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Higher order Kantorovich-type Szász–Mirakjan operators

Pembe Sabancigil[1*], Mustafa Kara[1] and Nazim I. Mahmudov[1]

*Correspondence:
pembe.sabancigil@emu.edu.tr
[1]Department of Mathematics, Eastern Mediterranean University, Mersin 10, 99628, Gazimagusa, T.R. Northern Cyprus, Turkey

Abstract

In this paper, we define new higher order Kantorovich-type Szász–Mirakjan operators, we give some approximation properties of these operators in terms of various moduli of continuity. We prove a local approximation theorem, a Korovkin-type theorem, and a Voronovskaja-type theorem. We also prove weighted approximation theorems for these new operators.

Keywords: Modulus of continuity; Higher order approximation; Szász–Mirakjan operators; Kantorovich operators; Voronovskaja-type theorem

1 Introduction and auxiliary results

The well-known Bernstein polynomials belonging to a function $f(x)$ defined on the interval $[0,1]$ are defined as follows:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \quad (n = 1,2,\ldots).$$

If $f(x)$ is continuous on $[0,1]$, the polynomials converge uniformly to $f(x)$. These polynomials have an important role in approximation theory and also in other fields of mathematics.

In 1950, for $f \in C[0,\infty)$, Szász [23] defined the operators

$$S_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty), n = 1,2,\ldots,$$

$$\text{where } p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

In [5], Dubey and Jain proposed the integral modification of the Szász–Mirakjan operators to approximate integrable functions on the interval $[0,\infty]$, and in [9], Gupta and Sinha studied some direct results on certain Szász–Mirakjan operators. Some related problems were considered by many authors, see for example [1, 2, 5, 10, 13–23] and the references therein.

An operator $L : C[0,1] \to C[0,1]$ is said to be convex of order $l-1$ if it preserves convexity of order $l-1$, $l \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. The classical Bernstein operator is an example of a mapping convex of all orders $l-1$, $l \in \mathbb{N}$. For an operator L being convex of order $l-1$, consider

$$I_l : C[0,1] \to C[0,1]$$

given by

$$I_l f = f, \quad \text{if } l = 0,$$
$$(I_l f)(x) = \int_0^x \frac{(x-t)^{l-1}}{(l-1)!} f(t)\,dt, \quad \text{if } l \geq 1.$$

Suppose that $L(C^l[0,1]) \subset C^l[0,1]$. Let

$$Q^l := D^l \circ L \circ I_l, \quad \text{where } D^l = \frac{d^l}{dx^l}.$$

Q^l may be considered as an lth order Kantorovich modification of L. The construction of positive operators Q^l, $l \geq 0$, is most useful in simultaneous approximation where for appropriate mappings L the difference

$$D^l L f - D^l f$$

is considered (see [6, 7, 11]).

On the other hand, we know that Kantorovich-type Szász–Mirakjan operators can be defined as follows:

$$K_n(f;x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_0^1 f\left(\frac{k+t}{n}\right) dt.$$

By using the lth order integral and the above definition of the Kantorovich-type Szász–Mirakjan operators, we define a new lth order Kantorovich-type Szász–Mirakjan operator as follows:

$$K_n^l(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^1 \cdots \int_0^1 f\left(\frac{k+t_1+\cdots+t_l}{n+l}\right) dt_1 \cdots dt_l,$$

where $p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$, $n \in \mathbb{N}$, $x \geq 0$, f is a real-valued continuous function defined on $[0,\infty)$.

The paper is organized as follows. In the preliminaries section we give some known results and we derive a recurrence formula for the lth order Szász–Mirakjan–Kantorovich operators $K_n^l(f;x)$. With the help of the derived recurrence formula, we calculate the moments $K_n^l(t^m;x)$ for $m = 0,1,2,3,4$ and we calculate the central moments $K_n^l((t-x)^m;x)$ for some m. In Sect. 3, we prove a local approximation theorem, a Korovkin-type approximation theorem, and a Voronovskaja-type theorem. We obtain the rate of convergence of these types of operators for Lipschitz-type maximal functions, second order modulus of smoothness and Peetre's K-functional. In Sect. 4, we investigate weighted approximation properties of the lth order Szász–Mirakjan–Kantorovich operators in terms of the modulus of continuity.

2 Preliminaries

We consider the following class of functions.

Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions f on $[0,\infty)$, endowed with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$.

$$C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}.$$

Let $B_m[0,\infty)$ be the set of all functions f satisfying the condition that $|f(x)| \le M_f(1+x^m)$, $x\in[0,\infty)$ with some constant M_f depending on f. Introduce

$$C_m[0,\infty) = \left\{f \in B_m[0,\infty) \cap C[0,\infty) : \|f\|_m := \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^m} < \infty\right\},$$
$$C_m^*[0,\infty) = \left\{f \in C_m[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^m} < \infty\right\}.$$

In the following lemma we give the moments of the Szász operator up to the fourth order.

Lemma 1 ([23]) *We have*

$$S_n(1,x) = 1,$$
$$S_n(t,x) = x,$$
$$S_n(t^2,x) = x^2 + \frac{x}{n},$$
$$S_n(t^3,x) = x^3 + \frac{3}{n}x^2 + \frac{1}{n^2}x,$$
$$S_n(t^4,x) = x^4 + \frac{6}{n}x^3 + \frac{7}{n^2}x^2 + \frac{1}{n^3}x.$$

In the following lemma we derive a recurrence formula for $K_n^l(t^m;x)$ which will be used to calculate moments of the lth order Kantorovich-type Szász–Mirakjan operators.

