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On a system of monotone variational inclusion problems with fixed-point constraint

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Abstract

In this paper, we study the problem of finding the solution of the system of monotone variational inclusion problems recently introduced by Chang et al. (Optimization 70(12):2511–2525, 2020) with the constraint of a fixed-point set of quasipseudocontractive mappings. We propose a new iterative method that employs an inertial technique with self-adaptive step size for approximating the solution of the problem in Hilbert spaces and prove a strong-convergence result for the proposed method under more relaxed conditions. Moreover, we apply our results to study related optimization problems. Finally, we present some numerical experiments to demonstrate the performance of our proposed method, compare it with a related method as well as experiment on the dependency of the key parameters on the performance of our method.

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1 Introduction

In recent years, the *split inverse problem* (SIP) has received much research attention (see [1, 11, 12, 20, 24, 50] and the references therein) because of its extensive applications, for example, in phase retrieval, signal processing, image recovery, intensity-modulated radiation therapy, data compression, among others (see [13, 14, 42] and the references therein). The SIP model is presented as follows: Find a point

$$\hat{x} \in H_1 \quad \text{that solves } \mathrm{IP}_1 \tag{1.1}$$

such that

$$\hat{y} := A\hat{x} \in H_2 \quad \text{solves } \mathrm{IP}_2, \tag{1.2}$$

where H_1 and H_2 are real Hilbert spaces, IP_1 denotes an inverse problem formulated in H_1 and IP_2 denotes an inverse problem formulated in H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

Censor and Elfving [14] in 1994 introduced the *split feasibility problem* (SFP), which was the first instance of the SIP for modeling inverse problems that arise from medical-image reconstruction. Since then, several authors have studied and developed different iterative methods for approximating the solution of the SFP. The SFP has wide areas of applications, for instance, in signal processing, approximation theory, control theory, geophysics, communications, biomedical engineering, etc. [13, 30]. The SFP is formulated as follows:

$$\text{find a point } \hat{x} \in C \text{ such that } \hat{y} = A\hat{x} \in Q, \tag{1.3}$$

where C and Q are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

Moudafi [33] introduced another instance of the SIP known as the *split monotone variational inclusion problem* (SMVIP). Let H_1, H_2 be real Hilbert spaces, $f_1 : H_1 \to H_1$, $f_2 : H_2 \to H_2$, are inverse strongly monotone mappings, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ are multivalued maximal monotone mappings. The SMVIP is formulated as follows:

$$\text{find a point } \hat{x} \in H_1 \text{ such that } 0 \in f_1(\hat{x}) + B_1(\hat{x}) \tag{1.4}$$

and

$$\hat{y} = A\hat{x} \in H_2 \quad \text{such that } 0 \in f_2(\hat{y}) + B_2(\hat{y}). \tag{1.5}$$

We point out that if (1.4) and (1.5) are considered separately, then each of (1.4) and (1.5) is a monotone variational inclusion problem (MVIP) with solution set $(B_1 + f_1)^{-1}(0)$ and $(B_2 + f_2)^{-1}(0)$, respectively. Moudafi in [33] showed that $x^* \in (B_1 + f_1)^{-1}(0)$ if and only if $x^* = J_\lambda^{B_1}(I - \lambda f_1)(x^*)$, for all $\lambda > 0$, where $J_\lambda^{B_1} : H_1 \to H_1$ is the resolvent operator associated with B_1 and λ defined by

$$J_\lambda^{B_1}(x) = (I + \lambda B_1)^{-1}x, \quad x \in H, \lambda > 0. \tag{1.6}$$

It is known that the resolvent operator $J_\lambda^{B_1}$ is single valued, nonexpansive and 1-inverse strongly monotone (see, e.g., [8]).

Moreover, it was shown in [33] that, if f_1 is an α-inverse strongly monotone mapping and B_1 is a maximal monotone mapping, then $J_\lambda^{B_1}(I - \lambda f_1)$ is averaged with $0 < \lambda < 2\alpha$. Consequently, $J_\lambda^{B_1}(I - \lambda f_1)$ is nonexpansive. Furthermore, $(B_1 + f_1)^{-1}(0)$ was shown to be closed and convex.

