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# Higher order difference equations with homogeneous governing functions nonincreasing in each variable with unbounded solutions

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**Abstract**

By using a comparison method and some difference inequalities we show that the following higher order difference equation

$$x_{n+k} = \frac{1}{f(x_{n+k-1}, \dots, x_n)}, \quad n \in \mathbb{N},$$

where $k \in \mathbb{N}$, $f : [0, +\infty)^k \to [0, +\infty)$ is a homogeneous function of order strictly bigger than one, which is nondecreasing in each variable and satisfies some additional conditions, has unbounded solutions, presenting a large class of such equations. The class can be used as a useful counterexample in dealing with the boundedness character of solutions to some difference equations. Some analyses related to such equations and a global convergence result are also given.

**MSC:** 39A10; 39A22

**Keywords:** Difference equation; Unbounded solutions; Homogeneous function

## 1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the sets of natural, whole, real, and complex numbers respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ = (0, +\infty)$. If $s, t \in \mathbb{Z}$, then we use the notation $r = \overline{s, t}$ instead of $s \le r \le t$, $r \in \mathbb{Z}$.

Difference equations have been attracting attention of scientists for centuries. Since the time of de Moivre, many equations have been investigated so far. For some classical results see, e.g., [3, 5, 10–12, 14] and the related references therein.

### 1.1 First order difference equation, monotonicity, and some known facts

The general first order difference equation

$$x_{n+1} = f(x_n), \quad n \in \mathbb{N}_0, \tag{1}$$

has been investigated for a long time, and nowadays many results on the equation and its special cases are known.

Here we recall some very basic facts on the behavior of solutions to equation (1) when the function $f$ is monotone. If the function $f$ is a self-map of an interval $I \subseteq \mathbb{R}$, then the case when $f$ is monotone is one of the basic ones. If $f$ is nondecreasing and $x_0 \le x_1 = f(x_0)$, then the sequence $(x_n)_{n\in\mathbb{N}_0}$ is nondecreasing, whereas if $x_0 \ge x_1 = f(x_0)$ then the sequence $(x_n)_{n\in\mathbb{N}_0}$ is nonincreasing. If $f$ is additionally bounded, then the sequence is convergent (see, e.g., [1, Problem 9.34]). If the function $f$ is nonincreasing, then the sequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n+1})_{n\in\mathbb{N}_0}$ are monotone, one of them is nondecreasing and the other is nonincreasing. If $x_0$ does not belong to the interval with the endpoints $x_1$ and $x_2$, then the sequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n+1})_{n\in\mathbb{N}_0}$ are convergent (see, e.g., [1, Problem 9.35]). These results are some of the basic ones, and along with the additional condition on the continuity of the function $f$ they are frequently used in determining convergence of the solutions to equation (1). Many examples can be found, e.g., in [1] and [11].

We prove the statement related to the case when the function $f$ is nonincreasing, for completeness and benefit of the reader, and as a good motivation and a suggestion for part of the arguments in the rest of the paper. If

$$x_0 \le f(x_0) = x_1, \tag{2}$$

then from (1), (2), and the monotonicity of $f$, we have $x_2 = f(x_1) \le f(x_0) = x_1$.

There are two cases to be considered.

*Case* 1. If $x_0 \le x_2 = f(f(x_0))$, then since in this case $x_0 \le x_2 \le x_1$, the monotonicity of $f$ implies $f(x_1) \le f(x_2) \le f(x_0)$, from which along with $x_0 \le x_2$ it follows that $x_0 \le x_2 \le x_3 \le x_1$. Assume that we have proved

$$x_0 \le x_2 \le \cdots \le x_{2n-2} \le x_{2n} \le x_{2n+1} \le x_{2n-1} \le \cdots \le x_3 \le x_1 \tag{3}$$

for some $n \in \mathbb{N}$. Then using the monotonicity of $f$, (1), (3), and $x_0 \le x_2$, we obtain

$$x_0 \le x_2 \le \cdots \le x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1} \le \cdots \le x_3 \le x_1,$$

from which by another application of the same procedure it follows that

$$x_0 \le x_2 \le \cdots \le x_{2n+2} \le x_{2n+3} \le x_{2n+1} \le \cdots \le x_3 \le x_1.$$

From this and by induction we have proved that (3) holds for every $n \in \mathbb{N}_0$.