Lemma 2 *For all $n\in\mathbb{N}$, $x\in[0,\infty)$, we have*

$$K_n^l(t^m;x) = \sum_{j_0+\cdots+j_l=m} \binom{m}{j_0,\ldots,j_l} \frac{n^{j_0}}{(n+l)^m(j_1+1)\cdots(j_l+1)} S_n(t^{j_0},x),$$

where $S_n(f,x)$ is the Szász–Mirakjan operator defined in [23].

Proof We can obtain the recurrence formula with the help of the following equality:

$$\left(\frac{k+t_1+\cdots+t_l}{n+l}\right)^m = \sum_{j_0+\cdots+j_l=m} \binom{m}{j_0,\ldots,j_l} \frac{k^{j_0}t_1^{j_1}\cdots t_l^{j_l}}{(n+l)^m} \int_0^1\cdots\int_0^1 \frac{k^{j_0}t_1^{j_1}\cdots t_l^{j_l}}{(n+l)^m}\,dt_1\cdots dt_l$$

$$= \frac{k^{j_0}}{(n+l)^m (j_1+1)\cdots(j_l+1)}.$$

Now by direct calculation we write

$$\begin{aligned}
K_n^l(t^m;x) &= \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^1 \cdots \int_0^1 f\left(\frac{k+t_1+\cdots+t_l}{n+l}\right)^m dt_1\cdots dt_l \\
&= \sum_{k=0}^{\infty} p_{n,k}(x) \sum_{j_0+\cdots+j_l=m} \binom{m}{j_0,\ldots,j_l} \int_0^1 \cdots \int_0^1 \frac{k^{j_0} t_1^{j_1}\cdots t_l^{j_l}}{(n+l)^m}{}^m \, dt_1\cdots dt_l \\
&= \sum_{k=0}^{\infty} p_{n,k}(x) \sum_{j_0+\cdots+j_l=m} \binom{m}{j_0,\ldots,j_l} \frac{k^{j_0}}{(n+l)^m (j_1+1)\cdots(j_l+1)} dt_1\cdots dt_l \\
&= \sum_{j_0+\cdots+j_l=m} \binom{m}{j_0,\ldots,j_l} \frac{n^{j_0}}{(n+l)^m (j_1+1)\cdots(j_l+1)} \sum_{k=0}^{\infty} \frac{k^{j_0}}{n^{j_0}} p_{n,k}(x) \\
&= \sum_{j_0+\cdots+j_l=m} \binom{m}{j_0,\ldots,j_l} \frac{n^{j_0}}{(n+l)^m (j_1+1)\cdots(j_l+1)} S_n\left(t^{j_0},x\right),
\end{aligned}$$

where $S_n(f,x)$ is the Szász–Mirakjan operator. □

Moments and central moments play an important role in approximation theory. In the following lemma we give explicit formulas for the mth ($m = 0,1,2,3,4,$) order moments of the lth order Kantorovich-type Szász–Mirakjan operators $K_n^l(f;x)$.

Lemma 3 *For all $n \in \mathbb{N}$ and $x \in [0,\infty)$, we have the following equalities*:

$$\begin{aligned}
K_n^l(1;x) &= 1, \\
K_n^l(t;x) &= \frac{n}{n+l}x + \frac{l}{2(n+l)}, \\
K_n^l\left(t^2;x\right) &= \left(\frac{n}{n+l}\right)^2 x^2 + \frac{n(l+1)}{(n+l)^2}x + \frac{l(3l+1)}{12(n+l)^2}, \\
K_n^l\left(t^3;x\right) &= \left(\frac{n}{n+l}\right)^3 x^3 + \frac{3n^2 l + 6n^2}{2(n+l)^3}x^2 + \frac{3nl^2+7nl+4n}{4(n+l)^3}x + \frac{l^3-l^2+2l}{8(n+l)^3}, \\
K_n^l\left(t^4;x\right) &= \left(\frac{n}{n+l}\right)^4 x^4 + \frac{2n^3 l + 6n^3}{(n+l)^4}x^3 + \frac{3n^2l^2+13n^2l+14n^2}{2(n+l)^4}x^2 \\
&\quad + \frac{nl^3+2nl^2+7nl+2n}{2(n+l)^4}x + \frac{15l^4-50l^3+185l^2-102l}{240(n+l)^4}.
\end{aligned}$$

Proof The proof is done by using the recurrence formula given in Lemma 2.

$$\begin{aligned}
&K_n^l(1;x) \quad \text{is obvious.} \\
&K_n^l(t;x) = \sum_{j_0+\cdots+j_l=1} \binom{1}{j_0,\ldots,j_l} \frac{n^{j_0}}{(n+l)(j_1+1)\cdots(j_l+1)} S_n\left(t^{j_0},x\right) \\
&\qquad = \binom{l}{1} \frac{1}{2(n+l)} + \frac{n}{n+l}x,
\end{aligned}$$

$$\begin{aligned}K_n^l(t^2;x) &= \sum_{j_0+\cdots+j_l=2}\binom{2}{j_0,\ldots,j_l}\frac{n^{j_0}}{(n+l)^2(j_1+1)\cdots(j_l+1)}S_n(t^{j_0},x)\\ &= \binom{l}{2}\frac{2}{4(n+l)^2}+\binom{l}{1}\frac{1}{3(n+l)^2}+\binom{l}{1}\frac{n}{(n+l)^2}x+\left(\frac{n}{n+l}\right)^2\left(x^2+\frac{x}{n}\right)\\ &= \left(\frac{n}{n+l}\right)^2x^2+\frac{n(l+1)}{(n+l)^2}x+\frac{l(3l+1)}{12(n+l)^2}.\end{aligned}$$

