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Some hyperstability and stability results for the Cauchy and Jensen equations

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Abstract

In this paper we give some hyperstability and stability results for the Cauchy and Jensen functional equations on restricted domains. We provide a simple and short proof for Brzdęk's result concerning a hyperstability result for the Cauchy equation.

MSC: 39B62; 39B82; 47H14

Keywords: Cauchy functional equation; Additive function; Restricted domain

1 Introduction

Let V and W be linear spaces. A function $f : V \to W$ is called

- additive if $f(x + y) = f(x) + f(y)$ for all $x, y \in V$;
- Jensen if $2f(\frac{x+y}{2}) = f(x) + f(y)$ for all $x, y \in V$.

The main motivation for the investigation of the stability of functional equations originated from a question of Ulam [21] concerning the stability of group homomorphisms. Hyers [9] gave an affirmative answer to the question of Ulam. The stability and hyperstability problems for various functional equations have been investigated by numerous mathematicians. For more information on this area of research and further references, see [1, 2, 4, 7, 10, 11, 13–15, 20].

Let us state the following theorem that is one of the classical results concerning the stability problem for the Cauchy functional equation $f(x + y) = f(x) + f(y)$.

Theorem 1.1 ([3, 5, 8, 9, 18, 19]) *Let $\varepsilon \geq 0$ and $f : X \to Y$, where X is a normed space and Y is a Banach space. Let $p \neq 1$ be a real number and*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon\left(\|x\|^p + \|y\|^p\right), \quad x, y \in X \setminus \{0\}.$$

Then, there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p, \quad x \in X \setminus \{0\}.$$

Rassias [16, 17] considered the case $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon\|x\|^p\|y\|^q$, where p, q are real numbers with $p + q \in [0, 1)$. Brzdęk [6, Theprem 1.3] provided a complement for this result in the case $p + q < 0$. His proof was based on a fixed-point theorem. We provide a simple and short proof for Brzdęk's result. In addition, some further results on the hyperstability of the Cauchy and Jensen functional equations are investigated.

2 Superstability

Denote by $\mathbb{N}$ the set of positive integers. A version of the following theorem is introduced by Brzdęk [6, Theorem 1.3] and its proof is based on a fixed-point theorem. A simple and brief proof is given here.

Theorem 2.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and $E \subseteq \mathcal{X} \setminus \{0\}$ be a nonempty set. Take $\varepsilon \geq 0$ and let p, q be real numbers with $p + q < 0$. Assume that for each $x \in E$ there exists a positive integer m_x such that $nx \in E$ for all $n \in \mathbb{N}$ with $n \geq m_x$. Then, every function $f : \mathcal{X} \to \mathcal{Y}$ satisfying the inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \varepsilon \|x\|^p \|y\|^q, \quad x, y \in E, x + y \in E, \tag{2.1}$$

is additive on E, that is

$$f(x+y) = f(x) + f(y), \quad x, y \in E, x + y \in E.$$

Proof Without loss of generality, we may assume that $q < 0$. Let $x, y \in E$ with $x + y \in E$. By assumption, there exists a positive integer m such that $nx, ny, n(x+y) \in E$ for all $n \geq m$. Then, (2.1) yields

$$\begin{aligned}
&\left\| f(x+nx) - f(x) - f(nx) \right\| \leq \varepsilon n^q \|x\|^{p+q}, \\
&\left\| f(y+ny) - f(y) - f(ny) \right\| \leq \varepsilon n^q \|y\|^{p+q}, \\
&\left\| f\bigl(x+y+n(x+y)\bigr) - f(x+y) - f\bigl(n(x+y)\bigr) \right\| \leq \varepsilon n^q \|x+y\|^{p+q}.
\end{aligned}$$

Letting $n \to \infty$ in the above inequalities, we obtain

$$\begin{aligned}
&f(x) = \lim_{n\to\infty} \bigl[f(x+nx) - f(nx)\bigr], \\
&f(y) = \lim_{n\to\infty} \bigl[f(y+ny) - f(ny)\bigr], \\
&f(x+y) = \lim_{n\to\infty} \bigl[f\bigl(x+y+n(x+y)\bigr) - f\bigl(n(x+y)\bigr)\bigr].
\end{aligned}$$

