = d(\delta,\gamma)$;
($d_3$) $d(\gamma,\delta) \le d(\gamma,\kappa) + d(\kappa,\delta)$,
is called a metric on $\Upsilon$ and the pair $(\Upsilon, d)$ is said to be a MS.

We start with a brief recollection of basic ideas and the facts of $F$-MS. First, let $\Xi$ be the set of functions $\mathcal{L} : (0,\infty) \to \mathbb{R}$ satisfying the following stipulations:
($\Xi1$) $\mathcal{L}$ is nondecreasing, i.e., $0 < \vartheta < \varsigma \Longrightarrow \mathcal{L}(\vartheta) \le \mathcal{L}(\varsigma)$.
($\Xi2$) For every sequence $\{\varsigma_\zeta\} \subset (0,\infty)$, we have

$$\lim_{\zeta\to\infty} \varsigma_\zeta = 0 \quad \Longleftrightarrow \quad \lim_{\zeta\to\infty} \mathcal{L}(\varsigma_\zeta) = -\infty.$$

**Definition 1.2** ([21]) Let $\Upsilon$ be a nonempty set and $\mathbb{Q} : \Upsilon \times \Upsilon \to [0,\infty)$ be a given mapping. We postulate that there exists $(\mathcal{L},\alpha) \in \Xi \times [0,\infty)$ such that,
($\mathbb{Q}1$) $(\gamma,\delta) \in \Upsilon \times \Upsilon$, $\mathbb{Q}(\gamma,\delta) = 0 \Longleftrightarrow \gamma = \delta$;
($\mathbb{Q}2$) $\mathbb{Q}(\gamma,\delta) = \mathbb{Q}(\delta,\gamma)$;
($\mathbb{Q}3$) for every $(\gamma,\delta) \in \Upsilon \times \Upsilon$, $\forall \nu \in \mathbb{N}$, $\nu \ge 2$, and $\forall\ (\varsigma_i)_{i=1}^{\nu} \subset \Upsilon$, $(\varsigma_1,\varsigma_\nu) = (\gamma,\delta)$, we have

$$\mathbb{Q}(\gamma,\delta) > 0 \quad \Longrightarrow \quad \mathcal{L}\big(\mathbb{Q}(\gamma,\delta)\big) \le \mathcal{L}\left(\sum_{i=1}^{\nu-1} \mathbb{Q}(\varsigma_i,\varsigma_{i+1})\right) + \alpha.$$

Then, $\mathbb{Q}$ is said to be an $F$-M on $\Upsilon$, and the pair $(\Upsilon,\mathbb{Q})$ is said to be an $F$-MS.

*Example* 1.3 ([21]) Let $\Upsilon = \mathbb{N}$, and let $\mathbb{Q} : \Upsilon \times \Upsilon \to [0,\infty)$ be the mapping defined by

$$\mathbb{Q}(\gamma,\delta) = \begin{cases} (\gamma-\delta)^2, & \text{if } (\gamma,\delta) \in [0,3] \times [0,3], \\ |\gamma-\delta| & \text{if } (\gamma,\delta) \notin [0,3] \times [0,3], \end{cases}$$

for all $(\gamma,\delta) \in \Upsilon \times \Upsilon$, with $\mathcal{L}(\varsigma) = \ln(\varsigma)$ and $\alpha = \ln(3)$. Then, $(\Upsilon,\mathbb{Q})$ is an $F$-MS.

*Example* 1.4 ([21]) Let $\Upsilon = \mathbb{N}$, and let $\mathbb{Q} : \Upsilon \times \Upsilon \to [0,\infty)$ be the mapping defined by

$$\mathbb{Q}(\gamma,\delta) = \begin{cases} e^{|\gamma-\delta|}, & \text{if } \gamma = \delta, \\ 0 & \text{if } \gamma \ne \delta, \end{cases}$$

for all $(\gamma,\delta) \in \Upsilon \times \Upsilon$, with $\mathcal{L}(\varsigma) = \frac{-1}{\varsigma}$ and $\alpha = 1$. Then, $(\Upsilon,\mathbb{Q})$ is an $F$-MS.

**Definition 1.5** ([21]) Let $(\Upsilon,\mathbb{Q})$ be an $F$-MS, then:
(i) Let $\{\gamma_\zeta\}$ be a sequence in $\Upsilon$; we say that $\{\gamma_\zeta\}$ is $F$-convergent to $\gamma \in \Upsilon$ if $\{\gamma_\zeta\}$ is convergent to $\gamma$ with respect to the $F$-MS $\mathbb{Q}$.
(ii) A sequence $\{\gamma_\zeta\}$ is $F$-Cauchy if $\lim_{\zeta,\eta\to\infty} \mathbb{Q}(\gamma_\zeta,\gamma_\eta) = 0$.
(iii) We say that $(\Upsilon,\mathbb{Q})$ is $F$-complete if every $F$-Cauchy sequence in $\Upsilon$ is $F$-convergent to an assured element in $\Upsilon$.

**Theorem 1.6** ([21]) *Let $(\Upsilon,\mathbb{Q})$ be an $F$-MS, and let $\Gamma : \Upsilon \to \Upsilon$ be a mapping. We postulate that the following affirmations hold*:

(i) $(\Upsilon,\mathbb{Q})$ *is* $F$*-complete*,
(ii) *there exists* $k\in(0,1)$ *such that*

$$\mathbb{Q}\big(\Gamma(\gamma),\Gamma(\delta)\big)\leq k\mathbb{Q}(\gamma,\delta),\quad \forall(\gamma,\delta)\in\Upsilon\times\Upsilon.$$

*Then*, $\Gamma$ *has a unique fixed point* $\gamma^*\in\Upsilon$. *Moreover, for any* $\gamma_0\in\Upsilon$, *the sequence* $\{\gamma_\zeta\}\subset\Upsilon$ *defined by* $\gamma_{\zeta+1}=\Gamma(\gamma_\zeta)$, $\zeta\in\mathbb{N}$ *is* $F$*-convergent*.

Many writers used the motif of $F$-MS to investigate powerful fixed-point results; for instance, Alnaser et al. [4] defined relation theoretic contractions and proved some generalized fixed-point theorems in $F$-metric spaces. Hussain and Kanwal [20] considered the notion of $\alpha$-$\psi$-contraction and presented some fixed- and coupled fixed-point results in the setting of $F$-MSs. Lateef and Ahmad [24] defined Dass and Gupta's contraction in the context of $F$-MSs and then proved some new fixed-point theorems to generalize and elaborate several known literature results. Mitrović et al. [26] proved certain common fixed-point theorems and some consequences to obtain the results of Banach, Jungck, Reich, and Berinde in $F$-MSs with an application for dynamic programming. Hussain [19] introduced the idea of fractional convex-type contraction and established some new fixed-point results for Reich-type $\alpha$-$\eta$-contraction and Kannan-type $\alpha$-$\eta$-contraction mappings in $F$-MS. He derived some consequences for Suzuki-type contractions, orbitally $T$-complete, and orbitally continuous mappings.

BCP [12] appeared in 1922 as the basis of functional analysis and plays a main role in several branches of mathematics and applied sciences, which asserts that every contraction mapping defined in complete MS has a fixed point. In many directions, this principle has been extended and generalized either by relaxing the contractive stipulations or imposing some more stipulations on space. Jungck [22] studied coincidence and common fixed points of commuting mappings and improved the BCP. In [35], coincidence-point and common fixed-point theorems for a class of Ćirić–Suzuki hybrid contractions involving a multivalued and two single-valued maps in an MS are obtained. Coincidence-point theorems for Geraghty contraction mappings have been introduced in different spaces [27–29, 33, 34, 37–39].

**Theorem 1.7** ([12]) *Let* $(\Upsilon,\mathbb{Q})$ *be a complete MS and* $\Gamma:\Upsilon\longrightarrow\Upsilon$ *be a contraction mapping, that is* $\forall\gamma,\delta\in\Upsilon$, *and* $k\in(0,1)$,

$$\mathbb{Q}(\Gamma\gamma,\Gamma\delta)\leq k\mathbb{Q}(\gamma,\delta).$$

*Then*, $\Gamma$ *has a unique fixed point*.

In 1973, Geraghty [17] generalized BCP and established its fixed-point results on complete MS.

**Theorem 1.8** ([17]) *Let* $(\Upsilon,\mathbb{Q})$ *be a complete MS and* $\Gamma:\Upsilon\longrightarrow\Upsilon$ *be a mapping such that* $\forall\ \gamma,\delta\in\Upsilon$, *and* $\beta\in\mho$,

$$\mathbb{Q}(\Gamma\gamma,\Gamma\delta)\leq\beta\big(\mathbb{Q}(\gamma,\delta)\big)\mathbb{Q}(\gamma,\delta),$$

*where $\mho$ is a class of functions $\beta : [0,\infty) \to [0,1)$ satisfying $\beta(\varsigma_\zeta) \to 1 \implies \varsigma_\zeta \to 0$ as $\zeta \to \infty$.*

Then, $\Gamma$ has a unique fixed point $\gamma^* \in \Upsilon$.