$K_n^l(t^3;x)$ and $K_n^l(t^4;x)$ can be done in a similar way. □

In the following lemma we give formulas for the mth order central moments of the lth order Kantorovich-type Szász–Mirakjan operators for $m = 1, 2, 4$.

Lemma 4 *For all $n \in \mathbb{N}$, we have the following central moments*:

$$\begin{aligned}K_n^l((t-x);x) &= \frac{l(1-2x)}{2(n+l)},\\ K_n^l((t-x)^2;x) &= \frac{l^2}{(n+l)^2}x^2+\frac{n-l^2}{(n+l)^2}x+\frac{3l^2+l}{12(n+l)^2},\\ K_n^l((t-x)^4;x) &= \left(\frac{l}{n+l}\right)^4x^4+\frac{6nl^2+2l^4}{(n+l)^4}x^3+\frac{-12nl^2+6n^2-8nl+3l^4+l^3}{2(n+l)^4}x^2\\ &\quad+\frac{3nl^2+5nl+2n-l^4+l^3-2l^2}{2(n+l)^4}x+\frac{15l^4-50l^3+185l^2-102l}{240(n+l)^4}.\end{aligned}$$

Proof The proof is done by using Lemma 3 and the linearity of the operators.

$$\begin{aligned}K_n^l((t-x);x) &= K_n^l(t;x)-x=\frac{n}{n+l}x+\frac{l}{2(n+l)}-x=\frac{l(1-2x)}{2(n+l)},\\ K_n^l((t-x)^2;x) &= K_n^l(t^2;x)-2xK_n^l(t;x)+x^2\\ &= \left(\frac{n}{n+l}\right)^2x^2+\frac{n(l+1)}{(n+l)^2}x+\frac{l(3l+1)}{12(n+l)^2}-2x\left(\frac{n}{n+l}x+\frac{l}{2(n+l)}\right)+x^2\\ &= \frac{l^2}{(n+l)^2}x^2+\frac{n-l^2}{(n+l)^2}x+\frac{3l^2+l}{12(n+l)^2},\end{aligned}$$

$$\begin{aligned}&K_n^l((t-x)^4;x)\\ &= K_n^l(t^4;x)-4xK_n^l(t^3;x)+6x^2K_n^l(t^2;x)-4x^3K_n^l(t;x)+x^4\\ &= \left(\frac{n}{n+l}\right)^4x^4+\frac{2n^3l+6n^3}{(n+l)^4}x^3+\frac{3n^2l^2+13n^2l+14n^2}{2(n+l)^4}x^2\\ &\quad+\frac{nl^3+2nl^2+7nl+2n}{2(n+l)^4}x+\frac{15l^4-50l^3+185l^2-102l}{240(n+l)^4}\\ &\quad-4x\left(\left(\frac{n}{n+l}\right)^3x^3+\frac{3n^2l+6n^2}{2(n+l)^3}x^2+\frac{3nl^2+7nl+4n}{4(n+l)^3}x+\frac{l^3-l^2+2l}{8(n+l)^3}\right)\\ &\quad+6x^2\left(\left(\frac{n}{n+l}\right)^2x^2+\frac{n(l+1)}{(n+l)^2}x+\frac{l(3l+1)}{12(n+l)^2}\right)-4x^3\left(\frac{n}{n+l}x+\frac{l}{2(n+l)}\right)+x^4\\ &= \left(\frac{l}{n+l}\right)^4x^4+\frac{6nl^2+2l^4}{(n+l)^4}x^3+\frac{-12nl^2+6n^2-8nl+3l^4+l^3}{2(n+l)^4}x^2\end{aligned}$$

$$+\frac{3nl^2+5nl+2n-l^4+l^3-2l^2}{2(n+l)^4}x+\frac{15l^4-50l^3+185l^2-102l}{240(n+l)^4}. \qquad \square$$

One of the main problems in approximation theory is to estimate the rate of convergence for sequences of positive linear operators. Voronovskaja-type formulas are one of the most important tools for studying their asymptotic behavior. In the following lemma we give two limits that later will be used to prove Voronovskaja-type theorem for the lth order Kantorovich-type Szász–Mirakjan operators.