Then,

$$\begin{aligned}
&\left\| f(x+y) - f(x) - f(y) \right\| \\
&\quad = \lim_{n\to\infty} \Bigl\| \bigl[f\bigl((n+1)(x+y)\bigr) - f\bigl(n(x+y)\bigr)\bigr] \\
&\qquad - \bigl[f\bigl((n+1)x\bigr) - f(nx)\bigr] - \bigl[f\bigl((n+1)y\bigr) - f(ny)\bigr] \Bigr\| \\
&\quad \leq \limsup_{n\to\infty} \left\| f\bigl((n+1)(x+y)\bigr) - f\bigl((n+1)x\bigr) - f\bigl((n+1)y\bigr) \right\| \\
&\qquad + \limsup_{n\to\infty} \left\| f\bigl(n(x+y)\bigr) - f(nx) - f(ny) \right\| \quad \text{(by (2.1))}
\end{aligned}$$

$$\le \limsup_{n\to\infty} \varepsilon\big[(n+1)^{p+q} + n^{p+q}\big]\|x\|^p\|y\|^q = 0.$$

Hence, $f(x+y) = f(x) + f(y)$ for all $x, y \in E$ with $x + y \in E$. This completes the proof. □

Remark 2.2 The assumption $p + q < 0$ is necessary in Theorem 2.1. For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ fulfilling $|f(x+y) - f(x) - f(y)| = 2|x||y|$ for all $x, y \in \mathbb{R}$. However, f is not additive.

The following theorem states a hyperstability result for the Jensen functional equation on a restricted domain.

Theorem 2.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and $E \subseteq \mathcal{X} \setminus \{0\}$ be a nonempty set. Take $\varepsilon \ge 0$ and let p, q be real numbers with $p + q < 0$. Assume that for each $x \in E$ there exists a positive integer m_x such that $\frac{nx}{2} \in E$ for all $n \in \mathbb{N}$ with $n \ge m_x$. Then, every function $f : \mathcal{X} \to \mathcal{Y}$ satisfying the inequality*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \le \varepsilon\|x\|^p\|y\|^q, \quad x, y \in E, \frac{x+y}{2} \in E, \tag{2.2}$$

is Jensen on E, that is

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad x, y \in E, \frac{x+y}{2} \in E.$$

Proof Without loss of generality, we may assume that $q < 0$. Let $x, y \in E$ with $\frac{x+y}{2} \in E$. By assumption, there exists a positive integer m such that $\{\frac{nx}{2}, \frac{nx}{2}, \frac{n(x+y)}{4}\} \subseteq E$ for all $n \ge m$. Then, (2.2) yields

$$\begin{aligned}
&\left\|2f\left(\frac{x+nx}{2}\right) - f(x) - f(nx)\right\| \le \varepsilon n^q\|x\|^{p+q}, \\
&\left\|2f\left(\frac{y+ny}{2}\right) - f(y) - f(ny)\right\| \le \varepsilon n^q\|y\|^{p+q}, \\
&\left\|2f\left(\frac{x+y+n(x+y)}{4}\right) - f\left(\frac{x+y}{2}\right) - f\left(\frac{n(x+y)}{2}\right)\right\| \le \varepsilon n^q\left\|\frac{x+y}{2}\right\|^{p+q}.
\end{aligned}$$

Letting $n \to \infty$ in the above inequalities, we obtain

$$\begin{aligned}
f(x) &= \lim_{n\to\infty}\left[2f\left(\frac{x+nx}{2}\right) - f(nx)\right], \\
f(y) &= \lim_{n\to\infty}\left[2f\left(\frac{y+ny}{2}\right) - f(ny)\right], \\
f\left(\frac{x+y}{2}\right) &= \lim_{n\to\infty}\left[2f\left(\frac{x+y+n(x+y)}{4}\right) - f\left(\frac{n(x+y)}{2}\right)\right].
\end{aligned}$$

Then,

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \\
&\quad = \lim_{n\to\infty}\left\|\left[4f\left(\frac{x+y+n(x+y)}{4}\right) - 2f\left(\frac{n(x+y)}{2}\right)\right] - \left[2f\left(\frac{x+nx}{2}\right) - f(nx)\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \left[2f\left(\frac{y+ny}{2}\right) - f(ny)\right]\Bigg\| \\
&\le 2\limsup_{n\to\infty}\left\|2f\left(\frac{x+y+n(x+y)}{4}\right) - f\left(\frac{x+nx}{2}\right) - f\left(\frac{y+ny}{2}\right)\right\| \\
&\quad + \limsup_{n\to\infty}\left\|2f\left(\frac{n(x+y)}{2}\right) - f(nx) - f(ny)\right\| \quad \text{(by (2.2))} \\
&\le \limsup_{n\to\infty}\varepsilon\left[2\left(\frac{n+1}{2}\right)^{p+q} + n^{p+q}\right]\|x\|^p\|y\|^q = 0.
\end{aligned}
$$