In 2013, Cho et al. [14] presented the notion of $\alpha$-Geraghty contraction-type mappings and deduced the unique fixed-point theorems for such mappings in a complete MS. In 2014, Popescu [31] opened a wide field in fixed-point theory by defining the concepts of $\alpha$-orbital and triangular $\alpha$-orbital admissible mappings and verified the unique fixed-point theorems for the said mappings, which are generalizations of $\alpha$-Geraghty contraction-type mappings. In 2012, Wardowski [36] introduced the definition of $F$-contraction and proved fixed-point results as a generalization of the BCP in a complete MS, see also [1, 2, 5–7, 11, 18].

**Definition 1.9** ([31]) Let $\Gamma : \Upsilon \to \Upsilon$ be a map and $\alpha : \Upsilon \times \Upsilon \to \mathbb{R}$ be a function. Then, $\Gamma$ is said to be $\alpha$-orbital admissible if $\alpha(\gamma, \Gamma\gamma) \geq 1$ implies $\alpha(\Gamma\gamma, \Gamma^2\gamma) \geq 1$.

**Definition 1.10** ([31]) Let $\Gamma : \Upsilon \to \Upsilon$ be a map and $\alpha : \Upsilon \times \Upsilon \to \mathbb{R}$ be a function. Then, $\Gamma$ is said to be triangular $\alpha$-orbital admissible if $\Gamma$ is $\alpha$-orbital admissible and $\alpha(\gamma,\delta) \geq 1$ and $\alpha(\delta, \Gamma\delta) \geq 1$ imply $\alpha(\gamma, \Gamma\delta) \geq 1$.

**Lemma 1.11** ([31]) *Let $\Gamma : \Upsilon \to \Upsilon$ be a triangular $\alpha$-orbital admissible mapping. Assume that there exists $\gamma_1 \in \Upsilon$ such that $\alpha(\gamma_1, \Gamma\gamma_1) \geq 1$. Define a sequence $\{\gamma_\zeta\}$ by $\gamma_{\zeta+1} = \Gamma\gamma_\zeta$. Then, we have $\alpha(\gamma_\zeta, \gamma_\eta) \geq 1$ for all $\zeta, \eta \in \mathbb{N}$ with $\zeta < \eta$.*

**Definition 1.12** ([23]) Let $\Upsilon$ be a set. Assume that $\Im : \Upsilon \to \Upsilon$ and $\Gamma : \Upsilon \to 2^\Upsilon$. If $w = \Im\gamma \in \Gamma\gamma$ for some $\gamma \in \Upsilon$, then $\gamma$ is called a coincidence point of $\Im$ and $\Gamma$, and $w$ is called a point of coincidence of $\Im$ and $\Gamma$.

Mappings $\Im$ and $\Gamma$ are called weakly compatible if $\Im\gamma \in \Gamma\gamma$ for some $\gamma \in \Upsilon$ implies $\Im\Gamma(\gamma) \subseteq \Gamma\Im(\gamma)$.

**Proposition 1.13** ([23]) *Let $\Upsilon$ be a set. Assume that $\Im : \Upsilon \to \Upsilon$ and $\Gamma : \Upsilon \to 2^\Upsilon$ are weakly compatible mappings. If $\Im$ and $\Gamma$ have a unique point of coincidence $w = \Im\gamma \in \Gamma\gamma$, then $w$ is the unique common fixed point of $\Im$ and $\Gamma$.*

**Definition 1.14** ([9]) Let $(\Upsilon, d)$ be an MS. Let $CB(\Upsilon)$ be the family of all nonempty closed and bounded subsets of $\Upsilon$. Let $H : CB(\Upsilon) \times CB(\Upsilon) \to [0,\infty)$ be a function defined by

$$H(A,B) = \max\left\{\sup_{\gamma\in A}\mathbb{Q}(\gamma,B), \sup_{\delta\in B}\mathbb{Q}(A,\delta)\right\} \quad \text{for all } A,B \in CB(\Upsilon),$$

where $\mathbb{Q}(\gamma,B) = \inf\{d(\gamma,\delta), \delta \in B\}$. Then, $H$ defines a metric on $CB(\Upsilon)$ called the Hausdorff metric induced by $d$.

Asif et al. [10] obtain fixed points and common fixed-point results for Reich-type $F$-contractions for both single and set-valued mappings in $F$-MSs. Alansari et al. [3] studied a few fuzzy fixed-point theorems and discussed the corresponding fixed-point theorems of multivalued and single-valued mappings on $F$-complete $F$-MSs.

**Lemma 1.15** ([3]) *Let $A$ and $B$ be nonempty closed and compact subsets of an $F$-metric space $(\Upsilon,\mathbb{Q})$. If $a \in A$, then $\mathbb{Q}(a,B) \le H_{\mathbb{Q}}(A,B)$.*

*Let $\Psi$ be the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$, $\forall t > 0$, where $\psi^n$ is the nth iterate of $\psi$.*

**Lemma 1.16** ([8]) *Let $\psi \in \Psi$. Then,*
1. $\psi(t) < t$, $\forall t > 0$;
2. $\psi(0) = 0$.

**Definition 1.17** ([25]) Let $\Lambda : (0,\infty) \longrightarrow (0,\infty)$ be a mapping verifying:
($\Phi$1) $\Lambda$ is nondecreasing;
($\Phi$2) for each positive sequence $\{t_n\}$,

$$\lim_{n\to\infty} \Lambda(t_n) = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} t_n = 0;$$

($\Phi$3) $\Lambda$ is continuous.

We denote by $\Phi$ the set of functions $\Lambda : (0,\infty) \longrightarrow (0,\infty)$ satisfying the conditions ($\Phi$1) – ($\Phi$3).

We modify the Definition 1.17 by adding a general condition ($\Phi$4) that is given in the following way:

($\Phi$4) $\Lambda(\sum_{i=1}^{n} A_i) \le \sum_{i=1}^{n} \Lambda(A_i)$, for all $A_i \in (0,\infty)$, $i = 1,2,\cdots,n$,
where $\Phi$ is the set of functions $\Lambda : (0,\infty) \longrightarrow (0,\infty)$ satisfying the conditions ($\Phi$1), ($\Phi$3), and ($\Phi$4).

*Example* 1.18 Define the following functions for all $t \in (0,\infty)$,
(1) $\Lambda(t) = at$, $a > 0$;
(2) $\Lambda(t) = |t|$.
Then $\Lambda \in \Phi$.

Now, we state and prove our main result.

## 2 Main results

In this section, we shall introduce a generalization of Geraghty contraction type mappings and establish some novel fixed-point theorems for $\alpha_*$-$\Lambda$-$\psi$-contraction multivalued mappings in the setting of $F$-MS.

**Definition 2.1** Let $(\Upsilon,\mathbb{Q})$ be an $F$-MS, $\alpha : \Upsilon \times \Upsilon \to [0,\infty)$ be a function. A mapping $\Gamma : \Upsilon \to CB(\Upsilon)$ is called a $\alpha_*$-$\Lambda$-$\psi$-contraction multivalued mapping if there exists $\beta \in \mho$, $\Lambda \in \Phi$ and $\psi \in \Psi$ such that

$$\Lambda\big(\alpha_*(\Gamma\gamma,\Gamma\delta)H_{\mathbb{Q}}(\Gamma\gamma,\Gamma\delta)\big) \le \psi\big[\Lambda\big(\beta\big(\aleph(\gamma,\delta)\big)\aleph(\gamma,\delta)\big)\big], \tag{2.1}$$

where

$$\aleph(\gamma,\delta) = \max\left\{\mathbb{Q}(\gamma,\delta), \mathbb{Q}(\gamma,\Gamma\gamma), \mathbb{Q}(\delta,\Gamma\delta), \frac{\mathbb{Q}(\gamma,\Gamma\delta)+\mathbb{Q}(\delta,\Gamma\gamma)}{2}\right\},$$

for all $\gamma,\delta \in \Upsilon$.

**Theorem 2.2** *Let $(\Upsilon, \mathbb{Q})$ be an F-complete F-MS, $\alpha : \Upsilon \times \Upsilon \to [0, \infty)$ be a function, and $\Gamma : \Upsilon \to CB(\Upsilon)$ a mapping. Postulating that the following affirmations hold*:

(1) $\Gamma$ *is* $\alpha_*$-$\Lambda$-$\psi$*-contraction*;

(2) $\Gamma$ *is triangular* $\alpha_*$*-orbital admissible*;

(3) *there exists an* $\gamma_0 \in \Upsilon$ *such that* $\alpha_*(\gamma_0, \Gamma\gamma_0) \geq 1$;

(4) $\Gamma$ *is continuous.*

*Then,* $\Gamma$ *has a unique fixed point* $\gamma^* \in \Upsilon$.