Moudafi [33], pointed out that the SMVIP (1.4) and (1.5) generalizes the split fixed-point problem, split feasibility problem, split variational inequality problem, split equilibrium problem, and split variational inclusion problem, which have been studied extensively by several researchers (e.g., see [3, 5, 10, 21, 25, 27, 36, 46, 49]). Moreover, it is applied in solving many real-life problems such as in sensor networks, in computerized tomography and data compression, modeling of inverse problems arising from phase retrieval [9, 19], and in modeling intensity-modulated radiation therapy treatment planning [13, 14].

If $f_1 \equiv 0 \equiv f_2$, then the SMVIP (1.4) and (1.5) reduces to the following *split variational inclusion problem* (SVIP):

$$\text{find a point } \hat{x} \in H_1 \text{ such that } 0 \in B_1(\hat{x}) \tag{1.7}$$

and

$$\hat{y} = A\hat{x} \in H_2 \quad \text{such that } 0 \in B_2(\hat{y}). \tag{1.8}$$

Moudafi [33], showed that the SVIP (1.7) and (1.8) includes the SFP (1.3) as a special case. Several authors have studied and proposed different iterative methods for solving SVIP (1.7) and (1.8), see for instance [22, 27], and the references therein. However, results on SMVIP (1.4) and (1.5) are relatively scanty in the literature.

Very recently, Yao et al. [48] proposed and studied the convergence of the following iterative method with an inertial extrapolation step for approximating the solution of SMVIP (1.4) and (1.5) in Hilbert spaces (Algorithm 1), where $F_1 := J_\lambda^{B_1}(I - \lambda f_1)$ and $F_2 := J_\lambda^{B_2}(I - \lambda f_2)$, $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are an α_1-inverse strongly monotone mapping and an α_2-inverse strongly monotone mapping, respectively, with $\alpha = \min\{\alpha_1, \alpha_2\}$, $A : H_1 \to H_2$ is a bounded linear operator with adjoint A^*, $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ are multivalued maximal monotone mappings. The authors were able to prove the weak-convergence result for the sequence generated by the proposed algorithm under the following conditions:

Algorithm 1

1: Select arbitrary points $x_0, x_1 \in H_1$ and $\theta \in [0, 1)$. Set $n = 1$.

2: Given the iterates x_{n-1} and $x_n, n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\mu_n}{\|x_n - x_{n-1}\|^2}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{if } x_n = x_{n-1}. \end{cases} \tag{1.9}$$

3: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1})$$

and

$$x_{n+1} = F_1\big(w_n + \gamma_n A^*(F_2 - I)Aw_n\big),$$

where

$$\gamma_n := \begin{cases} \tau_n \frac{\|(F_2 - I)Aw_n\|^2}{\|A^*(F_2 - I)Aw_n\|^2}, & \text{if } (F_2 - I)Aw_n \neq 0, \\ \gamma, & \text{if } (F_2 - I)Aw_n = 0, \end{cases} \tag{1.10}$$

where $0 < \tau_n < 1$ and $\gamma > 0$.

3: Set $n \leftarrow n + 1$ and *goto 2.*

(i) The solution set $\mathcal{F}$ is nonempty;
(ii) $\lambda \in (0, 2\alpha)$, $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < 1$;
(iii) $\{\mu_n\}_{n=1}^{\infty} \subset \ell_1$, i.e., $\sum_{n=1}^{\infty} |\mu_n| < \infty$.

Bauschke and Combettes [6] pointed out that in solving optimization problems, the strong convergence of iterative schemes is more desirable and useful than the weak-convergence counterparts. Therefore, when solving optimization problems the authors strive to construct algorithms that generate sequences that converge strongly to the solution of the problem under investigation.

Also, very recently, Chang et al. [16] introduced and studied the following system of monotone variational inclusion problems in Hilbert spaces: find a point $x^* \in H_1$ such that

$$\begin{cases} 0 \in h_i(x^*) + B_i(x^*), \quad i = 1, 2, \dots, m; \quad \text{and} \\ y^* = Ax^* \quad \text{solves } 0 \in g_j(y^*) + D_j(y^*), \quad j = 1, 2, \dots, k, \end{cases} \tag{1.11}$$

where for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k$, h_i and g_j are φ_i- and ϑ_j- inverse strongly monotone mappings on H_1 and H_2, respectively, where $\varphi_i > 0$ and $\vartheta_j > 0$, B_i and D_j are multivalued maximal monotone operators on H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. Set

$$\phi = \min\{\varphi_1, \varphi_2, \dots, \varphi_m; \vartheta_1, \vartheta_2, \dots, \vartheta_k\}, \tag{1.12}$$