Lemma 5 *For $x \in [0,\infty)$ and $n \to \infty$, we have the following limits*:

$$(i) \quad \lim_{n\to\infty} nK_n^l(t-x;x) = \frac{l(1-2x)}{2},$$
$$(ii) \quad \lim_{n\to\infty} nK_n^l\big((t-x)^2;x\big) = x.$$

Proof The proof is trivial with the use of the formulas $K_n^l(t-x;x)$ and $K_n^l((t-x)^2;x)$ given in Lemma 3,

$$(i) \quad \lim_{n\to\infty} nK_n^l(t-x;x) = \lim_{n\to\infty}\frac{nl(1-2x)}{2(n+l)} = \frac{l(1-2x)}{2},$$
$$(ii) \quad \lim_{n\to\infty} nK_n^l\big((t-x)^2;x\big) = \lim_{n\to\infty}\left\{\frac{nl^2}{(n+l)^2}x^2+\frac{n^2-nl^2}{(n+l)^2}x+\frac{3nl^2+nl}{12(n+l)^2}\right\}$$
$$= x. \qquad \square$$

3 Local approximation

In this section, we establish local approximation theorem for the lth order Kantorovich-type Szász–Mirakjan operators. We consider the Peetre's K-functional

$$K_2(f,\delta) := \inf\big\{\|f-g\|+\delta\|g''\| : g \in C_B^2[0,\infty)\big\}, \quad \delta \geq 0.$$

Then from the known result in [4], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \tag{1}$$

where

$$\omega_2(f,\sqrt{\delta}) := \sup_{0<h\leq\sqrt{\delta}}\ \sup_{x\pm h\in[0,\infty)} \big|f(x-h)-2f(x)+f(x+h)\big|$$

is the second modulus of smoothness of $f \in C_B[0,\infty)$.

In the following theorem we state the first main result for the local approximation of our operators $K_n^l(f;x)$.

Theorem 6 *There exists an absolute constant $C > 0$ such that*

$$\big|K_n^l(f;x)-f(x)\big| \leq C\omega_2\big(f,\sqrt{\delta_n(x)}\big)+\omega\big(f,\theta_n(x)\big),$$

where

$$f \in C_B[0,\infty], \quad \delta_n(x) = K_n^l\big((t-x)^2;x\big) + \big(K_n^l\big((t-x);x\big)\big)^2$$
$$= \frac{6l^2+l}{12(n+l)^2} + \frac{n-2l^2}{(n+l)^2}x + \frac{2l^2}{(n+l)^2}x^2,$$
$$\theta_n(x) = \big|K_n^l\big((t-x);x\big)\big| = \left|\frac{l(1-2x)}{2(n+l)}\right|, \quad 0 \le x < \infty.$$

Proof Let

$$\widetilde{K}_n^l(f;x) = K_n^l(f;x) + f(x) - f\big(\mu_n(x)\big),$$

where $f \in C_B[0,\infty]$, $\mu_n(x) = K_n^l((t-x);x) + x = \frac{l+2nx}{2(n+l)}$. Note that $\widetilde{K}_n^l((t-x);x) = 0$. By using Taylor's formula, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)\,ds, \quad g \in C_B^2[0,\infty).$$

Applying $\widetilde{K}_n^l$ to both sides of the above equation, we have

$$\begin{aligned}
&\widetilde{K}_n^l(g;x) - g(x) \\
&\quad = \widetilde{K}_n^l\big((t-x)g'(x);x\big) + \widetilde{K}_n^l\left(\int_x^t (t-s)g''(s)\,ds;x\right) \\
&\quad = g'(x)\widetilde{K}_n^l\big((t-x);x\big) + K_n^l\left(\int_x^t (t-s)g''(s)\,ds;x\right) - \int_x^{\mu_n(x)} \big(\mu_n(x)-s\big)g''(s)\,ds \\
&\quad = K_n^l\left(\int_x^t (t-s)g''(s)\,ds;x\right) - \int_x^{\mu_n(x)} \big(\mu_n(x)-s\big)g''(s)\,ds.
\end{aligned}$$

On the other hand,

$$\left|\int_x^t (t-s)g''(s)\,ds\right| \le \int_x^t (t-s)\big|g''(s)\big|\,ds \le \|g''\| \int_x^t (t-s)\,ds \le \|g''\|(t-s)^2$$

and

$$\left|\int_x^{\mu_n(x)} \big(\mu_n(x)-s\big)g''(s)\,ds\right| \le \|g''\|\big(\mu_n(x)-x\big)^2 = \|g''\|\big(K_n^l(t-x;x)\big)^2,$$

which implies

$$\begin{aligned}
\big|\widetilde{K}_n^l(g;x) - g(x)\big| &\le \left|K_n^l\left(\int_x^t (t-s)g''(s)\,ds;x\right)\right| + \left|\int_x^{\mu_n(x)} \big(\mu_n(x)-s\big)g''(s)\,ds\right| \\
&\le \|g''\|\big\{K_n^l\big((t-x)^2;x\big) + \big(K_n^l(t-x);x\big)^2\big\} \\
&= \|g''\|\delta_n(x). \qquad (2)
\end{aligned}$$

We also have

$$\big|\widetilde{K}_n^l(f;x)\big| \le \big|K_n^l(f;x)\big| + \big|f(x)\big| + \big|f\big(\mu_n(x)\big)\big| \le K_n^l\big(|f|;x\big) + 2\|f\| \le 3\|f\|.$$

Using (2) and the uniform boundedness of $\widetilde{K}_n^l$, we get

$$\begin{aligned}\left|K_n^l(f;x)-f(x)\right| &\leq \left|\widetilde{K}_n^l(f-g;x)\right| + \left|\widetilde{K}_n^l(g;x)-g(x)\right| + \left|f(x)-g(x)\right| + \left|f\left(\mu_n(x)\right)-f(x)\right| \\ &\leq 4\|f-g\| + \left\|g''\right\|\delta_n(x) + \omega\left(f,\theta_n(x)\right).\end{aligned}$$