*Case* 2. If $x_2 \le x_0$, then since $x_2 \le x_0 \le x_1$, the monotonicity of $f$ implies $f(x_1) \le f(x_0) \le f(x_2)$, from which we have $x_2 \le x_0 \le x_1 \le x_3$. Using the monotonicity of $f$, (1), (2), the fact that $x_2 \le x_0$, and the method of induction, we get

$$x_{2n} \le x_{2n-2} \le \cdots \le x_2 \le x_0 \le x_1 \le x_3 \le \cdots \le x_{2n-1} \le x_{2n+1}, \quad n \in \mathbb{N}_0.$$

The case when, instead of (2), it is assumed that $x_1 \le x_0$ is treated similarly. From this and the monotonicity of $f$, we have $x_1 = f(x_0) \le f(x_1) = x_2$.

*Case* 3. If $x_1 \le x_2 \le x_0$, then as above we obtain

$$x_1 \le x_3 \le \cdots \le x_{2n-1} \le x_{2n+1} \le x_{2n} \le x_{2n-2} \le \cdots \le x_2 \le x_0, \quad n \in \mathbb{N}_0.$$

*Case* 4. If $x_1 \le x_0 \le x_2$, then as above we obtain

$$x_{2n+1} \le x_{2n-1} \le \cdots \le x_3 \le x_1 \le x_0 \le x_2 \le \cdots \le x_{2n-2} \le x_{2n}, \quad n \in \mathbb{N}_0.$$

This simple and well-known analysis shows that for each solution $(x_n)_{n\in\mathbb{N}_0}$ to equation (1) its subsequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n+1})_{n\in\mathbb{N}_0}$ are monotone, as claimed. In the first and the third case from a well-known theorem it follows that the subsequences are convergent. Recall also that in this case $g = f \circ f$ is a nondecreasing function, from which the monotonicity and convergence results can be concluded from the ones in the case when $f$ is nondecreasing by noting that $x_{2n+2} = g(x_{2n})$ and $x_{2n+3} = g(x_{2n+1})$, $n \in \mathbb{N}_0$.

To say more about the long-term behavior of solutions to equation (1), some additional conditions on function $f$ should be posed.

### 1.2 A previous claim

Recent literature shows frequent applications of known global convergence results. The following statement was formulated in [13].

**Theorem 1** *Assume that $f$ has nonpositive partial derivatives and is homogeneous with degree $s$. Then the equation*

$$x_{n+1} = f(x_{n-k}, x_{n-m}), \quad n \in \mathbb{N}_0 \tag{4}$$

*has a unique positive equilibrium $x^*$, and every solution to equation (4) converges to $x^*$.*

To prove the statement in Theorem 1, paper [13] quotes Theorem 1.4.7 in [9] which deals with equation (4), but only when $k = 0$ and $m = 1$, which means that the result cannot be applied for other values of $k$ and $m$. Beside this, the proof of the theorem only checks the fact that from the associated two-dimensional algebraic system

$$l = f(L, L), \quad L = f(l, l), \tag{5}$$

it follows that $l = L$. But since under the conditions in Theorem 1 system (5) is

$$l = L^s f(1,1), \quad L = l^s f(1,1),$$

it is immediately obtained $l = L$, when $f(1,1) \ne 0$. It is claimed therein that this finishes the proof of Theorem 1.

Quite recently in [28] we have shown that the claim in Theorem 1 is not correct by presenting a counterexample in the class of difference equations with interlacing indices. The class of equations seems quite suitable for providing some counterexamples in the theory of difference equations (see, e.g., recent paper [4]).

### 1.3 Our aim

We show that there is a related global convergence result which can be applied for all values of $k, m \in \mathbb{N}_0$, but under some additional conditions. We also show that there is a large class of governing functions $f$ satisfying the conditions in the formulation of Theorem 1, such that the corresponding difference equations have solutions which are unbounded, showing that the statement in Theorem 1 is not correct for the large class of equations.

## 2 Preliminary analysis

In this section we conduct some analyses of difference equations whose governing functions are homogeneous and nonincreasing in all variables.

### 2.1 An instructive example, a product-type difference equation

Now we give a simple, but instructive, example which contains some ideas which are useful for the study.