Therefore, $2f(\frac{x+y}{2}) = f(x) + f(y)$ for all $x, y \in E$ with $\frac{x+y}{2} \in E$. This ends the proof. □

Example 2.4 Let $E = [1, +\infty)$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. It is easy to see that

$$
\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right| = \frac{|x-y|^2}{2} \le |x|^2|y|^2, \quad x, y \in E.
$$

Then, f satisfies (2.2) with $p + q > 0$. However, f is not Jensen on E.

In the following, we obtain other hyperstability results for the Cauchy and Jensen functional equations.

Theorem 2.5 *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and $E \subseteq \mathcal{X} \setminus \{0\}$ be a nonempty set. Take $\theta, \varepsilon \ge 0$ and let p, q, r be real numbers with $p + q + r < 0$ and $p + q + 2r < 0$. Assume that for each $x \in E$ there exists a positive integer m_x such that $nx \in E$ for all $n \in \mathbb{N}$ with $n \ge m_x$. Then, every function $f : \mathcal{X} \to \mathcal{Y}$ satisfying the inequality*

$$
\left\|f(x+y) - f(x) - f(y)\right\| \le \|x\|^p\|y\|^q\left(\varepsilon\|x+y\|^r + \theta\|x-y\|^r\right), \quad x, y \in E, x + y \in E, \tag{2.3}
$$

is additive on E.

Proof Put

$$
\varphi(x, y) := \|x\|^p\|y\|^q\left(\varepsilon\|x+y\|^r + \theta\|x-y\|^r\right).
$$

Since $p + q + 2r < 0$, we may assume that $q + r < 0$ without loss of generality. Let $x, y \in E$ with $x + y \in E$. By assumption, there exists a positive integer m such that $nx, ny, n(x+y) \in E$ for all $n \ge m$. By a similar argument as in the proof of Theorem 2.1, we obtain

$$
\begin{aligned}
f(x) &= \lim_{n\to\infty}\left[f(x+nx) - f(nx)\right], \\
f(y) &= \lim_{n\to\infty}\left[f(y+ny) - f(ny)\right], \\
f(x+y) &= \lim_{n\to\infty}\left[f\left(x+y+n(x+y)\right) - f\left(n(x+y)\right)\right].
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\left\|f(x+y) - f(x) - f(y)\right\| \\
&\quad = \lim_{n\to\infty}\left\|\left[f\left((n+1)(x+y)\right) - f\left(n(x+y)\right)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \big[f\big((n+1)x\big) - f(nx)\big] - \big[f\big((n+1)y\big) - f(ny)\big]\big\| \\
&\le \limsup_{n\to\infty} \big\|f\big((n+1)(x+y)\big) - f\big((n+1)x\big) - f\big((n+1)y\big)\big\| \\
&\quad + \limsup_{n\to\infty} \big\|f\big(n(x+y)\big) - f(nx) - f(ny)\big\| \quad \text{(by (2.3))} \\
&\le \limsup_{n\to\infty} \big[(n+1)^{p+q+r} + n^{p+q+r}\big]\varphi(x,y) = 0.
\end{aligned}
$$

Hence, $f(x+y) = f(x) + f(y)$ for all $x, y \in E$ with $x + y \in E$. This completes the proof. □

Example 2.6 Let $E = [1, +\infty)$ and f be a function defined by $f(x) = x^3$. It is clear that

$$
\big|f(x+y) - f(x) - f(y)\big| = 3|x||y||x+y| \le 3|x||y|\big(|x+y| + |x-y|\big), \quad x, y \in E.
$$

Then, f satisfies (2.3) with $p = q = r = 1$. However, f is not additive on E.