*Proof* Due to (3), we define a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ by assuming that $\gamma_1 \in \Gamma\gamma_0$ such that $\alpha(\gamma_0, \Gamma\gamma_0) = \alpha(\gamma_0, \gamma_1) \geq 1$ and $\gamma_2 \in \Gamma\gamma_1,\ \gamma_3 \in \Gamma\gamma_2, \ldots, \gamma_{\zeta+1} \in \Gamma\gamma_\zeta = \Gamma^\zeta\gamma_0$, from (2) and Lemma 1.11, we have $\alpha(\gamma_\zeta, \gamma_{\zeta+1}) \geq 1$ for all $\zeta \in \mathbb{N} \cup \{0\}$. Using Lemma 1.15, from (1) and $(\Phi_1)$, we have

$$\begin{aligned}\Lambda\big(\mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\big) &\leq \Lambda\big(H_{\mathbb{Q}}(\Gamma\gamma_{\zeta-1}, \Gamma\gamma_\zeta)\big)\\ &\leq \Lambda\big(\alpha_*(\Gamma\gamma_{\zeta-1}, \Gamma\gamma_\zeta)H_{\mathbb{Q}}(\Gamma\gamma_{\zeta-1}, \Gamma\gamma_\zeta)\big)\\ &\leq \psi\big(\Lambda\big[\beta\big(\aleph(\gamma_{\zeta-1}, \gamma_\zeta)\big)\aleph(\gamma_{\zeta-1}, \gamma_\zeta)\big]\big). \end{aligned}\tag{2.2}$$

We evaluate

$$\begin{aligned}\aleph(\gamma_{\zeta-1}, \gamma_\zeta) &= \max\left\{\begin{array}{c}\mathbb{Q}(\gamma_{\zeta-1}, \gamma_\zeta), \mathbb{Q}(\gamma_{\zeta-1}, \Gamma\gamma_{\zeta-1}),\\ \mathbb{Q}(\gamma_\zeta, \Gamma\gamma_\zeta), \frac{\mathbb{Q}(\gamma_{\zeta-1}, \Gamma\gamma_\zeta)+\mathbb{Q}(\gamma_\zeta, \Gamma\gamma_{\zeta-1})}{2}\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\mathbb{Q}(\gamma_{\zeta-1}, \gamma_\zeta), \mathbb{Q}(\gamma_{\zeta-1}, \gamma_\zeta), \mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\\ \frac{\mathbb{Q}(\gamma_{\zeta-1}, \gamma_{\zeta+1})+\mathbb{Q}(\gamma_\zeta, \gamma_\zeta)}{2}\end{array}\right\},\end{aligned}$$

since

$$\frac{\mathbb{Q}(\gamma_{\zeta-1}, \gamma_{\zeta+1})}{2} \leq \max\big\{\mathbb{Q}(\gamma_{\zeta-1}, \gamma_\zeta), \mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\big\},$$

we conclude that

$$\aleph(\gamma_{\zeta-1}, \gamma_\zeta) = \max\big\{\mathbb{Q}(\gamma_{\zeta-1}, \gamma_\zeta), \mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\big\}.$$

Now, if $\max\{\mathbb{Q}(\gamma_{\zeta-1}, \gamma_\zeta), \mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\} = \mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})$ for $\zeta \geq 1$, then from (2.2), we obtain

$$\Lambda\big(\mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\big) \leq \psi\big(\Lambda\big[\beta\big(\mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\big).\mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\big]\big),$$

since $\beta \in \mho$ and from $(\Phi_1)$, we have

$$\mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1}) < \mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1}),$$

which is a discrepancy as $\mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1}) \geq 0$. Therefore,

$$\max\big\{\mathbb{Q}(\gamma_{\zeta-1}, \gamma_\zeta), \mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1})\big\} = \mathbb{Q}(\gamma_{\zeta-1}, \gamma_\zeta),$$

by (2.2), we have

$$\begin{aligned}
\Lambda\big(\mathbb{Q}(\gamma_\zeta,\gamma_{\zeta+1})\big) &\leq \Lambda\big(\alpha_*(\Gamma\gamma_{\zeta-1},\Gamma\gamma_\zeta)H_{\mathbb{Q}}(\Gamma\gamma_{\zeta-1},\Gamma\gamma_\zeta)\big) \\
&\leq \psi\big(\Lambda\big[\beta\big(\mathbb{Q}(\gamma_{\zeta-1},\gamma_\zeta)\big).\mathbb{Q}(\gamma_{\zeta-1},\gamma_\zeta)\big]\big) \\
&\leq \psi\big(\Lambda\big[\beta\big(\mathbb{Q}(\gamma_{\zeta-1},\gamma_\zeta)\big).\big(\alpha_*(\Gamma\gamma_{\zeta-2},\Gamma\gamma_{\zeta-1})H_{\mathbb{Q}}(T\gamma_{\zeta-2},T\gamma_{\zeta-1})\big)\big]\big) \\
&\leq \psi^2\big(\Lambda\big[\beta\big(\mathbb{Q}(\gamma_{\zeta-1},\gamma_\zeta)\big)\beta\big(\mathbb{Q}(\gamma_{\zeta-2},\gamma_{\zeta-1})\big)\mathbb{Q}(\gamma_{\zeta-2},\gamma_{\zeta-1})\big]\big) \\
&\;\;\vdots \\
&\leq \psi^\zeta\big(\Lambda\big[\beta\big(\mathbb{Q}(\gamma_{\zeta-1},\gamma_\zeta)\big)\beta\big(\mathbb{Q}(\gamma_{\zeta-2},\gamma_{\zeta-1})\big)\ldots\beta\big(\mathbb{Q}(\gamma_0,\gamma_1)\big)\mathbb{Q}(\gamma_0,\gamma_1)\big]\big) \\
&= \psi^\zeta\left(\Lambda\left[\left(\prod_{i=1}^{\zeta}\beta\big(\mathbb{Q}(\gamma_{i-1},\gamma_i)\big)\right)\mathbb{Q}(\gamma_0,\gamma_1)\right]\right) \\
&< \psi^\zeta\big(\Lambda\big[\mathbb{Q}(\gamma_0,\gamma_1)\big]\big), \quad \text{for all } \zeta\in\mathbb{N}.
\end{aligned}$$

Let $\epsilon>0$ be fixed and $(\mathcal{L},a)\in\Xi\times[0,\infty)$ be such that ($\mathbb{Q}3$) is satisfied. By ($\Xi2$), there exists $\eth>0$ such that

$$0<\varsigma<\eth \quad \text{implies} \quad \mathcal{L}(\varsigma)<\mathcal{L}(\epsilon)-a. \tag{2.3}$$

Let $\ell(\epsilon)\in\mathbb{N}$ such that $0<\sum_{\zeta\geq\ell(\epsilon)}\psi^\zeta(\Lambda[\mathbb{Q}(\gamma_0,\gamma_1)])<\Lambda(\eth)$.

Hence, by using properties of $\psi$, (2.3) and ($\Xi1$), we have

$$\begin{aligned}
\mathcal{L}\left(\sum_{j=\zeta}^{\eta-1}\psi^j\big(\Lambda\big[\mathbb{Q}(\gamma_0,\gamma_1)\big]\big)\right) &\leq \mathcal{L}\left(\sum_{\zeta\geq\ell(\epsilon)}\psi^\zeta\big(\Lambda\big[\mathbb{Q}(\gamma_0,\gamma_1)\big]\big)\right) \\
&< \mathcal{L}\big(\Lambda(\epsilon)\big)-a,
\end{aligned} \tag{2.4}$$

where $\eta>\zeta>\ell(\epsilon)$ with $\mathbb{Q}(\gamma_\zeta,\gamma_\eta)>0$ using ($\mathbb{Q}3$) and (2.4), we have

$$\begin{aligned}
\mathcal{L}\big(\Lambda\big(\mathbb{Q}(\gamma_\zeta,\gamma_\eta)\big)\big) &\leq \mathcal{L}\left(\sum_{j=\zeta}^{\eta-1}\psi^j\big(\Lambda\big[\mathbb{Q}(\gamma_0,\gamma_1)\big]\big)\right)+a \\
&\leq \mathcal{L}\left(\sum_{\zeta\geq\ell(\epsilon)}\psi^\zeta\big(\Lambda\big[\mathbb{Q}(\gamma_0,\gamma_1)\big]\big)\right)+a \\
&< \mathcal{L}\big(\Lambda(\epsilon)\big)-a+a \\
&= \mathcal{L}\big(\Lambda(\epsilon)\big),
\end{aligned}$$

which implies by ($\Xi1$) and ($\Phi_1$) that

$$\mathbb{Q}(\gamma_\zeta,\gamma_\eta)<\epsilon, \quad \forall\eta>\zeta>\ell(\epsilon).$$