then all h_i and g_j are ϕ-inverse strongly monotone mappings. Moreover, the authors proposed the following inertial forward–backward splitting algorithm with the viscosity technique for approximating the solution of problem (1.11) in Hilbert spaces:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = U(I - \gamma A^*(I - T)A)w_n, \\ x_{n+1} = \alpha_n f(w_n) + (1 - \alpha_n)u_n, \end{cases} \tag{1.13}$$

where

$$\begin{cases} U := J_\lambda^{B_1}(I - \lambda h_1) \circ J_\lambda^{B_2}(I - \lambda h_2) \circ \dots J_\lambda^{B_m}(I - \lambda h_m), \\ T := J_\lambda^{D_1}(I - \lambda g_1) \circ J_\lambda^{D_2}(I - \lambda g_2) \circ \dots J_\lambda^{D_k}(I - \lambda g_k), \end{cases} \tag{1.14}$$

for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k$, h_i, g_j, B_i, D_j are as defined in (1.11), and $f : H_1 \to H_1$ is a contraction with contraction constant $\rho \in (\frac{1}{2}, 1)$. The authors proved the strong-convergence theorem for the proposed method under the following conditions:

(i) The solution set Γ is nonempty;
(ii) $\{\alpha_n\} \subset (0, 1)$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \gamma < \frac{1}{2\|A\|^2}$, $\lambda \in (0, 2\phi)$ with ϕ as defined in (1.12);
(iv) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$, $\theta_n \in [0, 1)$.

Remark 1.1 Observe that the problem (1.11) solved by Algorithm (1.13) is more general than the problem SMVIP (1.4) and (1.5) solved by Algorithm 1. The SMVIP (1.4) and (1.5) is a special case of the problem (1.11) when $i = j = 1$. We also point out that the term

$\theta_n(x_n - x_{n-1})$ in Algorithm 1 and Algorithm (1.13) above is referred to as the inertial term. It is employed in algorithm design to accelerate the rate of convergence. However, we note that condition (iii) of Algorithm 1 and condition (iv) of Algorithm (1.13) imposed to incorporate the inertial term are too restrictive. These might affect the implementation of the proposed methods. Some other drawbacks with Algorithm (1.13) are that the contraction constant ρ of the contraction f is restricted to the interval $(\frac{1}{2}, 1)$. Moreover, the implementation of the proposed algorithm requires knowledge of the operator norm, which is often very difficult to calculate or even estimate. On the other hand, while Algorithm 1 does not require knowledge of the operator norm for its implementation the authors were only able to obtain the weak-convergence result for the proposed algorithm.

From the above discourse, it is natural to ask the following question

> *Can we develop a new inertial iterative method with the viscosity technique that does not require knowledge of the operator norm for approximating the solution of system of monotone variational inclusion problems* (1.11)*, such that condition (iii) of Algorithm* 1 *and condition (iv) of Algorithm* (1.13) *are dispensed with and obtain a strong-convergence result? Can the contraction constant ρ of the contraction mapping f of Algorithm* (1.13) *be selected from a larger interval than $(\frac{1}{2}, 1)$?*

Some of our aims in this paper are to provide affirmative answers to the above questions.

Another problem we consider in this paper is the fixed-point problem (FPP). Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonlinear mapping. A point $\hat{x} \in C$ is called a *fixed point* of S if $S\hat{x} = \hat{x}$. We denote by $F(S)$, the set of all fixed points of S, i.e.,

$$F(S) = \{\hat{x} \in C : S\hat{x} = \hat{x}\}. \tag{1.15}$$

In recent years, the study of fixed-point theory for nonlinear mappings has flourished owing to its extensive applications in various fields like economics, compressed sensing, and other applied sciences (see [4, 17, 38] and the references therein).

Recently, optimization problems dealing with finding a common solution of the set of fixed points of nonlinear mappings and the set of solutions of SMVIP (see, for instance, [3, 22]) were considered. One of the motivations for studying such a common solution problem is in its potential application to mathematical models whose constraints can be expressed as FPPs and SMVIP. An instance of this is found in practical problems such as signal processing, network-resource allocation, and image recovery. One scenario is in the network bandwidth-allocation problem for two services in a heterogeneous wireless access networks where the bandwidth of the services are mathematically related (see, for instance, [26, 31] and the references therein).