**Example 1** Consider the following difference equation:

$$x_{n+1} = \frac{a}{x_n^\alpha}, \quad n \in \mathbb{N}_0, \tag{6}$$

where $a, \alpha \in \mathbb{R}_+$.

First note that in this case $f(x) = \frac{a}{x^\alpha}$, which is a continuous and decreasing function on $\mathbb{R}_+$ and maps it onto itself.

Note also that by using the change of variables $x_n = a^{\frac{1}{\alpha+1}} y_n$, $n \in \mathbb{N}_0$, equation (6) is transformed to the following one:

$$y_{n+1} = \frac{1}{y_n^\alpha}, \quad n \in \mathbb{N}_0.$$

Hence, we may assume that $a = 1$.

*Case* $\alpha = 1$. Assume that $\alpha = 1$. Then equation (6) becomes

$$x_{n+1} = \frac{1}{x_n}, \quad n \in \mathbb{N}_0. \tag{7}$$

Let $x_0 \in \mathbb{R}_+$. Then, by using (7) and a simple inductive argument, we have

$$x_{2n} = x_0 \quad \text{and} \quad x_{2n+1} = \frac{1}{x_0}, \quad n \in \mathbb{N}_0.$$

If $x_0 \in [1, \infty)$, then we have $x_1 = \frac{1}{x_0} \leq x_0 = x_2$. So, this situation corresponds to Case 3, and also Case 4 above. Moreover, if $x_0 \in (1, \infty)$, then the subsequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n+1})_{n\in\mathbb{N}_0}$ are convergent, but the whole sequence is not. If $x_0 = 1$, then $x_n = 1$ for every $n \in \mathbb{N}_0$, and then the sequence is convergent.

If $x_0 \in (0, 1]$, then we have $x_2 = x_0 \leq \frac{1}{x_0} = x_1$. So, this situation corresponds to Case 1 and also Case 2 above. Moreover, if $x_0 \in (0, 1)$, then the subsequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n+1})_{n\in\mathbb{N}_0}$ are convergent, but the sequence $(x_n)_{n\in\mathbb{N}_0}$ is not.

Now note that by using (6) with $a = 1$ twice, we obtain

$$x_{2n+2} = x_{2n}^{\alpha^2} \quad \text{and} \quad x_{2n+3} = x_{2n+1}^{\alpha^2}, \quad n \in \mathbb{N}_0. \tag{8}$$

From (8) it is easily obtained

$$x_{2n} = x_0^{\alpha^{2n}} \quad \text{and} \quad x_{2n+1} = x_1^{\alpha^{2n}} = \frac{1}{x_0^{\alpha^{2n+1}}}, \quad n \in \mathbb{N}_0. \tag{9}$$

There are two cases to be considered.

*Case* $\alpha \in (0,1)$. In this case the sequences $\alpha^{2n}$ and $\alpha^{2n+1}$ decreasingly converge to zero, and consequently

$$\lim_{n\to+\infty} x_{2n} = 1 \quad \text{and} \quad \lim_{n\to+\infty} x_{2n+1} = 1.$$

Moreover, if $x_0 \in (0,1)$, then $x_{2n}$ increases to one, whereas $x_{2n+1}$ decreases to one. Since in this case $x_0 < x_2 = x_0^{\alpha^2} < x_1 = \frac{1}{x_0^{\alpha}}$, it corresponds to Case 1 above. If $x_0 > 1$, then $x_{2n}$ decreases to one, whereas $x_{2n+1}$ increases to one. Since in this case $x_1 = \frac{1}{x_0^{\alpha}} < x_2 = x_0^{\alpha^2} < x_0$, we have the situation as in Case 3 above.

*Case* $\alpha \in (1,\infty)$. In this case the sequences $\alpha^{2n}$ and $\alpha^{2n+1}$ increasingly tend to $+\infty$, and consequently, if $x_0 \in (0,1)$, by letting $n \to +\infty$ in (9) we get

$$\lim_{n\to+\infty} x_{2n} = 0 \quad \text{and} \quad \lim_{n\to+\infty} x_{2n+1} = +\infty,$$

$x_{2n}$ is decreasing and $x_{2n+1}$ is increasing. Since $x_2 = x_0^{\alpha^2} < x_0 < \frac{1}{x_0^{\alpha}} = x_1$, we have the situation in Case 2. If $x_0 > 1$, then

$$\lim_{n\to+\infty} x_{2n} = +\infty \quad \text{and} \quad \lim_{n\to+\infty} x_{2n+1} = 0,$$

$x_{2n}$ is increasing and $x_{2n+1}$ is decreasing. Since $x_1 = \frac{1}{x_0^{\alpha}} < x_0 < x_0^{\alpha^2} = x_2$, we have the situation in Case 4.