If we take the infimum on the right hand side over all $g \in C_B^2[0,\infty)$, we obtain

$$\left|K_n^l(f;x)-f(x)\right| \leq 4K_2\left(f;\delta_n(x)\right) + \omega\left(f,\theta_n(x)\right),$$

which together with (1) gives the proof of the theorem. □

Corollary 7 *Let $A > 0$. Then, for each $f \in C[0,\infty)$, the sequence of operators $K_n^l(f;x)$ converges to f uniformly on $[0,A]$.*

Theorem 8 *Let $f \in C_2^*[0,\infty)$. Then $\lim_{n\to\infty} K_n^l(f;x) = f(x)$, uniformly on $[0,A]$.*

Proof Since

$$K_n^l(1;x) \to 1, \qquad K_n^l(t;x) \to x, \qquad K_n^l\left(t^2;x\right) \to x^2 \quad \text{as } n \to \infty,$$

uniformly in $[0,\infty)$. By the Korovkin theorem, $K_n^l(f;x)$ converges to $f(x)$ uniformly on $[0,A]$. □

Theorem 9 *Let $n \geq l^2$, $f \in C_2[0,\infty)$ and $\omega_{A+1}(f,\delta) = \sup_{|t-x|\leq\delta}\sup_{x,t\in[0,A+1]}|f(t)-f(x)|$ be the modulus of continuity on the interval $[0,A+1] \subset [0,\infty)$, where $A > 0$. Then we have*

$$\left\|K_n^l(f;x)-f(x)\right\|_{C[0,A]} \leq 4M_f\left(1+A^2\right)\alpha_n(A) + 2\omega_{A+1}\left(f,\sqrt{\alpha_n(A)}\right),$$

where $\alpha_n(A) = K_n^l((t-x)^2;A)$.

Proof For $x \in [0,A]$ and $t \geq 0$, we can get (see [8], Eq. 3.3)

$$\left|f(t)-f(x)\right| \leq 4M_f\left(1+A^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{A+1}(f,\delta).$$

Now, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|K_n^l(f;x)-f(x)\right| &\leq K_n^l\left(\left|f(t)-f(x)\right|;x\right) \\ &\leq 4M_f\left(1+A^2\right)K_n^l\left((t-x)^2;x\right) + \left(1+K_n^l\left(\frac{|t-x|}{\delta};x\right)\right)\omega_{A+1}(f,\delta) \\ &\leq 4M_f\left(1+A^2\right)K_n^l\left((t-x)^2;x\right) + \omega_{A+1}(f,\delta)\left(1+\frac{1}{\delta}\left(K_n^l\left((t-x)^2;x\right)\right)^{\frac{1}{2}}\right).\end{aligned}$$

For $x \in [0,A]$, using Lemma 4,

$$K_n^l\left((t-x)^2;x\right) = \frac{x^2l^2}{(n+l)^2} + \frac{x(n-l^2)}{(n+l)^2} + \frac{3l^2+l}{12(n+l)^2} \leq \alpha_n(A).$$

Thus we get

$$\left|K_n^l(f;x)-f(x)\right| \le 4M_f\left(1+A^2\right)\alpha_n(A)+\omega_{A+1}(f,\delta)\left(1+\frac{1}{\delta}\left(\alpha_n(A)\right)^{\frac{1}{2}}\right).$$

By taking $\delta = \sqrt{\alpha_n(A)}$, we get the desired result. □

In the following theorem we give a Voronovskaja-type result for the lth order Kantorovich-type Szász–Mirakjan operators.

Theorem 10 *For any* $f \in C_B^2[0,\infty)$, *the following asymptotic equality holds*:

$$\lim_{n\to\infty} n\left(K_n^l(f;x)-f(x)\right) = \frac{l(1-2x)}{2}f'(x)+\frac{1}{2}xf''(x)$$

uniformly on $[0,A]$.

Proof Let $f \in C_B^2[0,\infty)$ and $x \in [0,\infty)$ be fixed. By using Taylor's formula, we write

$$f(t) = f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \tag{3}$$

where the function $r(t,x)$ is the Peano form of the remainder, $r(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x} r(t,x) = 0$. Applying K_n^l to (3), we obtain

$$\begin{aligned} &n\left(K_n^l(f;x)-f(x)\right) \\ &\quad = nf'(x)K_n^l(t-x;x)+\frac{n}{2}f''(x)K_n^l\left((t-x)^2;x\right)+nK_n^l\left(r(t,x)(t-x)^2;x\right). \end{aligned}$$

By using the Cauchy–Schwarz inequality, we get

$$K_n^l\left(r(t,x)(t-x)^2;x\right) \le \sqrt{K_n^l\left(r^2(t,x);x\right)}\sqrt{K_n^l\left((t-x)^4;x\right)}. \tag{4}$$