Motivated by the above results and the current research interest in this direction, in this paper, we study the problem of finding the solution of the system of monotone variational inclusion problems (1.11) with the constraint of a fixed-point set of quasipseudocontractions. Precisely, we consider the following problem: find a point $x^* \in F(S)$ such that

$$\begin{cases} 0 \in h_i(x^*) + B_i(x^*), \quad i = 1, 2, \dots, m; \quad \text{and} \\ y^* = Ax^* \quad \text{solves} \quad 0 \in g_j(y^*) + D_j(y^*), \quad j = 1, 2, \dots, k, \end{cases} \tag{1.16}$$

where $S : H_1 \to H_1$ is a quasipseudocontractive mapping, for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k$, h_i and g_j are φ_i- and ϑ_j- inverse strongly monotone mappings on H_1 and H_2, respectively, where $\varphi_i > 0$ and $\vartheta_j > 0$, B_i and D_j are multivalued maximal monotone operators on H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

Moreover, we introduce a new inertial iterative method that employs the viscosity technique to approximate the solution of the problem in the framework of Hilbert spaces. Furthermore, under mild conditions we prove that the sequence generated by the proposed method converges strongly to a solution of the problem. We point out that the implementation of our algorithm does not require knowledge of the operator norm and the contraction constant of the contraction mapping employed in the viscosity technique can be selected in the interval $(0, 1)$; a larger interval than the restriction to interval $(\frac{1}{2}, 1)$ in Algorithm (1.13). In addition, we obtained a strong-convergence result dispensing with condition (iii) of Algorithm 1 and condition (iv) of Algorithm (1.13). We further apply our results to study other optimization problems and we provide some numerical experiments with graphical illustrations to demonstrate the implementability and efficiency of the proposed method in comparison with some methods in the current literature. Our results in this study improve and extend the recent ones announced by Yao et al. [48], Chang et al. [16], and many other results in the literature.

The paper is organized as follows: In Sect. 2, we recall basic definitions and lemmas employed in the convergence analysis. Section 3 presents the proposed algorithm and highlights some of its features, while in Sect. 4 we analyze the convergence of the proposed method. Section 5 presents applications of our results to some optimization problems. In Sect. 6, we provide some numerical examples with graphical illustrations and compare the performance of our proposed method with some of the existing methods in the literature. Finally, we give some concluding remarks in Sect. 7.

2 Preliminaries

In this section, we present some definitions and results, which will be needed in the following.

In what follows, we denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively, and $w_\omega(x_n)$ denotes the set of weak limits of $\{x_n\}$, that is,

$$\omega_w(x_n) := \bigl\{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\bigr\},$$

where H is a real Hilbert space. For a nonempty closed and convex subset C of H, the *metric projection* [37] $P_C : H \to C$ is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$\|x - P_C x\| = \inf\bigl\{\|x - z\| : z \in C\bigr\}.$$

The operator P_C is nonexpansive and has the following properties [34, 44]:

1. it is firmly nonexpansive, that is,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle \quad \text{for all } x, y \in C;$$

2. for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0 \quad \text{for all } y \in C; \tag{2.1}$$

3. for any $x \in H$ and $y \in C$,

$$\|P_C x - y\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2.$$

Definition 2.1 Let $T : H \to H$ be a nonlinear mapping and I be the identity mapping on H. The mapping $I - T$ is said to be demiclosed at zero, if for any sequence $\{x_n\} \subset H$ that converges weakly to x and $\|x_n - Tx_n\| \to 0$, then $x \in F(T)$.

Definition 2.2 Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be:

(1) *L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C;$$

if $L \in [0, 1)$, then T is called a *contraction*.

(2) *nonexpansive* if T is 1-*L*ipschitz continuous;

(3) *averaged* if it can be written as

$$T = (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$, $S : C \to C$ is nonexpansive and I is the identity mapping on C;

(4) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in C;$$

(5) *quasinonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, \text{and } p \in F(T);$$

(6) *firmly quasinonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \left\|(I - T)x\right\|^2, \quad \forall x \in C, \text{and } p \in F(T);$$

(7) *κ-strictly pseudocontractive* if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in C;$$

(8) *directed* if $F(T) \neq \emptyset$ and $\langle Tx - p, Tx - x\rangle \leq 0$, $\forall x \in C$, and $p \in F(T)$;

(9) *demicontractive* if $F(T) \neq \emptyset$ and there exists $\kappa \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \kappa\|Tx - x\|^2, \quad \forall x \in C \text{ and } p \in F(T);$$

(10) *monotone* if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(11) *L-inverse strongly monotone* (L-ism), if there exits $L > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq L\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Remark 2.3 As pointed out by Bauschke and Combettes [6], $T : C \to C$ is directed if and only if

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|Tx - x\|^2, \quad \forall x \in C, \text{and } p \in F(T).$$

In other words, the class of directed mappings coincides with the class of firmly quasinonexpansive mappings.