*Remark* 1 Equation (6) is one of the simplest product-type difference equations which is solvable in closed form. The solvability is essentially what enables the simple analysis given in Example 1. For some more complex solvable product-type difference equations and systems see, for example, [24, 29] and the related references therein. Some classical results on solvability can be found, e.g., in [1, 3, 5, 8, 10–12, 14], whereas some other recent ones can be found, e.g., in [2, 20, 23, 25] (see also the related references therein).

*Remark* 2 Note that the function $f(x) = \frac{1}{x^{\alpha}}$ in Example 1 is homogeneous with degree $-\alpha$. Recall also that it is decreasing. But, as we have seen in the analysis in the example, if $\alpha > 1$, then there are unbounded solutions to equation (6), although

$$l = \frac{1}{L^{\alpha}} \quad \text{and} \quad L = \frac{1}{l^{\alpha}}$$

implies $l = L$ in this case.

This homogeneous function of one variable strikingly suggests that similar situation should appear also in the case of homogeneous functions of several variables. This example along with the construction of difference equations with interlacing indices (for some examples of the equations see, e.g. [26, 27]) was used for constructing the counterexample in [28].

### 2.2 An example with a heuristic asymptotic approach

Here we consider a difference equation heuristically. The first higher order difference equation with non-interlacing indices, such that the governing function satisfies the conditions in the formulation of Theorem 1 and is related to the one-dimensional equation (6) with $a = 1$ that came to our mind, is the following second order one:

$$x_{n+2} = \frac{2}{x_{n+1}^2 + x_n^2}, \quad n \in \mathbb{N}, \tag{10}$$

with $x_1, x_2 \in \mathbb{R}_+$.

Consider the equation with

$$x_1 = x_2 = \varepsilon, \tag{11}$$

where

$$\varepsilon \in \left(0, \frac{1}{\sqrt[3]{2}}\right) \tag{12}$$

is very small.

From (10) and (11) we have

$$x_3 = \frac{2}{x_2^2 + x_1^2} = \frac{1}{\varepsilon^2}. \tag{13}$$

From (10), (11), (13) and by the Taylor formula with Peano's remainder, we have

$$x_4 = \frac{2}{x_3^2 + x_2^2} = \frac{2\varepsilon^4}{1 + \varepsilon^6} = 2\varepsilon^{2^2}\left(1 + O\left(\varepsilon^{2+2^2}\right)\right). \tag{14}$$

From (10), (13), (14) and by the Taylor formula with Peano's remainder, we have

$$x_5 = \frac{2}{x_4^2 + x_3^2} = \frac{2\varepsilon^4}{1 + 2^2\varepsilon^{2^2+2^3} + o(\varepsilon^{12})} = 2\varepsilon^{2^2}\left(1 + O\left(2^2\varepsilon^{2^2+2^3}\right)\right). \tag{15}$$

From (10), (14), (15) as above, we have

$$x_6 = \frac{2}{x_5^2 + x_4^2} = \frac{1}{2^2\varepsilon^{2^3}(1 + O(\varepsilon^{2+2^2}))} = \frac{1}{2^2\varepsilon^{2^3}}\left(1 + O\left(\varepsilon^{2+2^2}\right)\right). \tag{16}$$

From (10), (15), (16), we have

$$x_7 = \frac{2}{x_6^2 + x_5^2} = \frac{2^{1+2^2}\varepsilon^{2^4}}{1 + O(\varepsilon^{2+2^2})} = 2^{1+2^2}\varepsilon^{2^4}\left(1 + O\left(\varepsilon^{2+2^2}\right)\right). \tag{17}$$

From (10), (16), (17), we have

$$x_8 = \frac{2}{x_7^2 + x_6^2} = \frac{2^{1+2^2}\varepsilon^{2^4}}{1 + O(\varepsilon^{2+2^2})} = 2^{1+2^2}\varepsilon^{2^4}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big). \tag{18}$$