Theorem 2.7 *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces, and $E \subseteq \mathcal{X} \setminus \{0\}$ be a nonempty set. Take $\theta, \varepsilon \ge 0$ and let p, q, r be real numbers with $p + q + r < 0$ and $p + q + 2r < 0$. Assume that for each $x \in E$ there exists a positive integer m_x such that $\frac{nx}{2} \in E$ for all $n \in \mathbb{N}$ with $n \ge m_x$. Suppose that a function $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$
\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \|x\|^p \|y\|^q \big(\varepsilon \|x+y\|^r + \theta \|x-y\|^r\big) \tag{2.4}
$$

for all $x, y \in E$ with $\frac{x+y}{2} \in E$. Then, f is Jensen on E.

Proof Put

$$
\varphi(x,y) := \|x\|^p \|y\|^q \big(\varepsilon \|x+y\|^r + \theta \|x-y\|^r\big).
$$

Without loss of generality we may assume that $q + r < 0$. Let $x, y \in E$ with $\frac{x+y}{2} \in E$. By assumption, there exists a positive integer m such that $\{\frac{nx}{2}, \frac{ny}{2}, \frac{n(x+y)}{4}\} \subseteq E$ for all $n \ge m$. By a similar argument as in the proof of Theorem 2.3, we obtain

$$
\begin{aligned}
f(x) &= \lim_{n\to\infty} \left[2f\left(\frac{x+nx}{2}\right) - f(nx)\right], \\
f(y) &= \lim_{n\to\infty} \left[2f\left(\frac{y+ny}{2}\right) - f(ny)\right], \\
f\left(\frac{x+y}{2}\right) &= \lim_{n\to\infty} \left[2f\left(\frac{x+y+n(x+y)}{4}\right) - f\left(\frac{n(x+y)}{2}\right)\right].
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \\
&\quad = \lim_{n\to\infty} \left\| \left[4f\left(\frac{x+y+n(x+y)}{4}\right) - 2f\left(\frac{n(x+y)}{2}\right)\right] - \left[2f\left(\frac{x+nx}{2}\right) - f(nx)\right] \right. \\
&\qquad \left. - \left[2f\left(\frac{y+ny}{2}\right) - f(ny)\right] \right\|
\end{aligned}
$$

$$\begin{aligned}
&\leq 2\limsup_{n\to\infty}\left\|2f\left(\frac{x+y+n(x+y)}{4}\right)-f\left(\frac{x+nx}{2}\right)-f\left(\frac{y+ny}{2}\right)\right\| \\
&\quad+\limsup_{n\to\infty}\left\|2f\left(\frac{n(x+y)}{2}\right)-f(nx)-f(ny)\right\| \quad \text{(by (2.4))} \\
&\leq \limsup_{n\to\infty}\left[2\left(\frac{n+1}{2}\right)^{p+q+r}+n^{p+q+r}\right]\varphi(x,y)=0.
\end{aligned}$$

Therefore, $2f(\frac{x+y}{2})=f(x)+f(y)$ for all $x,y\in E$ with $\frac{x+y}{2}\in E$. This completes the proof. □

Example 2.8 Let $E=[1,+\infty)$ and f be a function defined by $f(x)=x^2$. It is clear that

$$\left|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right|=\frac{|x-y|^2}{2}\leq |x|^2|y|^2\big(|x+y|+|x-y|\big), \quad x,y\in E.$$

Then, f satisfies (2.4) with $p=q=2$ and $r=1$. However, f is not Jensen on E.

Theorem 2.9 *Assume that X is a linear space over the field $\mathbb{F}$, and $\mathcal{Y}$ is a normed space over the field $\mathbb{K}$. Let $a,b\in\mathbb{F}\setminus\{0\}$ and $\varphi:X\times X\to[0,+\infty)$ be a function such that*

$$\lim_{m\to\infty}\varphi\big(a^{-1}(m+1)x,-b^{-1}mx\big)=0, \qquad \lim_{m\to\infty}\varphi(mx,my)=0 \tag{2.5}$$

for all $x,y\in X\setminus\{0\}$. Let $A,B\in\mathbb{K}$, $C\in\mathcal{Y}$ and $f:X\to\mathcal{Y}$ satisfy

$$\big\|f(ax+by)-Af(x)-Bf(y)-C\big\|\leq\varphi(x,y) \tag{2.6}$$

for all $x,y\in E_d=\{z\in X:\|z\|\geq d\}$ for some $d>0$. Then, f satisfies

$$f(ax+by)=Af(x)+Bf(y)+C, \tag{2.7}$$

for all $x,y\in X$. Moreover,

$$(A+B)f(0)=Af(x)+Bf\big(-ab^{-1}x\big) \tag{2.8}$$

for all $x\in X$.