We observe that $r^2(x,x) = 0$ and $r^2(.,x) \in C_B[0,\infty)$. Now from Corollary 7 it follows that

$$\lim_{n\to\infty} K_n^l\left(r^2(t,x);x\right) = r^2(x,x) = 0 \tag{5}$$

uniformly with respect to $x \in [0,A]$. Finally, from (4), (5), and Lemma 5, we get immediately

$$\lim_{n\to\infty} nK_n^l\left(r(t,x)(t-x)^2;x\right) = 0,$$

which completes the proof. □

Theorem 11 *Let* $\alpha \in (0,1]$ *and* S *be any subset of the interval* $[0,\infty)$. *Then, if* $f \in C_B[0,\infty)$ *is locally Lip*(α), *i.e., the condition*

$$\left|f(y)-f(x)\right| \le L|y-x|^\alpha, \quad y \in S \text{ and } x \in [0,\infty) \tag{6}$$

holds, then, for each $x \in [0,\infty)$, *we have*

$$\left|K_n^l(f;x) - f(x)\right| \le L\left\{\lambda_n^{\frac{\alpha}{2}}(x) + 2\big(d(x,S)\big)^\alpha\right\},$$

where $\lambda_n(x) = \frac{3l^2+l}{12(n+l)^2} + \frac{n-l^2}{(n+l)^2}x + \frac{l^2}{(n+l)^2}x^2$, L *is a constant depending on* α *and* f, *and* $d(x,S)$ *is the distance between* x *and* S *defined as*

$$d(x,S) = \inf\left\{|t-x| : t \in S\right\}.$$

Proof Let $\bar{S}$ be the closure of S in $[0,\infty)$. Then there exists a point $x_0 \in \bar{S}$ such that $|x-x_0| = d(x,S)$. By the triangle inequality

$$\left|f(t) - f(x)\right| \le \left|f(t) - f(x_0)\right| + \left|f(x) - f(x_0)\right|$$

and by (6), we get

$$\begin{aligned}\left|K_n^l(f;x) - f(x)\right| &\le K_n^l\left(\left|f(t) - f(x_0)\right|;x\right) + K_n^l\left(\left|f(x) - f(x_0)\right|;x\right)\\ &\le L\left\{K_n^l\left(|t-x_0|^\alpha;x\right) + |x-x_0|^\alpha\right\}\\ &\le L\left\{K_n^l\left(|t-x|^\alpha + |x-x_0|^\alpha;x\right) + |x-x_0|^\alpha\right\}\\ &\le L\left\{K_n^l\left(|t-x|^\alpha;x\right) + 2|x-x_0|^\alpha\right\}.\end{aligned}$$

Now, by using the Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned}\left|K_n^l(f;x) - f(x)\right| &\le L\left\{\left[K_n^l\left(|t-x|^{\alpha p};x\right)\right]^{\frac{1}{p}}\left[K_n^l\left(1^q;x\right)\right]^{\frac{1}{q}} + 2\big(d(x,S)\big)^\alpha\right\}\\ &= L\left\{\left[K_n^l\left(|t-x|^2;x\right)\right]^{\frac{\alpha}{2}} + 2\big(d(x,S)\big)^\alpha\right\}\\ &= L\left\{\left[\frac{3l^2+l}{12(n+l)^2} + \frac{n-l^2}{(n+l)^2}x + \frac{l^2}{(n+l)^2}x^2\right]^{\frac{\alpha}{2}} + 2\big(d(x,S)\big)^\alpha\right\}\\ &= L\left\{\big(\lambda_n(x)\big)^{\frac{\alpha}{2}} + 2\big(d(x,S)\big)^\alpha\right\},\end{aligned}$$

and the proof is completed. □

4 Weighted approximation

In this section, we give weighted approximation theorems for the lth order Kantorovich-type Szász–Mirakjan operators. We will use the following two lemmas which can be found in [3] and [12].

Lemma 12 *For* $m \in \mathbb{N}$, *we have*

$$S_n\left(t^{j_0};x\right) = \sum_{j=1}^{j_0} a_{j_0,j}\frac{x^j}{n^{j_0-j}}, \tag{7}$$

where

$$a_{j_0+1,j} = j a_{j_0,j} + a_{j_0,j-1}, \quad j_0 \ge 0, j \ge 1,$$

$$a_{0,0}=1,\qquad a_{j_0,0}=0,\quad j_0>0,\qquad a_{j_0,j}=0,\quad j_0<j.$$

Lemma 13 *Let $m\in\mathbb{N}\cup\{0\}$ and $l\in\mathbb{Z}^+$ be fixed. Then there exists a positive constant $C_m(l)$ such that*

$$\left\|K_n^l\left(1+t^m;x\right)\right\|_m\le C_m(l),\quad n\in\mathbb{N}.\tag{8}$$

Moreover, for every $f\in C_2^*[0,\infty)$, we have

$$\left\|K_n^l(f;x)\right\|_m\le C_m(l)\|f\|_m,\quad n\in\mathbb{N}.\tag{9}$$

Thus K_n^l is a linear positive operator from $C_m^*[0,\infty)$ into $C_m^*[0,\infty)$ for any $m\in\mathbb{N}\cup\{0\}$.