Therefore, $\{\gamma_\zeta\}$ is an $F$-Cauchy sequence in $(\Upsilon,\mathbb{Q})$. Since $\Upsilon$ is $F$-complete, there exists $\gamma^*\in\Upsilon$ such that $\gamma_\zeta\longrightarrow\gamma^*$ as $\zeta\longrightarrow\infty$, implies

$$\lim_{\zeta\to\infty}\mathbb{Q}\big(\gamma^*,\gamma_\zeta\big)=0. \tag{2.5}$$

Now, to show that $\gamma^* \in \Gamma\gamma^*$ is a fixed point of $\Gamma$, presume that $\mathbb{Q}(\gamma^*, \Gamma\gamma) > 0$ such that $\gamma^* \notin \Gamma\gamma^*$ with $\alpha(\gamma^*, \gamma_\zeta) \geq 1$, $\zeta \in \mathbb{N}$. By ($\mathbb{Q}$3) and ($\Phi$4), we have

$$\begin{aligned}
\mathcal{L}(\Lambda(\mathbb{Q}(\Gamma\gamma^*, \gamma^*))) &\leq \mathcal{L}(\Lambda(\mathbb{Q}(\Gamma\gamma^*, \Gamma\gamma_\zeta) + \mathbb{Q}(\Gamma\gamma_\zeta, \gamma^*))) + a \\
&\leq \mathcal{L}(\Lambda(\mathbb{Q}(\Gamma\gamma^*, \Gamma\gamma_\zeta)) + \Lambda(\mathbb{Q}(\Gamma\gamma_\zeta, \gamma^*))) + a \\
&\leq \mathcal{L}(\Lambda(\alpha_*(\Gamma\gamma^*, \Gamma\gamma_\zeta) H_{\mathbb{Q}}(\Gamma\gamma^*, \Gamma\gamma_\zeta)) + \Lambda(\mathbb{Q}(\Gamma\gamma_\zeta, \gamma^*))) + a \\
&\leq \mathcal{L}(\psi(\Lambda[\beta(\aleph(\gamma^*, \gamma_\zeta))\aleph(\gamma^*, \gamma_\zeta)]) + \Lambda(\mathbb{Q}(\Gamma\gamma_\zeta, \gamma^*))) + a \\
&\leq \mathcal{L}(\psi(\Lambda[\aleph(\gamma^*, \gamma_\zeta)]) + \Lambda(\mathbb{Q}(\Gamma\gamma_\zeta, \gamma^*))) + a \\
&< \mathcal{L}(\Lambda[\aleph(\gamma^*, \gamma_\zeta)] + \Lambda(\mathbb{Q}(\Gamma\gamma_\zeta, \gamma^*))) + a,
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
\aleph(\gamma^*, \gamma_\zeta) &= \max\left\{ \begin{array}{c} \mathbb{Q}(\gamma^*, \gamma_\zeta), \mathbb{Q}(\gamma^*, \Gamma\gamma^*), \mathbb{Q}(\gamma_\zeta, \Gamma\gamma_\zeta), \\ \frac{\mathbb{Q}(\gamma^*, \Gamma\gamma_\zeta) + \mathbb{Q}(\Gamma\gamma^*, \gamma_\zeta)}{2} \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} \mathbb{Q}(\gamma^*, \gamma_\zeta), \mathbb{Q}(\gamma^*, \Gamma\gamma^*), \mathbb{Q}(\gamma_\zeta, \gamma_{\zeta+1}), \\ \frac{\mathbb{Q}(\gamma^*, \gamma_{\zeta+1}) + \mathbb{Q}(\Gamma\gamma^*, \gamma_\zeta)}{2} \end{array} \right\},
\end{aligned}$$

for all $\aleph(\gamma^*, \gamma_\zeta)$ and using (2.5), ($\Phi$2), and ($\Xi$2), we obtain

$$\lim_{\zeta\to\infty} \mathcal{L}(\Lambda(\aleph(\gamma^*, \gamma_\zeta)) + \Lambda(\mathbb{Q}(\Gamma\gamma_\zeta, \gamma^*))) + a = -\infty,$$

which is a discrepancy. Hence, we have

$$\mathbb{Q}(\gamma^*, \Gamma\gamma) = 0, \quad \text{that is } \gamma^* \in \Gamma\gamma^*. \tag{2.7}$$

For uniqueness, we postulate that $\gamma^*$ and $\delta^*$ are two fixed points of $\Gamma$ in $\Upsilon$ such that $\gamma^* \neq \delta^*$. Then,

$$\begin{aligned}
\Lambda(\mathbb{Q}(\gamma^*, \delta^*)) &= \Lambda(\mathbb{Q}(\Gamma\gamma^*, \Gamma\delta^*)) \\
&\leq \Lambda(\alpha_*(\Gamma\gamma^*, \Gamma\delta^*) H_{\mathbb{Q}}(\Gamma\gamma^*, \Gamma\delta^*)) \\
&\leq \psi(\Lambda[\beta(\aleph(\gamma^*, \delta^*))\aleph(\gamma^*, \delta^*)]) \\
&< \psi(\Lambda[\aleph(\gamma^*, \delta^*)]) \\
&< \Lambda[\aleph(\gamma^*, \delta^*)],
\end{aligned}$$

where

$$\aleph(\gamma^*, \delta^*) = \max\left\{ \begin{array}{c} \mathbb{Q}(\gamma^*, \delta^*), \mathbb{Q}(\gamma^*, \Gamma\gamma^*), \mathbb{Q}(\delta^*, \Gamma\delta^*), \\ \frac{\mathbb{Q}(\gamma^*, \Gamma\delta^*) + \mathbb{Q}(\delta^*, \Gamma\gamma^*)}{2} \end{array} \right\} = \mathbb{Q}(\gamma^*, \delta^*).$$

From ($\Phi$1), this yields that

$$\mathbb{Q}(\gamma^*, \delta^*) < \mathbb{Q}(\gamma^*, \delta^*),$$

a discrepancy. Therefore, $\gamma^* = \delta^*$ and $\Gamma$ has a unique fixed point $\gamma^* \in \Upsilon$. □

*Example* 2.3 Let $\Upsilon = \mathbb{R}$ be $F$-M and $\mathbb{Q}$ given by

$$\mathbb{Q}(\gamma,\delta) = \begin{cases} (\gamma-\delta)^2 & \text{if } (\gamma,\delta)\in[0,3]\times[0,3], \\ |\gamma-\delta| & \text{if } (\gamma,\delta)\notin[0,3]\times[0,3], \end{cases}$$

with $\mathcal{L}(\varsigma) = \ln(\varsigma)$ and $a = \ln(3)$. Then, $(\Upsilon,\mathbb{Q})$ is an $F$-complete $F$- MS. Define $\Gamma:\Upsilon\to CB(\Upsilon)$ by

$$\Gamma\gamma = \begin{cases} \{\frac{\gamma+1}{e^{10}}\}, & \text{if } \gamma\in[0,\infty), \\ \{0\} & \text{otherwise}, \end{cases}$$

and $\alpha:\Upsilon\times\Upsilon\to[0,\infty)$ by

$$\alpha(\gamma,\delta) = \begin{cases} \frac{1}{\gamma}+1 & \text{if } \gamma,\delta\in(0,\infty), \\ 0 & \text{otherwise}, \end{cases}$$

let $\beta:\Upsilon\times\Upsilon\to[0,1)$ be as $\beta(\gamma,\delta) = \frac{4(\gamma+1+e^{10})}{e^{20}(\gamma+1)}$, $\Lambda(t) = t$ and $\psi(t) = \frac{3}{4}t$.

Now, for all $(\gamma,\delta)\in[0,3]\times[0,3]$, then

$$\begin{aligned} &\Lambda\big(\alpha_*(\Gamma\gamma,\Gamma\delta)H_{\mathbb{Q}}(\Gamma\gamma,\Gamma\delta)\big) \\ &\quad= \Lambda\left[\frac{\gamma+1+e^{10}}{\gamma+1}\max\Big(\sup_{a\in\Gamma\delta}\mathbb{Q}(a,\Gamma\delta),\sup_{b\in\Gamma\delta}\mathbb{Q}(\Gamma\gamma,b)\Big)\right] \\ &\quad= \Lambda\left[\frac{\gamma+1+e^{10}}{\gamma+1}\max\left(\sup_{a\in\Gamma\gamma}\mathbb{Q}\left(a,\left\{\frac{\delta+1}{e^{10}}\right\}\right),\sup_{b\in\Gamma\delta}\mathbb{Q}\left(\left\{\frac{\gamma+1}{e^{10}}\right\},b\right)\right)\right] \\ &\quad= \Lambda\left[\frac{\gamma+1+e^{10}}{\gamma+1}\max\left(\mathbb{Q}\left(\frac{\gamma+1}{e^{10}},\left\{\frac{\delta+1}{e^{10}}\right\}\right),\mathbb{Q}\left(\left\{\frac{\gamma+1}{e^{10}}\right\},\frac{\delta+1}{e^{10}}\right)\right)\right] \\ &\quad= \Lambda\left[\frac{\gamma+1+e^{10}}{\gamma+1}\max\left(\mathbb{Q}\left(\frac{\gamma+1}{e^{10}},\frac{\delta+1}{e^{10}}\right),\mathbb{Q}\left(\frac{\gamma+1}{e^{10}},\frac{\delta+1}{e^{10}}\right)\right)\right] \\ &\quad\le \frac{3}{4}\Lambda\left[\frac{4(\gamma+1+e^{10})}{e^{20}(\gamma+1)}(\gamma-\delta)^2\right] \\ &\quad\le \psi\big[\Lambda\big(\beta\big(\aleph(\gamma,\delta)\big)\aleph(\gamma,\delta)\big)\big]. \end{aligned}$$

Otherwise, we have

$$\Lambda\big(\alpha_*(\Gamma\gamma,\Gamma\delta)H_{\mathbb{Q}}(\Gamma\gamma,\Gamma\delta)\big) = 0 \le \psi\big[\Lambda\big(\beta\big(\aleph(\gamma,\delta)\big)\aleph(\gamma,\delta)\big)\big].$$

Now, for $(\gamma,\delta)\in(0,3]\times(0,3]$, $\alpha_*(\gamma,\Gamma\gamma)\ge 1$ implies $\alpha_*(\Gamma\gamma,\Gamma^2\gamma)\ge 1$, then $\Gamma$ is $\alpha_*$-orbital admissible and $\alpha(\gamma,\delta)\ge 1$ and $\alpha_*(\delta,\Gamma\delta)\ge 1$ imply $\alpha_*(\gamma,\Gamma\delta)\ge 1$, therefore $\Gamma$ is triangular $\alpha_*$-orbital admissible. Hence, all affirmations of Theorem 2.2 are satisfied and $\gamma^* = \frac{1}{e^{10}-1}\in\Upsilon$ is the fixed point of $\Gamma$.