From (10), (17), (18), we have

$$x_9 = \frac{2}{x_8^2 + x_7^2} = \frac{1}{2^{2+2^3}\varepsilon^{2^5}(1 + O(\varepsilon^{2+2^2}))} = \frac{1}{2^{2+2^3}\varepsilon^{2^5}}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big). \tag{19}$$

Formulas (13)–(19) suggest that the following relations hold:

$$\begin{aligned} x_{3n} &= \frac{1}{2^{2+2^3+\cdots+2^{2n-3}}\varepsilon^{2^{2n-1}}}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big) \\ &= \frac{1 + O(\varepsilon^{2+2^2})}{2^{\frac{2^{2n-1}-2}{3}}\varepsilon^{2^{2n-1}}} = \frac{\sqrt[3]{4}(1 + O(\varepsilon^{2+2^2}))}{(\sqrt[3]{2}\varepsilon)^{2^{2n-1}}}, \end{aligned} \tag{20}$$

$$\begin{aligned} x_{3n+1} &= 2^{1+2^2+\cdots+2^{2(n-1)}}\varepsilon^{2^{2n}}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big) \\ &= 2^{\frac{2^{2n}-1}{3}}\varepsilon^{2^{2n}}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big) = \frac{(\sqrt[3]{2}\varepsilon)^{2^{2n}}}{\sqrt[3]{2}}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big), \end{aligned} \tag{21}$$

$$\begin{aligned} x_{3n+2} &= 2^{1+2^2+\cdots+2^{2(n-1)}}\varepsilon^{2^{2n}}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big) \\ &= 2^{\frac{2^{2n}-1}{3}}\varepsilon^{2^{2n}}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big) = \frac{(\sqrt[3]{2}\varepsilon)^{2^{2n}}}{\sqrt[3]{2}}\big(1 + O\big(\varepsilon^{2+2^2}\big)\big) \end{aligned} \tag{22}$$

for $n \le n_0$ for some large but fixed $n_0$.

However, since the calculation errors are accumulated from one step to another, we will not conduct further our analysis in the direction nor try to prove (20)–(22), but will simply leave it as a heuristic asymptotic analysis.

Note that if (20)–(22) were true, then by using assumption (12), i.e., $\varepsilon\sqrt[3]{2} \in (0, 1)$, in (20)–(22), it would follow that

$$\lim_{n\to\infty} x_{3n} = +\infty, \qquad \lim_{n\to\infty} x_{3n+1} = \lim_{n\to\infty} x_{3n+2} = 0,$$

which would show that the solution to equation (10) satisfying (11) is unbounded.

The heuristic proof suggests that another method should be employed. Since we do not have exact applicable formulas, one of the ideas is to compare some of the solutions to the equation with solutions to another equation for which it is possible to find the solutions in closed form. This idea will be used in the next section.

*Remark* 3 If $x_1 = x_2 = 0$, then the solution to equation (10) is not well defined. If $x_1 \neq 0$ or $x_2 \neq 0$, then $x_3 > 0$, and a simple inductive argument shows that $x_n > 0$ for every $n \in \mathbb{N}_0$. Hence, all solutions except the one obtained for $x_1 = x_2 = 0$ are well defined.

*Remark* 4 The only real equilibrium of equation (10) is $x^* = 1$. Hence, the equation has a bounded solution

$$x_n \equiv 1, \quad n \in \mathbb{N},$$

which is, of course, convergent. The solution is obtained for $x_1 = x_2 = 1$.

Assume that $x_{n_0+1} = x_{n_0+2} = 1$ for some $n_0 \geq 3$. Then we have

$$1 = x_{n_0+2} = \frac{2}{x_{n_0+1}^2 + x_{n_0}^2} = \frac{2}{1 + x_{n_0}^2},$$

from which it follows that $x_{n_0} = 1$ (by Remark 3, $x_n > 0$, $n \geq 3$ for each solution to equation (10)). By a simple inductive argument, we get $x_n = 1$ for $3 \leq n \leq n_0 + 2$. If $x_3 = x_4 = 1$, then $x_2^2 = 1$ is similarly obtained, that is, $|x_2| = 1$, from which in the same way $|x_1| = 1$ is obtained. Hence there are four solutions which are eventually equal to one, namely those satisfying the conditions

$$|x_1| = |x_2| = 1.$$

## 3 Main results

Here we prove the main results in this paper, which give some answers to the questions posed in the introduction and show that the statement in Theorem 1 is not true by presenting a large class of difference equations possessing unbounded solutions.