Proof Inequality (8) is obvious for $m=0$. Let $m\ge 1$. Then, by Lemma 12, we have

$$\begin{aligned}&\frac{1}{1+x^m}K_n^l\left(1+t^m;x\right)\\&\quad=\frac{1}{1+x^m}+\frac{1}{1+x^m}\sum_{j_0+\cdots+j_l=m}\binom{m}{j_0,\cdots j_l}\frac{n^{j_0}}{(n+l)^m(j_1+1)\cdots(j_l+1)}\sum_{j=1}^{j_0}a_{j_0,j}\frac{x^j}{n^{j_0-j}}.\end{aligned}$$

Thus

$$\frac{1}{1+x^m}K_n^l\left(1+t^m;x\right)\le 1+k_m(l)=C_m(l),$$

where $C_m(l)$ is a positive constant depending on m and l. On the other hand,

$$\left\|K_n^l(f;x)\right\|_m\le\|f\|_m\left\|K_n^l\left(1+t^m;x\right)\right\|_m$$

for every $f\in C_m^*[0,\infty)$. By applying (8), we obtain (9). □

Theorem 14 *For each $f\in C_2^*[0,\infty)$, one has*

$$\lim_{n\to\infty}\left\|K_n^l(f;x)-f(x)\right\|_2=0.$$

Proof To prove this theorem, we need to use a Korovkin-type theorem on weighted approximation. That is, it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\|K_n^l\left(t^m;x\right)-x^m\right\|_2=0,\quad m=0,1,2.$$

For $m=0$, it is obvious. For $m=1$, we have

$$\begin{aligned}\lim_{n\to\infty}\left\|K_n^l(t;x)-x\right\|_2&=\sup_{x\ge0}\frac{|K_n^l(t;x)-x|}{1+x^2}\\&=\sup_{x\ge0}\frac{1}{1+x^2}\left|\frac{l}{2(n+l)}+\frac{n}{(n+l)}x-x\right|\\&\le\frac{l}{2(n+l)}\sup_{x\ge0}\frac{1}{1+x^2}+\frac{l}{(n+l)}\sup_{x\ge0}\frac{x}{1+x^2}\end{aligned}$$

$$\leq \frac{l}{2(n+l)} + \frac{l}{(n+l)} = \frac{3l}{2(n+l)},$$

and by a similar way, we can write

$$\begin{aligned}
&\lim_{n\to\infty} \left\| K_n^l\left(t^2;x\right) - x \right\|_2 \\
&\quad = \sup_{x\geq 0} \frac{|K_n^l(t^2;x) - x^2|}{1+x^2} \\
&\quad = \sup_{x\geq 0} \frac{1}{1+x^2} \left| \frac{n^2}{(n+l)^2}x^2 + \frac{n(l+1)}{(n+l)^2}x + \frac{3l^2+l}{12(n+l)^2} - x^2 \right| \\
&\quad \leq \left| \frac{-l^2-2nl}{(n+l)^2} \right| \sup_{x\geq 0} \frac{x^2}{1+x^2} + \frac{n(l+1)}{(n+l)^2} \sup_{x\geq 0} \frac{x}{1+x^2} + \frac{3l^2+l}{12(n+l)^2} \sup_{x\geq 0} \frac{1}{1+x^2} \\
&\quad \leq \frac{l^2+2nl}{(n+l)^2} + \frac{n(l+1)}{(n+l)^2} + \frac{3l^2+l}{12(n+l)^2},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\| K_n^l\left(t^m;x\right) - x^m \right\|_2 = 0, \quad m = 0,1,2. \qquad \square$$

Theorem 15 *For each* $f \in C_2^*[0,\infty)$ *and all* $\beta > 0$, *one has*

$$\lim_{n\to\infty} \sup_{x\geq 0} \frac{|K_n^l(f;x) - f(x)|}{(1+x^2)^{1+\beta}} = 0.$$

Proof For any fixed $0 < A < \infty$ and by Lemma 13, we have

$$\begin{aligned}
\sup_{x\geq 0} \frac{|K_n^l(f;x) - f(x)|}{(1+x^2)^{1+\beta}} &= \sup_{x\leq A} \frac{|K_n^l(f;x) - f(x)|}{(1+x^2)^{1+\beta}} + \sup_{x\geq A} \frac{|K_n^l(f;x) - f(x)|}{(1+x^2)^{1+\beta}} \\
&\leq \sup_{x\leq A} \left| K_n^l(f;x) - f(x) \right| + \sup_{x\geq A} \frac{|K_n^l(f;x)| + |f(x)|}{(1+x^2)^{1+\beta}} \\
&\leq \left\| K_n^l(f) - f \right\|_{C[0,A]} + \|f\|_2 \sup_{x\geq A} \frac{|K_n^l(1+t^2;x|}{(1+x^2)^{1+\beta}} \\
&\quad + \sup_{x\geq A} \frac{|f(x)|}{(1+x^2)^{1+\beta}} \\
&= J_1 + J_2 + J_3. \qquad (10)
\end{aligned}$$

Using Theorem 9, we can see that J_1 goes to zero as $n \to \infty$.