## 3 Consequences

In this part, some consequences are discussed in $F$-MS.

**Theorem 3.1** *Let $(\Upsilon,\mathbb{Q})$ be an F-MS, set $\Im:\Upsilon\to\Upsilon$ and $\Gamma:\Upsilon\to CB(\Upsilon)$. Presume that there exist functions $\beta\in\mho$, $\Lambda\in\Phi$, and $\psi\in\Psi$ such that $\forall\gamma,\delta\in\Upsilon$,*

$$\Lambda\big(H_{\mathbb{Q}}(\Gamma\gamma,\Gamma\delta)\big)\leq\psi\big(\Lambda\big[\beta\big(\aleph_{\Im}(\gamma,\delta)\big)\aleph_{\Im}(\gamma,\delta)\big]\big), \tag{3.1}$$

*where*

$$\aleph_{\Im}(\gamma,\delta)=\max\left\{\mathbb{Q}(\Im\gamma,\Im\delta),\mathbb{Q}(\Im\gamma,\Gamma\gamma),\mathbb{Q}(\Im\delta,\Gamma\delta),\frac{\mathbb{Q}(\Im\gamma,\Gamma\delta)+\mathbb{Q}(\Im\delta,\Gamma\gamma)}{2}\right\}.$$

*If for any $\gamma\in\Upsilon$, $\Gamma\Upsilon\subseteq\Im\Upsilon$ and $\Im\Upsilon$ is an F-complete subspace of $\Upsilon$.*

Then, $\Gamma$ and $\Im$ have a unique point of coincidence. Indeed, if $\Gamma$ and $\Im$ are weakly compatible, then $\Gamma$ and $\Im$ have a unique common fixed point $\gamma^*\in\Upsilon$.

*Proof* Let $\gamma_0\in\Upsilon$, since $\Gamma\Upsilon\subseteq\Im\Upsilon$, we can construct a sequence $\{\delta_\zeta\}_{\zeta\in\mathbb{N}}$ by

$$\delta_\zeta\in\Gamma\gamma_{\zeta-1}=\Im\gamma_\zeta,\quad\forall\zeta\in\mathbb{N}. \tag{3.2}$$

Now, if there exists some $\zeta_0\in\mathbb{N}$ such that $\mathbb{Q}(\delta_{\zeta_0},\delta_{\zeta_0+1})=0$, then $\delta_{\zeta_0}=\delta_{\zeta_0+1}$, which implies that $\Im\gamma_{\zeta_0}=\Gamma\gamma_{\zeta_0}$, thus $\gamma_{\zeta_0}$ is a coincidence point of $\Gamma$ and $\Im$, so $w_0\in\Im\gamma_{\zeta_0}=\Gamma\gamma_{\zeta_0}$ is the point of coincidence of $\Gamma$ and $\Im$. We postulate that $\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})>0$ $\forall\zeta\in\mathbb{N}$. From (3.1) and (3.2), we have

$$\begin{aligned}\Lambda\big(\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})\big)&\leq\Lambda\big(H_{\mathbb{Q}}(\Gamma\gamma_{\zeta-1},\Gamma\gamma_\zeta)\big)\\&\leq\psi\big(\Lambda\big(\beta\big(\aleph_{\Im}(\gamma_{\zeta-1},\gamma_\zeta)\big)\aleph_{\Im}(\gamma_{\zeta-1},\gamma_\zeta)\big)\big),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\aleph_{\Im}(\gamma_{\zeta-1},\gamma_\zeta)&=\max\left\{\begin{array}{c}\mathbb{Q}(\Im\gamma_{\zeta-1},\Im\gamma_\zeta),\mathbb{Q}(\Im\gamma_{\zeta-1},\Gamma\gamma_{\zeta-1}),\\\mathbb{Q}(\Im\gamma_\zeta,\Gamma\gamma_\zeta),\frac{\mathbb{Q}(\Im\gamma_{\zeta-1},\Gamma\gamma_\zeta)+\mathbb{Q}(\Im\gamma_\zeta,\Gamma\gamma_{\zeta-1})}{2}\end{array}\right\}\\&=\max\left\{\begin{array}{c}\mathbb{Q}(\delta_{\zeta-1},\delta_\zeta),\mathbb{Q}(\delta_{\zeta-1},\delta_\zeta),\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1}),\\\frac{\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta+1})+\mathbb{Q}(y_\zeta,y_\zeta)}{2}\end{array}\right\}\\&=\max\big\{\mathbb{Q}(\delta_{\zeta-1},\delta_\zeta),\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})\big\}.\end{aligned}$$

We conclude that

$$\aleph_{\Im}(\gamma_{\zeta-1},\gamma_\zeta)=\max\big\{\mathbb{Q}(\delta_{\zeta-1},\delta_\zeta),\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})\big\}.$$

Now, if $\max\{\mathbb{Q}(\delta_{\zeta-1},\delta_\zeta),\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})\}=\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})$ for $\zeta\geq 1$, then from (3.2), we obtain

$$\Lambda\big(\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})\big)\leq\psi\big[\Lambda\big(\beta\big(\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})\big)\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})\big)\big].$$

Since $\beta\in\mho$ and from ($\Phi$1), we have

$$\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1})<\mathbb{Q}(\delta_\zeta,\delta_{\zeta+1}),$$

which is a discrepancy as $\mathbb{Q}(\delta_{\zeta},\delta_{\zeta+1}) \geq 0$. Therefore,

$$\max\{\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta}),\mathbb{Q}(\delta_{\zeta},\delta_{\zeta+1})\} = \mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta}), \tag{3.4}$$

by (3.3) and (3.4), we have

$$\begin{aligned}
&\Lambda\bigl(\mathbb{Q}(\delta_{\zeta},\delta_{\zeta+1})\bigr) \\
&\quad= \Lambda\bigl(\mathbb{Q}(\Gamma\gamma_{\zeta-1},\Gamma\gamma_{\zeta})\bigr) \leq \Lambda\bigl(H_{\mathbb{Q}}(\Gamma\gamma_{\zeta-1},\Gamma\gamma_{\zeta})\bigr) \\
&\quad\leq \psi\bigl[\Lambda\bigl(\beta\bigl(\mathbb{Q}(\Im\gamma_{\zeta-1},\Im\gamma_{\zeta})\bigr).\mathbb{Q}(\Im\gamma_{\zeta-1},\Im\gamma_{\zeta})\bigr)\bigr] \\
&\quad= \psi\bigl[\Lambda\bigl(\beta\bigl(\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta})\bigr).\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta})\bigr)\bigr] \\
&\quad= \psi\bigl[\Lambda\bigl(\beta\bigl(\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta})\bigr).\mathbb{Q}(\Gamma\gamma_{\zeta-2},\Gamma\gamma_{\zeta-1})\bigr)\bigr] \\
&\quad\leq \psi\bigl[\Lambda\bigl(\beta\bigl(\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta})\bigr).H_{\mathbb{Q}}(\Gamma\gamma_{\zeta-2},\Gamma\gamma_{\zeta-1})\bigr)\bigr] \\
&\quad\leq \psi^2\bigl[\Lambda\bigl(\beta\bigl(\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta})\bigr)\beta\bigl(\mathbb{Q}(\delta_{\zeta-2},\delta_{\zeta-1})\bigr)\mathbb{Q}(\Gamma\gamma_{\zeta-3},\Gamma\gamma_{\zeta-2})\bigr)\bigr] \\
&\quad= \psi^2\bigl[\Lambda\bigl(\beta\bigl(\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta})\bigr)\beta\bigl(\mathbb{Q}(\delta_{\zeta-2},\delta_{\zeta-1})\bigr)\mathbb{Q}(\delta_{\zeta-2},\delta_{\zeta-1})\bigr)\bigr] \\
&\qquad\cdots \\
&\quad\leq \psi^{\zeta}\bigl[\Lambda\bigl(\beta\bigl(\mathbb{Q}(\Im\gamma_{\zeta-1},\gamma_{\zeta})\bigr)\beta\bigl(\mathbb{Q}(\gamma_{\zeta-2},\gamma_{\zeta-1})\bigr)\ldots\beta\bigl(\mathbb{Q}(\gamma_0,\gamma_1)\bigr)\mathbb{Q}(\gamma_0,\gamma_1)\bigr)\bigr] \\
&\quad= \psi^{\zeta}\bigl[\Lambda\bigl(\beta\bigl(\mathbb{Q}(\delta_{\zeta-1},\delta_{\zeta})\bigr)\beta\bigl(\mathbb{Q}(\delta_{\zeta-2},\delta_{\zeta-1})\bigr)\ldots\beta\bigl(\mathbb{Q}(\delta_0,\delta_1)\bigr)\mathbb{Q}(\delta_0,\delta_1)\bigr)\bigr] \\
&\quad= \psi^{\zeta}\left[\Lambda\left(\left[\prod_{i=1}^{\zeta}\beta\bigl(\mathbb{Q}(\delta_{i-1},\delta_i)\bigr)\right]\mathbb{Q}(\delta_0,\delta_1)\right)\right] \\
&\quad< \psi^{\zeta}\bigl[\Lambda\bigl(\mathbb{Q}(\delta_0,\delta_1)\bigr)\bigr],
\end{aligned}$$

for all $\zeta \in \mathbb{N}$. Let $\epsilon > 0$ be fixed and $(\mathcal{L}, a) \in \Xi \times [0,\infty)$ be such that ($\mathbb{Q}$3) is satisfied. By ($\Xi$2), there exists $\eth > 0$ such that

$$0 < \varsigma < \eth \quad \text{implies } \mathcal{L}(\varsigma) < \mathcal{L}(\epsilon) - a. \tag{3.5}$$