Remark 2.4 From the definitions above, we observe that the class of demicontractive mappings includes several other classes of nonlinear mappings such as the directed mappings, the quasinonexpansive mappings, and the strictly pseudocontractive mappings with fixed points as special cases. Also, it is well known that every L-ism mapping is $\frac{1}{L}$-Lipschitz continuous and monotone, and every Lipschitz continuous operator is uniformly continuous but the converse of these statements are not always true (see, for example [41]).

Definition 2.5 A nonlinear operator $T : C \to C$ is called *pseudocontractive* if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

The interest of pseudocontractive mappings lies in their connection with monotone mappings, that is, T is a pseudocontraction if and only if $I - T$ is a monotone mapping. It is well known that T is pseudocontractive if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

Definition 2.6 An operator $T : C \to C$ is said to be *quasipseudocontractive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|Tx - x\|^2, \quad \forall x \in C, p \in F(T).$$

It is obvious that the class of quasipseudocontractive mappings includes the class of demicontractive mappings and the class of pseudocontractive mappings with a nonempty fixed-point set.

We have the following result on L-Lipschitz quasipseudocontractive mappings.

Lemma 2.7 ([15]) *Let H be a real Hilbert space and $T : H \to H$ be an L-Lipschitzian mapping with $L \geq 1$. Denote*

$$G := (1 - \psi)I + \psi T\big((1 - \eta)I + \eta T\big).$$

If $0 < \psi < \eta < \frac{1}{1+\sqrt{1+L^2}}$, *then the following conclusions hold*:

(1) $F(T) = F(T((I-\eta)I + \eta T)) = F(G)$.

(2) *If* $I - T$ *is demiclosed at* 0, *then* $I - G$ *is also demiclosed at* 0.

(3) *In addition, if* $T : H \to H$ *is quasipseudocontractive, then the mapping* G *is quasinonexpansive.*

Lemma 2.8 ([49]) (*Demiclosedness Principle*). *Let* T *be a nonexpansive mapping on a closed convex subset* C *of a real Hilbert space* H. *Then,* $I - T$ *is demiclosed at any point* $y \in H$, *that is, if* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to y \in H$, *then* $x - Tx = y$.

Lemma 2.9 ([18]) *Let* H *be a real Hilbert space. Then, the following results hold for all* $x, y \in H$ *and* $\delta \in \mathbb{R}$:

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;

(ii) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;

(iii) $\|\delta x + (1-\delta)y\|^2 = \delta\|x\|^2 + (1-\delta)\|y\|^2 - \delta(1-\delta)\|x-y\|^2$.

Lemma 2.10 ([40]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers,* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\{b_n\}$ *be a sequence of real numbers. Assume that*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n b_n, \quad \textit{for all } n \ge 1,$$

if $\limsup_{k\to\infty} b_{n_k} \le 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \ge 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11 ([32]) *Let* $\{a_n\}, \{c_n\} \subset \mathbb{R}_+$, $\{\sigma_n\} \subset (0,1)$, *and* $\{b_n\} \subset \mathbb{R}$ *be sequences such that*

$$a_{n+1} \le (1-\sigma_n)a_n + b_n + c_n \quad \textit{for all } n \ge 0.$$

Assume $\sum_{n=0}^{\infty} |c_n| < \infty$. *Then, the following results hold*:

(i) *If* $b_n \le \beta\sigma_n$ *for some* $\beta \ge 0$, *then* $\{a_n\}$ *is a bounded sequence.*

(ii) *If we have*

$$\sum_{n=0}^{\infty} \sigma_n = \infty \quad \textit{and} \quad \limsup_{n\to\infty} \frac{b_n}{\sigma_n} \le 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 ([7, 47]) *Let* H *be a real Hilbert space and let* $A, S, T, V : H \to H$ *be given operators.*