The following result on global convergence of solutions to a difference equation is folklore and one of many existing in the literature (see, e.g., [6, 7, 9, 15, 16, 18, 21]). The proof is standard and essentially given in [9, Theorem A.0.8]. We present it for the completeness and the benefit of the reader.

**Theorem 2** *Let $k \in \mathbb{N}$ and $f : [a,b]^k \to [a,b]$ be a continuous function which is nonincreasing in each variable, and such that from*

$$f(l,\dots,l) = L \quad \text{and} \quad f(L,\dots,L) = l, \tag{23}$$

*where $l, L \in [a,b]$, it follows that $l = L$.*

*Then the following difference equation*

$$x_{n+k} = f(x_{n+k-1},\dots,x_n), \quad n \in \mathbb{N}, \tag{24}$$

*has a unique equilibrium $x^* \in [a,b]$ and every solution to (24) converges to $x^*$.*

*Proof* Let $g(x) = f(x,\dots,x) - x$. Since $g(a) \geq a$ and $g(b) \leq b$, the continuity of $g$ implies that there is $x^* \in [a,b]$ such that $g(x^*) = 0$. Assume that there is $y^* \in [a,b]$, $x^* \neq y^*$ such that $g(y^*) = 0$. We may assume that $x^* > y^*$, since the other case is dual. Then from the relation $f(x^*,\dots,x^*) - f(y^*,\dots,y^*) = x^* - y^*$ and monotonicity of $f$ in each variable we have

$$0 < x^* - y^* = f\left(x^*,\dots,x^*\right) - f\left(y^*,\dots,y^*\right) \leq 0,$$

which is a contradiction. Hence, it must be $x^* = y^*$, proving the uniqueness.

Let $l_1 = a$, $L_1 = b$,

$$L_{n+1} = f(l_n,\dots,l_n) \quad \text{and} \quad l_{n+1} = f(L_n,\dots,L_n), \quad n \in \mathbb{N}. \tag{25}$$

Then from (25) and by using induction it is routinely obtained

$$l_1 \le l_2 \le \cdots \le l_{n-1} \le l_n \le L_n \le L_{n-1} \le \cdots \le L_2 \le L_1 \tag{26}$$

for every $n \in \mathbb{N}$, and for each solution $(x_n)_{n\in\mathbb{N}}$ to (24) we have

$$l_n \le x_j \le L_n, \quad \text{for } j \ge k(n-1) + 1. \tag{27}$$

From (26) we have

$$\widehat{l} = \lim_{n\to+\infty} l_n \quad \text{and} \quad \widehat{L} = \lim_{n\to+\infty} L_n$$

for some $\widehat{l}, \widehat{L} \in [a, b]$. By letting $n \to +\infty$ in (25) we get (23) with $l = \widehat{l}$ and $L = \widehat{L}$, which implies $\widehat{l} = \widehat{L}$. From this and by letting $n \to +\infty$ in (27) we get

$$\lim_{n\to+\infty} x_n = \widehat{l} = \widehat{L} = x^*,$$

finishing the proof. □

*Remark* 5 Theorem 2 is the result which can be applied in the case when the function $f$ is nonincreasing in each variable, whether or not the function $f(t_1, \dots, t_k)$ depends on all the variables. It could have been applied in the proof of Theorem 3.3 in [13], but only if all its conditions are verified, which was not the case therein.

The following theorem is the main result in the paper. It shows that there is a large class of functions satisfying the conditions in Theorem 1, such that the corresponding difference equations have solutions which are unbounded, showing that the claim in Theorem 1 is not correct. In the proof of the theorem we use a comparison argument. For some related comparison arguments see, e.g., [17, 19, 22].