Proof Replacing x by $a^{-1}(m+1)x$ and y by $-b^{-1}mx$ in (2.6), we obtain

$$\big\|f(x)-Af\big(a^{-1}(m+1)x\big)-Bf\big(-b^{-1}mx\big)-C\big\|\leq\varphi\big(a^{-1}(m+1)x,-b^{-1}mx\big), \tag{2.9}$$

for all $x\in X\setminus\{0\}$ and positive integers $m\geq n$, where $a^{-1}(n+1)x, b^{-1}nx\in E_d$. Letting $m\to\infty$ in (2.9) and using (2.5), we obtain

$$f(x)=\lim_{m\to\infty}\big[Af\big(a^{-1}(m+1)x\big)+Bf\big(-b^{-1}mx\big)+C\big], \quad x\in X\setminus\{0\}. \tag{2.10}$$

Let $x\in X\setminus\{0\}$, then (2.5) and (2.10) yield

$$\begin{aligned}
&\big\|(A+B)f(0)-Af(x)-Bf\big(-ab^{-1}x\big)\big\| \\
&\quad=\lim_{m\to\infty}\big\|(A+B)f(0)-A^2f\big(a^{-1}(m+1)x\big)-ABf\big(-b^{-1}mx\big)-AC
\end{aligned}$$

$$
\begin{aligned}
&- ABf\big(-b^{-1}(m+1)x\big) - B^2 f\big(ab^{-2}mx\big) - BC\big\| \\
&\le |A| \lim_{m\to\infty} \big\|f(0) - Af\big(a^{-1}(m+1)x\big) - Bf\big(-b^{-1}(m+1)x\big) - C\big\| \\
&\quad + |B| \lim_{m\to\infty} \big\|f(0) - Af\big(-b^{-1}mx\big) - Bf\big(ab^{-2}mx\big) - C\big\| \\
&\le |A| \lim_{m\to\infty} \varphi\big(a^{-1}(m+1)x, -b^{-1}(m+1)x\big) + |B| \lim_{m\to\infty} \varphi\big(-b^{-1}mx, ab^{-2}mx\big) = 0.
\end{aligned}
$$

Hence, we obtain

$$
(A+B)f(0) = Af(x) + Bf\big(-ab^{-1}x\big), \quad x \in X. \tag{2.11}
$$

If we replace x by bmx and y by $-amx$ in (2.6), we obtain

$$
\big\|f(0) - Af(bmx) - Bf(-amx) - C\big\| \le \varphi(bmx, -amx), \tag{2.12}
$$

for all $x \in X \setminus \{0\}$ and positive integers $m \ge n$, where $anx, bnx \in E_d$. Therefore,

$$
f(0) = \lim_{m\to\infty} \big[Af(bmx) + Bf(-amx) + C\big] \tag{2.13}
$$

for all $x \in X \setminus \{0\}$. Replacing x by bmx in (2.11) and letting $m \to \infty$, we obtain from (2.13) that

$$
(1 - A - B)f(0) = C.
$$

Therefore, (2.7) holds true for $x = y = 0$, and (2.10) holds for all $x \in X$. To prove (2.7), let $x, y \in X$ with $(x, y) \neq (0, 0)$. Then,

$$
\begin{aligned}
&\big\|f(ax+by) - Af(x) - Bf(y) - C\big\| \\
&= \lim_{m\to\infty} \big\|Af\big(a^{-1}(m+1)(ax+by)\big) + Bf\big(-b^{-1}m(ax+by)\big) \\
&\quad - A^2 f\big(a^{-1}(m+1)x\big) - ABf\big(-b^{-1}mx\big) - AC \\
&\quad - ABf\big(a^{-1}(m+1)y\big) - B^2 f\big(-b^{-1}my\big) - BC\big\| \\
&\le |A| \lim_{m\to\infty} \big\|f\big(a^{-1}(m+1)(ax+by)\big) - Af\big(a^{-1}(m+1)x\big) - Bf\big(a^{-1}(m+1)y\big) - C\big\| \\
&\quad + |B| \lim_{m\to\infty} \big\|f\big(-b^{-1}m(ax+by)\big) - Af\big(-b^{-1}mx\big) - Bf\big(-b^{-1}my\big) - C\big\| \\
&\le |A| \lim_{m\to\infty} \varphi\big(a^{-1}(m+1)x, -a^{-1}(m+1)y\big) + |B| \lim_{m\to\infty} \varphi\big(-b^{-1}mx, -b^{-1}my\big) = 0.
\end{aligned}
$$

Therefore, f satisfies (2.7) for all $x, y \in X$. □

In the following corollaries $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces.