(i) *If* $T = (1-\alpha)S + \alpha V$ *for some* $\alpha \in (0,1)$, *where* $S : H \to H$ *is* β*-averaged and* $V : H \to H$ *is nonexpansive, then* T *is* $\alpha + (1-\alpha)\beta$*-averaged.*

(ii) *The composite of finitely many averaged mappings is averaged. In particular, if* T_i *is* α_i*-averaged, where* $\alpha_i \in (0,1)$ *for* $i = 1, 2$, *then the composite* $T_1 \circ T_2$ *is* α*-averaged, where* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

(iii) *If the mappings* $\{T_i\}_{i=1}^N$ *are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} F(T_i) = F(T_1 \circ T_2 \circ \cdots \circ T_N).$$

(iv) *If A is β-ism and $\gamma \in (0,\beta]$, then $T := I - \gamma A$ is firmly nonexpansive.*
(v) *T is nonexpansive if and only if its complement $I - T$ is $\frac{1}{2}$-ism.*
(vi) *If T is β-ism, then for $\gamma > 0$, γT is $\frac{\beta}{\gamma}$-ism.*
(vii) *T is averaged if and only if its complement $I - T$ is β-ism for some $\beta > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*
(viii) *T is firmly nonexpansive if and only if its complement $I - T$ is firmly nonexpansive.*

Lemma 2.13 ([45]) *Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a nonzero bounded linear operator with adjoint A^* and let $T : H_2 \to H_2$ be a nonexpansive mapping. Then, $A^*(I - T)A$ is $\frac{1}{2\|A\|^2}$-ism.*

Lemma 2.14 ([44]) *Let H be a real Hilbert space, $r > 0, f : H \to H$ be a μ-ism mapping and $B : H \to 2^H$ be a maximal monotone mapping. Then,*
(I) *the following conclusions are equivalent:*
  (i) $x^* \in H$ *such that* $0 \in f(x^*) + B(x^*)$;
  (ii) $x^* \in F(J_r^B(I - rf))$.
(II) *If $r \in (0, 2\mu)$, then $J_r^B(I - rf)$ is averaged.*

3 Proposed method

In this section, we present our proposed algorithm and highlight some of its important features. We assume that:
(1) H_1 and H_2 are real Hilbert spaces;
(2) For each $i = 1,2,\dots,m$ and $j = 1,2,\dots,k$, h_i and g_j are φ_i- and ϑ_j- inverse strongly monotone mappings on H_1 and H_2, respectively, where $\varphi_i > 0$ and $\vartheta_j > 0$, B_i and D_j are multivalued maximal monotone operators on H_1 and H_2, respectively. Let $\phi = \min\{\varphi_1,\varphi_2,\dots,\varphi_m;\vartheta_1,\vartheta_2,\dots,\vartheta_k\}$, then all h_i and g_j are ϕ-inverse strongly monotone mappings;
(3) $A : H_1 \to H_2$ is a bounded linear operator with adjoint A^* and $f : H_1 \to H_1$ is a contraction with contraction constant $\rho \in (0,1)$;
(4) $S : H_1 \to H_1$ is a K-Lipschitz continuous quasipseudocontractive mapping such that $I - S$ is demiclosed at zero and with $K \geq 1$;
(5) The solution set $\Omega = \Gamma \cap F(S)$ is nonempty, where

$$\Gamma := \left\{x \in H_1 : x \in \bigcap_{i=1}^{m}\big((h_i + B_i)^{-1}(0)\big) \cap \left(A^{-1}\left(\bigcap_{j=1}^{k}(g_j + D_j)^{-1}(0)\right)\right)\right\}. \tag{3.1}$$

(6) We denote

$$\begin{cases} U := J_\lambda^{B_1}(I - \lambda h_1) \circ J_\lambda^{B_2}(I - \lambda h_2) \circ \dots J_\lambda^{B_m}(I - \lambda h_m), \\ T := J_\lambda^{D_1}(I - \lambda g_1) \circ J_\lambda^{D_2}(I - \lambda g_2) \circ \dots J_\lambda^{D_k}(I - \lambda g_k). \end{cases} \tag{3.2}$$

It was shown in [16] that the operators U and T defined above are averaged mappings.