**Theorem 3** *Consider the difference equation*

$$x_{n+k} = \frac{1}{f(x_{n+k-1}, \dots, x_n)}, \quad n \in \mathbb{N}, \tag{28}$$

*with* $x_j \in \mathbb{R}_+$, $j = \overline{1,k}$, *where the function* $f : [0, +\infty)^k \to [0, +\infty)$ *is homogeneous of order* $\alpha > 1$, *that is*, $f(\lambda t_1, \dots, \lambda t_k) = \lambda^\alpha f(t_1, \dots, t_k)$ *for every* $\lambda \in [0, +\infty)$, *nondecreasing in each variable*, $f(1, \dots, 1) = 1$, *and that*

$$q := \min\bigl\{f(1, 0, \dots, 0), f(0, 1, 0, \dots, 0), \dots, f(0, \dots, 0, 1)\bigr\} > 0.$$

*Then every solution to equation* (28) *such that*

$$0 < \max_{j=\overline{1,k}} x_j < q^{\frac{1}{\alpha^2-1}} \tag{29}$$

*is unbounded.*

*Remark* 6 If $f(1,\dots,1)\neq 1$, then by using the change of variables

$$x_n=\frac{y_n}{(f(1,\dots,1))^{1/(1+\alpha)}},\quad n\in\mathbb{N},$$

the sequence $(y_n)_{n\in\mathbb{N}}$ satisfies the equation

$$y_{n+k}=\frac{f(1,\dots,1)}{f(y_{n+k-1},\dots,y_n)},\quad n\in\mathbb{N},$$

and the function

$$\widetilde{f}(t_1,\dots,t_k):=\frac{f(t_1,\dots,t_k)}{f(1,\dots,1)}$$

satisfies the condition $\widetilde{f}(1,\dots,1)=1$. Hence, we may assume $f(1,\dots,1)=1$.

*Proof of Theorem* 3 Let

$$m_0:=\max_{j=\overline{1,k}}x_j>0.$$

Then, from (28) and the conditions of the theorem, we have

$$x_{k+1}=\frac{1}{f(x_k,\dots,x_1)}\geq\frac{1}{f(m_0,\dots,m_0)}=\frac{1}{m_0^\alpha f(1,\dots,1)}=\frac{1}{m_0^\alpha}.\tag{30}$$

Let

$$M_1:=\frac{1}{m_0^\alpha}.\tag{31}$$

Then, from (28), (30), and the conditions of the theorem, we have

$$\begin{aligned}
x_{k+2}&=\frac{1}{f(x_{k+1},\dots,x_2)}\leq\frac{1}{f(M_1,0,\dots,0)}=\frac{1}{M_1^\alpha f(1,0,\dots,0)}\leq\frac{1}{qM_1^\alpha},\\
x_{k+3}&=\frac{1}{f(x_{k+2},x_{k+1},\dots,x_3)}\leq\frac{1}{f(0,M_1,0,\dots,0)}=\frac{1}{M_1^\alpha f(0,1,0,\dots,0)}\leq\frac{1}{qM_1^\alpha},\\
&\vdots\\
x_{2k+1}&=\frac{1}{f(x_{2k},\dots,x_{k+1})}\leq\frac{1}{f(0,\dots,0,M_1)}=\frac{1}{M_1^\alpha f(0,\dots,0,1)}\leq\frac{1}{qM_1^\alpha}.
\end{aligned}\tag{32}$$

Let

$$m_1:=\frac{1}{qM_1^\alpha}.$$

Then, from (28), (32), and the conditions of the theorem, we have

$$x_{2k+2}=\frac{1}{f(x_{2k+1},\dots,x_{k+2})}\geq\frac{1}{f(m_1,\dots,m_1)}=\frac{1}{m_1^\alpha f(1,\dots,1)}=\frac{1}{m_1^\alpha}.$$

Let $(M_n)_{n\in\mathbb{N}}$ and $(m_n)_{n\in\mathbb{N}_0}$ be sequences defined as follows:

$$M_{n+1} = \frac{1}{m_n^\alpha}, \qquad m_{n+1} = \frac{1}{qM_{n+1}^\alpha}, \quad n \in \mathbb{N}_0. \tag{33}$$