Corollary 2.10 *Let $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $C \in Y$ and let $f : X \to Y$ be a function. Take $\theta, \varepsilon \ge 0$ and let p, q, r be real numbers. Then, f satisfies*

$$
f(ax+by) = Af(x) + Bf(y) + C, \qquad (A+B)f(0) = Af(x) + Bf\big(-ab^{-1}x\big), \quad x, y \in X
$$

if one of the following conditions holds:

(*i*) $p+q+r<0$ *and*

$$\|f(ax+by)-Af(x)-Bf(y)-C\| \le \|x\|^p\|y\|^q\big(\varepsilon\|x+y\|^r+\theta\|x-y\|^r\big);$$

(*ii*) $p+r<0, q+r<0$ *and*

$$\|f(ax+by)-Af(x)-Bf(y)-C\| \le \big(\|x\|^p+\|y\|^q\big)\big(\varepsilon\|x+y\|^r+\theta\|x-y\|^r\big);$$

(*iii*) $p,q<0$ *and*

$$\|f(ax+by)-Af(x)-Bf(y)-C\| \le \varepsilon\|x\|^p+\theta\|y\|^q;$$

for all $x,y\in E_d=\{z\in X:\|z\|\ge d\}$ *for some* $d>0$.

Corollary 2.11 *Every function* $f:X\to Y$ *satisfies one of the following assertions*:
(*i*) $f(ax+by)=Af(x)+Bf(y)+C, x,y\in X$.
(*ii*) $\limsup_{\min\{\|x\|,\|y\|\}\to\infty}\|f(ax+by)-Af(x)-Bf(y)-C\|\|x\|^r\|y\|^s=+\infty$ *for all real numbers* r, s *with* $r+s>0$.

Corollary 2.12 *Every function* $f:X\to Y$ *satisfies one of the following assertions*:
(*i*) $f(ax+by)=Af(x)+Bf(y)+C, x,y\in X$.
(*ii*) $\limsup_{\min\{\|x\|,\|y\|\}\to\infty}\frac{\|x\|^r\|y\|^s}{\|x\|^r+\|y\|^s}\|f(ax+by)-Af(x)-Bf(y)-C\|=+\infty$ *for all real nonnegative numbers* r, s.

3 Stability on restricted domains

Jung [12] proved the stability of Jensen's functional equation on a restricted and unbounded domain. In the following theorem, we improve the bound and thus the result of Jung [12] by obtaining sharper estimates.

Theorem 3.1 *Let* $\mathcal{X}$ *be a normed space and* $\mathcal{Y}$ *a Banach space. Take* $\varepsilon\ge 0$ *and let a function* $f:\mathcal{X}\to\mathcal{Y}$ *satisfy the inequality*

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\le\varepsilon \tag{3.1}$$

for all $x,y\in E_d=\{z\in X:\|z\|\ge d\}$ *for some* $d>0$. *Then, there exists a unique additive function* $T:\mathcal{X}\to\mathcal{Y}$ *such that*

$$\|f(x)-T(x)-f(0)\|\le\frac{3}{2}\varepsilon,\quad x\in\mathcal{X}.$$

Proof Letting $y=-x$ in (3.1), we obtain

$$\|2f(0)-f(x)-f(-x)\|\le\varepsilon,\quad \|x\|\ge d. \tag{3.2}$$

Letting $y=-3x$ in (3.1), we obtain

$$\|2f(-x)-f(x)-f(-3x)\|\le\varepsilon,\quad \|x\|\ge d. \tag{3.3}$$

Now, adding (3.2) and (3.3), we have

$$\left\|f(-3x)-3f(-x)+2f(0)\right\|\le 2\varepsilon,\quad \|x\|\ge d.$$

Then,

$$\left\|f(3x)-3f(x)+2f(0)\right\|\le 2\varepsilon,\quad \|x\|\ge d. \tag{3.4}$$

It is easy to see that

$$\left\|\frac{f(3^n x)}{3^n}-\frac{f(3^m x)}{3^m}+\sum_{i=m}^{n-1}\frac{2f(0)}{3^{i+1}}\right\|\le\sum_{i=m}^{n-1}\frac{2\varepsilon}{3^{i+1}},\quad \|x\|\ge d. \tag{3.5}$$