Let $\ell(\epsilon) \in \mathbb{N}$ such that $0 < \sum_{\zeta\geq\ell(\epsilon)}\psi^{\zeta}[\Lambda(\mathbb{Q}(\delta_0,\delta_1))] < \Lambda(\eth)$. Hence, by using the properties of $\psi$, (3.5), and ($\Xi$1), we have

$$\begin{aligned}
\mathcal{L}\left(\sum_{j=\zeta}^{\eta-1}\psi^j\bigl[\Lambda\bigl(\mathbb{Q}(\delta_0,\delta_1)\bigr)\bigr]\right) &\leq \mathcal{L}\left(\sum_{\zeta\geq\ell}\psi^{\zeta}\bigl[\Lambda\bigl(\mathbb{Q}(\delta_0,\delta_1)\bigr)\bigr]\right) \\
&< \mathcal{L}\bigl(\Lambda(\epsilon)\bigr) - a,
\end{aligned} \tag{3.6}$$

where $\eta > \zeta > \ell$ with $\mathbb{Q}(\delta_{\zeta},\delta_{\eta}) > 0$, using ($\mathbb{Q}$3) and (3.6), we have

$$\begin{aligned}
\mathcal{L}\bigl(\Lambda\bigl(\mathbb{Q}(\delta_{\zeta},\delta_{\eta})\bigr)\bigr) &\leq \mathcal{L}\left(\sum_{j=\zeta}^{\eta-1}\psi^j\bigl[\Lambda\bigl(\mathbb{Q}(\delta_0,\delta_1)\bigr)\bigr]\right) + a \\
&\leq \mathcal{L}\left(\sum_{\zeta\geq\ell}\psi^{\zeta}\bigl[\Lambda\bigl(\mathbb{Q}(\delta_0,\delta_1)\bigr)\bigr]\right) + a
\end{aligned}$$

$$
\begin{aligned}
&< \mathcal{L}\big(\Lambda(\epsilon)\big) - a + a \\
&= \mathcal{L}\big(\Lambda(\epsilon)\big),
\end{aligned}
$$

which yields by ($\Xi 1$) and ($\Phi 1$) that

$$\mathbb{Q}(\delta_\zeta, \delta_\eta) < \epsilon, \quad \forall \eta > \zeta > \ell.$$

Therefore, $\{\delta_\zeta\} = \{\Im\gamma_\zeta\}$ is an $F$-Cauchy sequence in $\Im\Upsilon$. Since $\Im\Upsilon$ is $F$-complete, there exists $v^*, u^* \in \Upsilon$ such that $v^* = \Im u^*$, which implies

$$\lim_{\zeta\to\infty} \mathbb{Q}\big(v^*, \delta_\zeta\big) = 0 = \lim_{\eta,\zeta\to\infty} \mathbb{Q}(\delta_\eta, \delta_\zeta) = \lim_{\zeta\to\infty} \mathbb{Q}\big(\Im u^*, \delta_\zeta\big) = 0. \tag{3.7}$$

Now, we show that $v^* \in \Gamma u^*$. Postulating that $\mathbb{Q}(v^*, \Gamma u^*) > 0$, by (3.1), we have

$$
\begin{aligned}
\Lambda\big(\mathbb{Q}\big(\delta_\zeta, \Gamma u^*\big)\big) &= \Lambda\big(\mathbb{Q}\big(\Gamma\gamma_{\zeta-1}, \Gamma u^*\big)\big) \le \Lambda\big(H_{\mathbb{Q}}\big(\Gamma\gamma_{\zeta-1}, \Gamma u^*\big)\big) \\
&\le \psi\big(\Lambda\big[\beta\big(\aleph_\Im\big(\gamma_{\zeta-1}, u^*\big)\big).\aleph_\Im\big(\gamma_{\zeta-1}, u^*\big)\big]\big),
\end{aligned} \tag{3.8}
$$

where

$$
\begin{aligned}
\aleph_\Im(\gamma_{\zeta-1}, u) &= \max\left\{\begin{array}{c} \mathbb{Q}(\Im\gamma_{\zeta-1}, \Im u^*), \mathbb{Q}(\Im\gamma_{\zeta-1}, \Gamma\gamma_{\zeta-1}), \\ \mathbb{Q}(\Im u^*, \Gamma u^*), \frac{\mathbb{Q}(\Im\gamma_{\zeta-1}, \Gamma u^*) + \mathbb{Q}(\Im u^*, \Gamma\gamma_{\zeta-1})}{2} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \mathbb{Q}(\delta_{\zeta-1}, v^*), \mathbb{Q}(\delta_{\zeta-1}, \delta_\zeta), \mathbb{Q}(v^*, \Gamma u^*), \\ \frac{\mathbb{Q}(\delta_{\zeta-1}, \Gamma u^*) + \mathbb{Q}(v^*, \delta_\zeta)}{2} \end{array}\right\}.
\end{aligned} \tag{3.9}
$$

Since $\beta \in \mho$, from ($\Phi 1$), letting $\zeta \to \infty$ in (3.8) and applying (3.9), we obtain

$$\mathbb{Q}\big(v^*, \Gamma u^*\big) < \mathbb{Q}\big(v^*, \Gamma u^*\big),$$

which is a discrepancy. Therefore, $\mathbb{Q}(v^*, \Gamma u^*) = 0$, which implies that $v^* \in \Gamma u^*$. Thus, $v^* = \Im u^* \in \Gamma u^*$, and hence $\Gamma$ and $\Im$ have a coincidence point $u^*$, and $v^*$ is a point of coincidence of $\Gamma$ and $\Im$. By ($\mathbb{Q}1$), we have $\mathbb{Q}(v^*, v^*) = 0$. Postulating that $v_1^*$ is another point of coincidence of $\Gamma$ and $\Im$ such that we can find $u_1^* \in \Upsilon$, such that $v_1^* = \Im u_1^* \in \Gamma u_1^*$ and by ($\mathbb{Q}1$), $\mathbb{Q}(v_1^*, v_1^*) = 0$. Now, we prove that $\mathbb{Q}(v^*, v_1^*) = 0$ by contrast. Assume that $\mathbb{Q}(v^*, v_1^*) > 0$, from (3.1)

$$
\begin{aligned}
\Lambda\big(\mathbb{Q}\big(v^*, v_1^*\big)\big) &\le \Lambda\big(\mathbb{Q}\big(\Gamma u^*, \Gamma u_1^*\big)\big) \le \Lambda\big(H_{\mathbb{Q}}\big(\Gamma u^*, \Gamma u_1^*\big)\big) \\
&\le \psi\big(\Lambda\big[\beta\big(\aleph_\Im\big(u^*, u_1^*\big)\big)\aleph_\Im\big(u^*, u_1^*\big)\big]\big),
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\aleph_\Im\big(u^*, u_1^*\big) &= \max\left\{\begin{array}{c} \mathbb{Q}(\Im u^*, \Im u_1^*), \mathbb{Q}(\Im u^*, \Gamma u^*), \\ \mathbb{Q}(\Im u_1^*, \Gamma u_1^*), \frac{\mathbb{Q}(\Im u^*, \Gamma u_1^*) + \mathbb{Q}(\Gamma u^*, \Im u_1^*)}{2} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \mathbb{Q}(v^*, v_1^*), \mathbb{Q}(v^*, v^*), \mathbb{Q}(v_1^*, v_1^*), \\ \frac{\mathbb{Q}(v^*, v_1^*) + \mathbb{Q}(v^*, v_1^*)}{2} \end{array}\right\} \\
&= \mathbb{Q}\big(v^*, v_1^*\big).
\end{aligned} \tag{3.11}
$$

Since $\beta \in \mho$, from ($\Phi$1), (3.10), and (3.11), we obtain $\mathbb{Q}(\nu^*, \nu_1^*) < \mathbb{Q}(\nu^*, \nu_1^*)$, which is a discrepancy. Therefore, $\mathbb{Q}(\nu^*, \nu_1^*) = 0$ implies that $\nu^* = \nu_1^*$. Thus, $\Gamma$ and $\Im$ have a unique point of coincidence. Moreover, since $\Gamma$ and $\Im$ are weakly compatible, we have $\Im\nu^* = \Gamma\nu^*$. Now, let $w = \Im\nu^* \in \Gamma\nu^*$. From the uniqueness of the point of coincidence, we have $w = \nu = \Im\nu^* \in \Gamma\nu^*$. Therefore, $\Gamma$ and $\Im$ have a unique common fixed point. □