Assume that for some $l \in \mathbb{N}$ we have proved that

$$x_{(k+1)l+j} \le m_l, \quad j = \overline{1,k}, \tag{34}$$

$$x_{(k+1)(l+1)} \ge M_{l+1}. \tag{35}$$

Then from (28), (35), and the conditions of the theorem we have

$$\begin{aligned} x_{(k+1)(l+1)+1} &= \frac{1}{f(x_{(k+1)(l+1)}, \dots, x_{(k+1)l+2})} \le \frac{1}{f(M_{l+1}, 0, \dots, 0)} \\ &= \frac{1}{M_{l+1}^\alpha f(1, 0, \dots, 0)} \le \frac{1}{qM_{l+1}^\alpha} = m_{l+1}, \\ &\vdots \\ x_{(k+1)(l+1)+k} &= \frac{1}{f(x_{(k+1)(l+1)+k-1}, \dots, x_{(k+1)(l+1)})} \le \frac{1}{f(0, \dots, 0, M_{l+1})} \\ &= \frac{1}{M_{l+1}^\alpha f(0, \dots, 0, 1)} \le \frac{1}{qM_{l+1}^\alpha} = m_{l+1}. \end{aligned} \tag{36}$$

Then from (28), (36), and the conditions of the theorem we have

$$\begin{aligned} x_{(k+1)(l+2)} &= \frac{1}{f(x_{(k+1)(l+1)+k}, \dots, x_{(k+1)(l+1)+1})} \ge \frac{1}{f(m_{l+1}, \dots, m_{l+1})} \\ &= \frac{1}{m_{l+1}^\alpha f(1, \dots, 1)} = \frac{1}{m_{l+1}^\alpha} = M_{l+2}. \end{aligned} \tag{37}$$

From this and by induction, we see that the inequalities in (34) and (35) hold for every $l \in \mathbb{N}_0$.

From the equations in (33) we have

$$M_n = \frac{1}{m_{n-1}^\alpha} = \left(qM_{n-1}^\alpha\right)^\alpha = q^\alpha M_{n-1}^{\alpha^2}, \quad n \ge 2. \tag{38}$$

Iterating equation (38) yields

$$M_n = q^\alpha M_{n-1}^{\alpha^2} = q^\alpha \left(q^\alpha M_{n-2}^{\alpha^2}\right)^{\alpha^2} = q^{\alpha(1+\alpha^2)} M_{n-2}^{(\alpha^2)^2}, \quad n \ge 3.$$

By a simple inductive argument we have

$$M_n = q^{\alpha(1+\alpha^2+\cdots+\alpha^{2n-4})} M_1^{\alpha^{2n-2}}, \quad n \in \mathbb{N},$$

from which it follows that

$$M_n = q^{\alpha \frac{\alpha^{2n-2}-1}{\alpha^2-1}} M_1^{\alpha^{2n-2}}, \quad n \in \mathbb{N}. \tag{39}$$

From (39) together with (31) we have

$$M_n = q^{\frac{\alpha^{2n-1}-\alpha}{\alpha^2-1}} m_0^{-\alpha^{2n-1}} = \left(\frac{q^{\frac{1}{\alpha^2-1}}}{m_0}\right)^{\alpha^{2n-1}} q^{\frac{\alpha}{1-\alpha^2}}, \quad n \in \mathbb{N}. \tag{40}$$

Employing (40) in the second equation in (33) we get

$$m_n = \frac{1}{qM_n^\alpha} = \left(\frac{q^{\frac{1}{\alpha^2-1}}}{m_0}\right)^{-\alpha^{2n}} q^{\frac{1}{\alpha^2-1}}. \tag{41}$$

Let $m_0 \in (0, q^{\frac{1}{\alpha^2-1}})$. Letting $n \to +\infty$ in (40) and (41) and using the fact $q^{\frac{1}{\alpha^2-1}}/m_0 > 1$ and the assumption $\alpha > 1$, we obtain

$$\lim_{n\to+\infty} m_n = 0 \tag{42}$$

and

$$\lim_{n\to+\infty} M_n = +\infty. \tag{43}$$

From (34), (35), (42), and (43), we see that for each solution to equation (28) satisfying condition (29) we have

$$\lim_{l\to+\infty} x_{(k+1)l+j} = 0, \quad j = \overline{1,k},$$

and

$$\lim_{l\to+\infty} x_{(k+1)l} = +\infty. \tag{44}$$

The relation in (44) shows that each solution with such chosen initial values is unbounded, finishing the proof of the theorem. □

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## Declarations

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
SS initiated the investigation, proposed some preliminary ideas, and conducted some detailed investigations. AEA, BI, and WK analyzed the proposed ideas, made some calculations, and gave many comments. All authors read and approved the final manuscript.

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