This implies that the sequence $\{\frac{f(3^n x)}{3^n}\}_n$ is Cauchy for all $x\in\mathcal{X}$. Define $T:\mathcal{X}\to\mathcal{Y}$ by

$$T(x):=\lim_{n\to\infty}\frac{f(3^n x)}{3^n},\quad x\in\mathcal{X}.$$

It is clear that $T(0)=0$ and $T(3x)=3T(x)$ for all $x\in\mathcal{X}$. In view of the definition of T, (3.1) yields

$$2T\left(\frac{x+y}{2}\right)=T(x)+T(y),\quad x,y\in\mathcal{X}\setminus\{0\}. \tag{3.6}$$

Putting $y=3x$ in (3.6) and using $T(3x)=3T(x)$, we infer that $T(2x)=2T(x)$ for all $x\in\mathcal{X}$. Hence, (3.6) implies that T is Jensen (additive) on $\mathcal{X}$. Letting $m=0$ and taking the limit as $n\to\infty$ in (3.5), one obtains

$$\left\|T(x)-f(x)+f(0)\right\|\le\varepsilon,\quad \|x\|\ge d. \tag{3.7}$$

To extend (3.7) to the whole $\mathcal{X}$, let $z\in\mathcal{X}\setminus\{0\}$ and choose a positive integer n such that $\|nz\|\ge d$. Take $x=2(n+1)z$ and $y=-2nz$. Then, (3.7) yields

$$\left\|f(y)-T(y)-f(0)\right\|\le\varepsilon\quad\text{and}\quad\left\|f(x)-T(x)-f(0)\right\|\le\varepsilon.$$

Using these inequalities together with (3.1), we obtain

$$\left\|2f\left(\frac{x+y}{2}\right)-T(x)-T(y)-2f(0)\right\|\le 3\varepsilon.$$

Since T is Jensen and $z=\frac{x+y}{2}$, we obtain

$$\left\|f(z)-T(z)-f(0)\right\|\le\frac{3}{2}\varepsilon.$$

This inequality is valid for $z=0$ because of $T(0)=0$. The uniqueness of T follows easily from the last inequality. □

Remark 3.2 Since $E_d \times E_d \subseteq \{(x,y) \in X \times X : \|x\| + \|y\| \geq d\}$, Theorems 2.9 and 3.1 are valid when (2.6) and (3.1) hold for all $x, y \in X$ with $\|x\| + \|y\| \geq d$.

Theorem 3.3 *Let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ a Banach space. Take $\varepsilon \geq 0$ and let a function $f : \mathcal{X} \to \mathcal{Y}$ satisfy the inequality (3.1) for all $x, y \in \mathcal{X}$ with $\|x+y\| \geq d$ for some $d > 0$. Then, there exists a unique additive function $T : \mathcal{X} \to \mathcal{Y}$ such that*

$$\left\|f(x) - T(x) - f(0)\right\| \leq \frac{3}{2}\varepsilon, \quad x \in \mathcal{X}.$$

Proof Letting $y = 0$ in (3.1), we obtain

$$\left\|2f\left(\frac{x}{2}\right) - f(x) - f(0)\right\| \leq \varepsilon, \quad \|x\| \geq d.$$

It is easy to see that

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m} + \sum_{i=m}^{n-1} \frac{f(0)}{2^{i+1}}\right\| \leq \sum_{i=m}^{n-1} \frac{\varepsilon}{2^{i+1}}, \quad \|x\| \geq d. \tag{3.8}$$

Then, the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is Cauchy for all $x \in \mathcal{X}$. Define $T : \mathcal{X} \to \mathcal{Y}$ by

$$T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in \mathcal{X}.$$

It is clear that $T(0) = 0$ and $T(2x) = 2T(x)$ for all $x \in \mathcal{X}$. In view of the definition of T, (3.1) yields

$$2T\left(\frac{x+y}{2}\right) = T(x) + T(y), \quad x + y \neq 0. \tag{3.9}$$

Let T_e and T_o be the even part and the odd part of T. Then, T_e and T_o satisfy (3.9) for all $x, y \in \mathcal{X}$ with $x + y \neq 0$. Since T_o is odd, (3.9) yields that T_o is additive on $\mathcal{X}$. It follows from (3.9) that $2T(\frac{x}{2}) = T(x)$ for all $x \neq 0$, and then

$$T_e(x-y) = 2T_e\left(\frac{x-y}{2}\right) = T_e(x) + T_e(y) = 2T_e\left(\frac{x+y}{2}\right) = T_e(x+y), \quad x \pm y \neq 0.$$