**Corollary 3.2** *Let* $(\Upsilon, \mathbb{Q})$ *be an F-complete F-MS,* $\alpha : \Upsilon \times \Upsilon \to [0, \infty)$ *be a function. A mapping* $\Gamma : \Upsilon \to \Upsilon$ *is called an improved* $\alpha$*-Geraghty contraction mapping if there exist* $\beta \in \mho$ *such that for all* $\gamma, \delta \in \Upsilon$,

$$\alpha(\gamma,\delta)\mathbb{Q}(\Gamma\gamma,\Gamma\delta) \le \beta\big(\aleph(\gamma,\delta)\big).\aleph(\gamma,\delta),$$

*where*

$$\aleph(\gamma,\delta) = \max\left\{\mathbb{Q}(\gamma,\delta), \mathbb{Q}(\gamma,\Gamma\gamma), \mathbb{Q}(\delta,\Gamma\delta), \frac{\mathbb{Q}(\gamma,\Gamma\delta)+\mathbb{Q}(\delta,\Gamma\gamma)}{2}\right\},$$

*for all* $\gamma, \delta \in \Upsilon$, *satisfying the following stipulations*:

(1) $\Gamma$ *is an improved* $\alpha$*-Geraghty contraction*;
(2) $\Gamma$ *is triangular* $\alpha$*-orbital admissible*;
(3) *there exists an* $\gamma_0 \in \Upsilon$ *such that* $\alpha(\gamma_0, \Gamma\gamma_0) \ge 1$;
(4) $\Gamma$ *is continuous*.

Then, $\Gamma$ has a unique fixed point $\gamma^* \in \Upsilon$.

**Corollary 3.3** *Let* $(\Upsilon, \mathbb{Q})$ *be an F-MS, and* $\Gamma, \Im : \Upsilon \to \Upsilon$ *be two mappings with* $\Gamma\Upsilon \subseteq \Im\Upsilon$ *and* $\Im\Upsilon$ *is F-complete. The pair* $(\Gamma, \Im)$ *is an improved Geraghty contraction if there exists* $\beta \in \mho$ *such that for all* $\gamma, \delta \in \Upsilon$,

$$\mathbb{Q}(\Gamma\gamma,\Gamma\delta) \le \beta\big(\aleph_\Im(\gamma,\delta)\big).\aleph_\Im(\gamma,\delta),$$

*where*

$$\aleph_\Im(\gamma,\delta) = \max\left\{\mathbb{Q}(\Im\gamma,\Im\delta), \mathbb{Q}(\Im\gamma,\Gamma\gamma), \mathbb{Q}(\Im\delta,\Gamma\delta), \frac{\mathbb{Q}(\Im\gamma,\Gamma\delta)+\mathbb{Q}(\Im\delta,\Gamma\gamma)}{2}\right\}.$$

*Then,* $\Gamma$ *and g have a unique point of coincidence. Indeed, if* $\Gamma$ *and* $\Im$ *are weakly compatible, then* $\Gamma$ *and* $\Im$ *have a unique common fixed point* $\gamma^* \in \Upsilon$.

*Example* 3.4 Let $\Upsilon = [0, \infty)$ and $F$-M $\mathbb{Q}$ given by

$$\mathbb{Q}(\gamma,\delta) = \begin{cases} e^{|\gamma-\delta|} & \text{if } \gamma \neq \delta, \\ 0 & \text{if } \gamma = \delta, \end{cases}$$

with $\mathcal{L}(\varsigma) = \frac{-1}{\varsigma}$ and $a = 1$. Then, $(\Upsilon, \mathbb{Q})$ is $F$-complete $F$-MS. Define $\Im : \Upsilon \to \Upsilon$ and $\Gamma : \Upsilon \to CB(\Upsilon)$ by

$$\Gamma\gamma = \begin{cases} \{\frac{\gamma}{8}\}, & \text{if } \gamma \in \mathbb{N}\cup\{0\}, \\ \{0\} & \text{otherwise,} \end{cases} \quad \text{and} \quad \Im\gamma = \begin{cases} \frac{3\gamma}{2} & \text{if } \gamma \in \mathbb{N}\cup\{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for all $\gamma \in \mathbb{N} \cup \{0\}$, $\Gamma(\Upsilon) \subseteq \Im(\Upsilon)$ and $\Im(\Upsilon)$ is an $F$-complete subset of $\Upsilon$; let $\beta : \Upsilon \times \Upsilon \to [0,1)$ be as $\beta(\gamma,\delta) = \frac{1}{2}$, $\Lambda(t) = t$ and $\psi(t) = \frac{2}{3}t$. Now, for all $(\gamma,\delta) \in \mathbb{N} \cup \{0\}$ with $\gamma \neq \delta$, then

$$\begin{aligned}
\Lambda\bigl(H_{\mathbb{Q}}(\Gamma\gamma,\Gamma\delta)\bigr) &= \Lambda\bigl(\max\bigl(\sup_{a\in\Gamma\gamma}\mathbb{Q}(a,\Gamma\delta), \sup_{b\in\Gamma\delta}\mathbb{Q}(\Gamma\gamma,b)\bigr)\bigr) \\
&= \Lambda\left(\max\left(\sup_{a\in\Gamma\gamma}\mathbb{Q}\left(a,\left\{\frac{\delta}{8}\right\}\right), \sup_{b\in\Gamma\delta}\mathbb{Q}\left(\left\{\frac{\gamma}{8}\right\},b\right)\right)\right) \\
&= \Lambda\left(\max\left(\mathbb{Q}\left(\frac{\gamma}{8},\left\{\frac{\delta}{8}\right\}\right),\mathbb{Q}\left(\left\{\frac{\gamma}{8}\right\},\frac{\delta}{8}\right)\right)\right) \\
&= \Lambda\left(\max\left(\mathbb{Q}\left(\frac{\gamma}{8},\frac{\delta}{8}\right),\mathbb{Q}\left(\frac{\gamma}{8},\frac{\delta}{8}\right)\right)\right) \\
&= \Lambda\left(\mathbb{Q}\left(\frac{\gamma}{8},\frac{\delta}{8}\right)\right) \\
&= \Lambda\bigl(e^{|\frac{\gamma}{8}-\frac{\delta}{8}|}\bigr) = \Lambda\bigl(e^{\frac{1}{4}|\frac{\gamma}{2}-\frac{\delta}{2}|}\bigr) \\
&\leq \frac{2}{3}\Lambda\left(\frac{1}{2}e^{|\frac{3\gamma}{2}-\frac{3\delta}{2}|}\right) \\
&\leq \psi\bigl(\Lambda\bigl[\beta\bigl(\aleph_{\Im}(\gamma,\delta)\bigr).\aleph_{\Im}(\gamma,\delta)\bigr]\bigr).
\end{aligned}$$

If $\gamma = \delta$, then we have

$$\Lambda\bigl(H_{\mathbb{Q}}(\Gamma\gamma,\Gamma\delta)\bigr) = 0 \leq \psi\bigl(\Lambda\bigl[\beta\bigl(\aleph_{\Im}(\gamma,\delta)\bigr).\aleph_{\Im}(\gamma,\delta)\bigr]\bigr).$$

Otherwise, we have that (3.1) trivially holds. Therefore, all stipulations of Theorem 3.1 are satisfied. Since $\Upsilon 0 = \Im 0 = 0$, thus $\gamma = 0$ is a common fixed point of $\Gamma$ and $\Im$.

## 4 Application for the existence of a solution to a functional equation

In this section, we use our main results to verify the existence and uniqueness of a solution to the functional equation:

$$\varrho(\gamma) = \sup_{\delta\in\Upsilon}\bigl\{\digamma(\gamma,\delta) + \Pi\bigl(\gamma,\delta,\varrho\bigl(\mu(\gamma,\delta)\bigr)\bigr)\bigr\}, \quad \gamma \in \eth, \tag{4.1}$$

where $\digamma : \eth \times \Upsilon \to \mathbb{R}$ and $\Pi : \eth \times \Upsilon \times \mathbb{R} \to \mathbb{R}$ are bounded, $\mu : \eth \times \Upsilon \to \eth$, $\eth$ and $\Upsilon$ are BSs. Equations of the type (4.1) have applications in mathematical optimization, computer programming, and in dynamic programming, giving tools for solutions to boundary value problems arising in engineering and physical sciences. Bhakta and Mitra [13] introduced the existence theorems that proved the existence and uniqueness of the solution of a functional equation under certain conditions in Banach spaces. Deepmala [15] utilized the fixed-point theorems to establish the existence, uniqueness, and iterative approximation of the solution for a functional equation in Banach spaces and complete metric spaces. In [30, 32], common solutions of certain functional equations arising in dynamic programming and common fixed-point theorems for a quadruple of self-mappings satisfying weak compatibility and $JH$-operator pairs on a complete metric space were discussed.