By Theorem 14, we can get

$$\begin{aligned}
J_2 &= \|f\|_2 \lim_{n\to\infty} \sup_{x\geq A} \frac{|K_n^l(1+t^2;x|}{(1+x^2)^{1+\beta}} \\
&= \sup_{x\geq A} \frac{\|f\|_2}{(1+x^2)^{\beta}} \leq \frac{\|f\|_2}{(1+A^2)^{\beta}}.
\end{aligned}$$

Since $|f(x)| \leq M_f(1+x^2)$,

$$J_3 = \sup_{x\geq A}\frac{|f(x)|}{(1+x^2)^{1+\beta}} \leq \sup_{x\geq A}\frac{M_f}{(1+x^2)^{\beta}} \leq \frac{M_f}{(1+A^2)^{\beta}}.$$

If we choose A large enough, we get

$$J_2 \to 0 \quad \text{and} \quad J_3 \to 0 \quad \text{as } n \to \infty.$$

Hence by (10) we obtain the desired result

$$\lim_{n\to\infty}\sup_{x\geq 0}\frac{|K_n^l(f;x)-f(x)|}{(1+x^2)^{1+\beta}} = 0.$$

□

For $f \in C_2^*[0,\infty)$, the weighted modulus of continuity is defined as

$$\Omega_m(f,\delta) = \sup_{x\geq 0, 0<h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^m}.$$

Lemma 16 *If $f \in C_m^*[0,\infty)$, $m \in \mathbb{N}$, then*
(i) *$\Omega_m(f,\delta)$ is a monotone increasing function of δ,*
(ii) $\lim_{\delta\to\infty}\Omega_m(f,\delta) = 0$,
(iii) *for any $\rho \in [0,\infty)$, $\Omega_m(f,\rho\delta) \leq (1+\rho)\Omega_m(f,\delta)$.*

Theorem 17 *If $f \in C_m^*[0,\infty)$, then*

$$\left\|K_n^l(f)-f\right\|_{m+1} \leq k\Omega_m\left(f,\frac{1}{\sqrt{n+l}}\right),$$

where k is a constant independent of f and n.

Proof From the definition of $\Omega_m(f,\delta)$ and Lemma 16, we may write

$$\begin{aligned}\left|f(t)-f(x)\right| &\leq \left(1+\left(x+|t-x|\right)^m\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta)\\ &\leq \left(1+(2x+t)^m\right)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta).\end{aligned}$$

Then we have

$$\begin{aligned}\left|K_n^l(f;x)-f(x)\right| &\leq K_n^l\left|\left(f(t)-f(x)\right|;x\right)\\ &\leq \Omega_m(f,\delta)K_n^l\left(1+(2x+t)^m;x\right)+K_n^l\left(\left(1+(2x+t)^m\right);x\right)\\ &= \Omega_m(f,\delta)K_n^l\left(1+(2x+t)^m;x\right)+I_1.\end{aligned}$$

Applying the Cauchy–Schwarz inequality to I_1, we get

$$I_1 \leq K_n^l\left(\left(1+(2x+t)^m\right)^2;x\right)^{1/2}\left(K_n^l\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}.$$

Therefore,

$$\begin{aligned}\left|K_n^l(f;x)-f(x)\right| &\leq \Omega_m(f,\delta)K_n^l\big(1+(2x+t)^m;x\big)\\ &\quad + K_n^l\big(\big(1+(2x+t)^m\big)^2;x\big)\big)^{1/2}\left(K_n^l\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}.\end{aligned}$$

From Lemmas 13 and 12, we have

$$K_n^l\big(1+(2x+t)^m;x\big) \leq C_m(l)\big(1+x^m\big),$$

$$K_n^l\big(\big(1+(2x+t)^m\big)^2;x\big)\big)^{1/2}\left(K_n^l\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2} \leq C_m^1(l)\big(1+x^m\big).$$

Also, from Lemma 4, we have

$$\begin{aligned}\left(K_n^l\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2} &\leq \frac{1}{\delta}\sqrt{\frac{x^2l^2}{(n+l)^2}+\frac{x(n-l^2)}{(n+l)^2}+\frac{3l^2+l}{12(n+l)^2}}\\ &\leq \frac{l(1+x)}{\delta\sqrt{(n+l)}}.\end{aligned}$$

So, if we combine all these results, we get

$$\begin{aligned}\left|K_n^l(f;x)-f(x)\right| &\leq \Omega_m(f,\delta)\left(C_m(l)\big(1+x^m\big)+C_m^1(l)\frac{(1+x^m)(1+x)l}{\delta\sqrt{(n+l)}}\right)\\ &= \Omega_m(f,\delta)\left(C_m(l)\big(1+x^m\big)+C_m^1(l)C_1\frac{l(1+x^{m+1})}{\delta\sqrt{(n+l)}}\right),\end{aligned}$$

where

$$C_1=\sup_{x\geq 0}\frac{1+x^m+x+x^{m+1}}{1+x^{m+1}}.$$

In the above inequality, if we substitute $\frac{1}{\sqrt{n+l}}$ instead of δ, we obtain the desired result. □

5 Conclusion

In this paper, by using the lth order integration and the definition of the Kantorovich type Szász–Mirakjan operators, we defined a new lth order Kantorovich-type Szász–Mirakjan operator. We derived a recurrence formula, and with the help of this formula we calculated the moments $K_n^l(t^m;x)$ for $m=0,1,2,3,4$ and we calculated the central moments $K_n^l((t-x)^m;x)$ for $m=1,2,4$. We established a local approximation theorem, a Korovkin-type approximation theorem, and a Voronovskaja-type theorem. We obtained the rate of convergence of these types of operators for Lipschitz-type maximal functions, second order modulus of smoothness, and Peetre's K-functional. At last we investigated weighted approximation properties of the lth order Szász–Mirakjan–Kantorovich operators in terms of the modulus of continuity.

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Author contributions
PS made the major analysis and the orijinal draft preparation. MK contributed with weighted approximation and NM reviewed and edited the manuscript. All authors read and approved the final manuscript.

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