Putting $y = 3x$ and using $T_e(2x) = 2T_e(x)$, we infer that $T_e(x) = 0$ for all $x \in \mathcal{X}$. Hence, (3.9) implies that T is Jensen (additive) on $\mathcal{X}$. Letting $m = 0$ and taking the limit as $n \to \infty$ in (3.8), one obtains

$$\left\|T(x) - f(x) + f(0)\right\| \leq \varepsilon, \quad \|x\| \geq d. \tag{3.10}$$

To extend (3.7) to the whole $\mathcal{X}$, let $z \in \mathcal{X} \setminus \{0\}$ and choose a positive integer n such that $\|nz\| \geq d$. Take $x = 2(n+1)z$ and $y = -2nz$. Then, (3.7) yields

$$\left\|f(y) - T(y) - f(0)\right\| \leq \varepsilon \quad \text{and} \quad \left\|f(x) - T(x) - f(0)\right\| \leq \varepsilon.$$

Using these inequalities together with (3.1), we obtain

$$\left\| 2f\left(\frac{x+y}{2}\right) - T(x) - T(y) - 2f(0) \right\| \le 3\varepsilon.$$

Since T is Jensen and $z = \frac{x+y}{2}$, we obtain

$$\|f(z) - T(z) - f(0)\| \le \frac{3}{2}\varepsilon.$$

This inequality is valid for $z = 0$ because of $T(0) = 0$. The uniqueness of T follows easily from the last inequality. □

Corollary 3.4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear normed spaces. For a function $f : \mathcal{X} \to \mathcal{Y}$ the following conditions are equivalent*:

(*i*) $\lim_{\|x+y\|\to\infty}[2f(\frac{x+y}{2}) - f(x) - f(y)] = 0$;
(*ii*) $\lim_{\|x\|+\|y\|\to\infty}[2f(\frac{x+y}{2}) - f(x) - f(y)] = 0$;
(*iii*) $\lim_{\min\{\|x\|,\|y\|\}\to\infty}[2f(\frac{x+y}{2}) - f(x) - f(y)] = 0$;
(*iv*) $\lim_{\max\{\|x\|,\|y\|\}\to\infty}[2f(\frac{x+y}{2}) - f(x) - f(y)] = 0$;
(*v*) $2f(\frac{x+y}{2}) = f(x) + f(y)$, $x, y \in X$.

Proof The implications $(v) \Rightarrow (ii) \Rightarrow (iii)$ and $(v) \Rightarrow (iv) \Rightarrow (i)$ are obvious. It is enough to prove the implications $(i) \Rightarrow (v)$ and $(iii) \Rightarrow (v)$.

To prove $(i) \Rightarrow (v)$, let $\varepsilon > 0$ be an arbitrary real number. By (i) there exists $d_\varepsilon > 0$ such that

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \varepsilon, \qquad \|x+y\| \ge d_\varepsilon.$$

Let $\widetilde{\mathcal{Y}}$ be the completion of $\mathcal{Y}$. In view of Theorem 3.3 there exists a unique additive function $A_\varepsilon : \mathcal{X} \to \widetilde{\mathcal{Y}}$ such that

$$\|f(x) - A_\varepsilon(x) - f(0)\| \le \frac{3}{2}\varepsilon, \quad x \in \mathcal{X}.$$

Then,

$$\begin{aligned}
&\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \\
&\quad\le \left\| 2f\left(\frac{x+y}{2}\right) - 2A_\varepsilon\left(\frac{x+y}{2}\right) - 2f(0) \right\| \\
&\qquad + \|f(x) - A_\varepsilon(x) - f(0)\| + \|f(y) - A_\varepsilon(y) - f(0)\| \\
&\quad\le 6\varepsilon, \quad x, y \in \mathcal{X}.
\end{aligned}$$

Since ε is arbitrary, we obtain that f satisfies (v). Using a similar argument, the implication $(iii) \Rightarrow (v)$ is obtained by Theorem 3.1. Hence, the proof is complete. □

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Author contribution
MBM and AN contributed to the study conception and design. The first draft of the manuscript was written by AN and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript. MBM supervised the project.

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