Let $\text{Ⱡ}(\eth)$ denote the set of all bounded real-valued functions on $\eth$. The pair $(\text{Ⱡ}(\eth), \|.\|)$, where $\|h\| = \sup_{\varsigma\in\eth} |h(\varsigma)|,\ h \in \text{Ⱡ}(\eth)$, is a BS along with the metric $\mathbb{Q}$ given by

$$\mathbb{Q}(h,k) = \sup_{\varsigma\in\eth}\big|h(\varsigma) - k(\varsigma)\big| = \|h - k\|.$$

To show the existence of a solution to (4.1), we put in place the following stipulations:
($S1$) $F$ and $\Pi$ are bounded,
($S2$) for all $h \in \text{Ⱡ}(\eth)$ and $\gamma \in \eth$, we define the operator $\Gamma : \text{Ⱡ}(\eth) \to \text{Ⱡ}(\eth)$ as

$$(\Gamma h)(\gamma) = \sup_{\delta\in\Upsilon}\big\{F(\gamma,\delta) + \Pi\big(\gamma,\delta,h\big(\mu(\gamma,\delta)\big)\big)\big\}. \tag{4.2}$$

Undoubtedly, $\Gamma$ is well defined since $F$ and $\Pi$ are bounded,
($S3$) for $a > 1$, $(\gamma,\delta) \in \eth \times \Upsilon,\ h,k \in \text{Ⱡ}(\eth)$ and $\varsigma \in \eth$, we have

$$\big|\Pi\big(\gamma,\delta,h(\varsigma)\big) - \Pi\big(\gamma,\delta,k(\varsigma)\big)\big| \le e^{-a}\aleph(h,k), \tag{4.3}$$

where

$$\aleph(h,k) = \max\left\{\mathbb{Q}(h,k), \mathbb{Q}(h,\Gamma h), \mathbb{Q}(k,\Gamma k), \frac{\mathbb{Q}(k,\Gamma h) + \mathbb{Q}(h,\Gamma k)}{2}\right\}.$$

We shall verify the following theorem.

**Theorem 4.1** *Postulate that the stipulations* ($S1$) – ($S3$) *hold, then the functional Eq.* (4.1) *has a bounded solution.*

*Proof* Let $\lambda > 0$ be arbitrary, $\gamma \in \eth$ and $h,k \in \text{Ⱡ}(\eth)$. The space $(\text{Ⱡ}(\eth), \mathbb{Q})$ is an $F$-complete $F$-MS. There exist $\delta_1, \delta_2 \in \Upsilon$ such that

$$(\Gamma h)(\gamma) < F(\gamma,\delta_1) + \Pi\big(\gamma,\delta_1,h\big(\mu(\gamma,\delta_1)\big)\big) + \lambda, \tag{4.4}$$

$$(\Gamma k)(\gamma) < F(\gamma,\delta_2) + \Pi\big(\gamma,\delta_2,k\big(\mu(\gamma,\delta_2)\big)\big) + \lambda, \tag{4.5}$$

$$(\Gamma h)(\gamma) \ge F(\gamma,\delta_2) + \Pi\big(\gamma,\delta_2,h\big(\mu(\gamma,\delta_2)\big)\big), \tag{4.6}$$

$$(\Gamma k)(\gamma) \ge F(\gamma,\delta_1) + \Pi\big(\gamma,\delta_1,k\big(\mu(\gamma,\delta_1)\big)\big). \tag{4.7}$$

Then from (4.4) and (4.7), we obtain

$$\begin{aligned}(\Gamma h)(\gamma) - (\Gamma k)(\gamma) &< \Pi\big(\gamma,\delta_1,h\big(\mu(\gamma,\delta_1)\big)\big) - \Pi\big(\gamma,\delta_1,k\big(\mu(\gamma,\delta_1)\big)\big) + \lambda \\ &\le \big|\Pi\big(\gamma,\delta_1,h\big(\mu(\gamma,\delta_1)\big)\big) - \Pi\big(\gamma,\delta_1,k\big(\mu(\gamma,\delta_1)\big)\big)\big| + \lambda \\ &\le e^{-a}\aleph(h,k) + \lambda. \end{aligned} \tag{4.8}$$

Similarly from (4.5) and (4.6), we obtain

$$\begin{aligned}(\Gamma k)(\gamma) - (\Gamma h)(\gamma) &< \Pi\big(\gamma,\delta_2,k\big(\mu(\gamma,\delta_2)\big)\big) - \Pi\big(\gamma,\delta_2,h\big(\mu(\gamma,\delta_2)\big)\big) + \lambda \\ &\le \big|\Pi\big(\gamma,\delta_2,k\big(\mu(\gamma,\delta_2)\big)\big) - \Pi\big(\gamma,\delta_2,h\big(\mu(\gamma,\delta_2)\big)\big)\big| + \lambda\end{aligned}$$

$$\leq e^{-a}\aleph(h,k) + \lambda. \tag{4.9}$$

Combining (4.8) and (4.9), we obtain

$$\left|(\Gamma h)(\gamma) - (\Gamma k)(\gamma)\right| \leq e^{-a}\aleph(h,k) + \lambda,$$

which implies for $\lambda > 0$ and $\gamma \in \eth$ such that

$$e \times \mathbb{Q}(\Gamma h, \Gamma k) \leq \frac{1}{e^{a-1}}\aleph(h,k).$$

Taking $\alpha(h,k) = e \geq 1$ and $\beta(h,k) = \frac{1}{e^{a-1}} \in [0,1)$, we have

$$\alpha(h,k)\mathbb{Q}(\Gamma h, \Gamma k) \leq \beta\big(\aleph(h,k)\big)\aleph(h,k).$$

All the stipulations of Corollary 3.2 are fulfilled, and $\Gamma$ has a unique fixed point, so Eq. (4.1) has a bounded solution. □

*Example* 4.2 Let $\eth = \Upsilon = \mathbb{R}$ be a BS with the standard norm $\|\gamma\| = |\gamma|$, for all $\gamma \in \eth$. Postulating that $S = [0,1] \subseteq \eth$ is the state space and $D = [0,\infty) \subseteq \Upsilon$ the decision space. Define $\mu : S \times D \to S$ and $F : S \times D \to \mathbb{R}$ by

$$\mu(\gamma,\delta) = \frac{\gamma\delta^2}{1+\delta^2} \quad \text{and} \quad F(\gamma,\delta) = 0, \quad \forall \gamma \in S, \text{ and } \delta \in D.$$

Define $\varrho : S \to \mathbb{R}$ by

$$\varrho(\gamma) = \frac{1}{32}A(\gamma), \quad A(\gamma) \in Ł(S) \text{ and } \gamma \in S.$$

Now, for all $h,k \in Ł(S)$, $\gamma \in S$, we define a map $\Gamma : Ł(S) \to Ł(S)$ as,

$$\Gamma h(\gamma) = \sup_{\delta\in D}\big\{F(\gamma,\delta) + \Pi\big(\gamma,\delta,h\big(\mu(\gamma,\delta)\big)\big)\big\},$$
$$\Gamma k(\gamma) = \sup_{\delta\in D}\big\{F(\gamma,\delta) + \Pi\big(\gamma,\delta,k\big(\mu(\gamma,\delta)\big)\big)\big\},$$

where $\Pi : S \times D \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\Pi(\gamma,\delta,\varsigma) = \frac{1}{32}\varsigma \sin\left(\frac{\delta}{\delta+2}\right).$$

Hence,

$$\begin{aligned}\Gamma h(\gamma) &= \sup_{\delta\in D}\big\{F(\gamma,\delta) + \Pi\big(\gamma,\delta,h\big(\mu(\gamma,\delta)\big)\big)\big\} \\ &= \sup_{\delta\in D}\Pi\left(\gamma,\delta,h\left(\frac{\gamma\delta^2}{1+\delta^2}\right)\right) \\ &= \sup_{\delta\in D}\frac{1}{32}\left[h\left(\frac{\gamma\delta^2}{1+\delta^2}\right)\sin\left(\frac{\delta}{\delta+2}\right)\right]\end{aligned}$$

$$= \frac{1}{32}h(\gamma) = \varrho_1(\gamma); \quad \gamma \in S, h \in Ł(S).$$

Similarly,

$$\Gamma k(\gamma) = \frac{1}{32}k(\gamma) = \varrho_2(\gamma); \quad \gamma \in S, k \in Ł(S).$$

Note that $\Pi$ and $F$ are bounded; this implies that stipulations $(S_1)$ and $(S_2)$ of Theorem 4.1 are satisfied. Now,

$$\begin{aligned}\left|\Pi\big(\gamma,\delta,h(\varsigma)\big) - \Pi\big(\gamma,\delta,k(\varsigma)\big)\right| &= \left|\frac{1}{32}h(\varsigma)\sin\left(\frac{\delta}{\delta+2}\right) - \frac{1}{32}h(\varsigma)\sin\left(\frac{\delta}{\delta+2}\right)\right| \\ &= \frac{1}{32}\left|\sin\left(\frac{\delta}{\delta+2}\right)\right|\left|h(\varsigma)-k(\varsigma)\right| \\ &\le \frac{1}{32}\left|h(\varsigma)-k(\varsigma)\right| \\ &\le \frac{1}{e^2}\|h-k\|.\end{aligned}$$

Thus, all the assertions of Theorem 4.1 are satisfied and the functional Eq. (4.1) has a bounded solution in $Ł(S)$.

## 5 Conclusion

In this paper, we introduced a new notion of $\alpha_*$-$\psi$-$\Lambda$-contraction multivalued mappings and proved some novel fixed-point theorems for such contraction in $F$-MSs. Some consequences are studied to investigate coincidence-point results for this contraction in $F$-MSs. Also, we gave some examples to clarify our obtained results; we utilized the main results to discuss the existence and uniqueness of a solution to a functional equation. The new concepts lead to further investigations and applications.

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