We establish the strong-convergence result for the proposed algorithm under the following conditions on the control parameters:
(C1) $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2) $\{\beta_n\}, \{\delta_n\}, \{\xi_n\} \subset (0,1)$ such that $0 < a \leq \beta_n, \delta_n, \xi_n \leq b < 1$;

Algorithm 2

Step 0. Let $x_0, x_1 \in H_1$ be two arbitrary initial points and set $n = 1$.

Step 1. Given the $(n-1)$th and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ being defined by

$$\hat{\theta}_n = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.3}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$u_n = U\big(w_n + \gamma_n A^*(T-I)Aw_n\big),$$

where

$$\gamma_n := \begin{cases} \tau_n \frac{\|(T-I)Aw_n\|^2}{\|A^*(T-I)Aw_n\|^2}, & \text{if } Aw_n \neq TAw_n, \\ \gamma, & \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases} \tag{3.4}$$

Step 4. Compute

$$v_n = \beta_n w_n + (1-\beta_n)\mathbb{V}u_n,$$

where

$$\mathbb{V} = (1-\eta)I + \eta S\big((1-\mu)I + \mu S\big).$$

Step 5. Compute

$$x_{n+1} = \alpha_n f(w_n) + \delta_n u_n + \xi_n v_n.$$

Set $n := n + 1$ and return to *Step* 1.

(C3) $\{\epsilon_n\}$ is a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$;

(C4) $\theta > 0$, $\lambda \in (0, 2\phi)$, $0 < \eta < \mu < \frac{1}{1+\sqrt{1+K^2}}$ and $0 < \tau_1 \le \tau_n \le \tau_2 < 1$.

Now, our main algorithm is presented in Algorithm 2.

Remark 3.1

- We point out that the step size of the proposed method defined in (3.4) does not depend on the norm of the bounded linear operator. This makes our algorithm easy to implement, unlike the methods proposed in [16, 22, 33, 50], which require knowledge of the operator norm for their implementation.

- Step 1 of the algorithm can be implemented since the value of $\|x_n - x_{n-1}\|$ is known prior to choosing θ_n. Also, observe that in incorporating the inertial term our method does not require stringent conditions, like we have in condition (iii) of Algorithm 1 and condition (iv) of Algorithm (1.13).
- We note that unlike in Algorithm (1.13), the viscosity technique employed in *Step* 5 of our algorithm accommodates a larger class of contraction mappings since the contraction constant $\rho \in (0,1)$.

Remark 3.2 By conditions (C1) and (C3), it follows from (3.3) that

$$\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0. \tag{3.5}$$

Remark 3.3 We note that in (3.4), the choice of the step size γ_n is independent of the operator norm $\|A\|$. Also, the value of γ has no effect on the proposed algorithm but was introduced for clarity. Now, we show that the step size of the algorithm in (3.4) is well defined.

Lemma 3.4 *The step sizes* $\{\gamma_n\}$ *of the Algorithm* 2 *defined by* (3.4) *are well defined.*

Proof Let $p \in \Omega$. Then, by Lemma 2.14(I) we have that $p \in \bigcap_{i=1}^{m}((h_i + B_i)^{-1}(0))$ and $Ap \in \bigcap_{j=1}^{k}((g_j + D_j)^{-1}(0))$. From Lemma 2.12(iii) we have $p \in F(U)$ and $Ap \in F(T)$. Applying the fact that T is averaged together with Lemma 2.12(vii), we have

$$\begin{aligned} \big\|A^*(I-T)Aw_n\big\|\|w_n - p\| &\geq \big\langle A^*(I-T)Aw_n, w_n - p\big\rangle \\ &= \big\langle (I-T)Aw_n - (I-T)Ap, Aw_n - Ap\big\rangle \\ &\geq \beta\big\|(I-T)Aw_n\big\|^2, \end{aligned}$$

for some $\beta > \frac{1}{2}$. This shows that $\|A^*(I-T)Aw_n\| > 0$ when $\|(I-T)Aw_n\| \neq 0$. Thus, $\{\gamma_n\}$ is well defined. □

4 Convergence analysis

First, we establish some lemmas before proving the strong-convergence theorem for the proposed algorithm.

Lemma 4.1 *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 2. *Then,* $\{x_n\}$ *is bounded.*

Proof Observe that the mapping $P_\Omega \circ f$ is a contraction. Then, by the Banach Contraction Principle there exists an element $p \in H$ such that $p = P_\Omega \circ f(p)$. It follows that $p \in \Omega$, $Sp = p$, $Up = p$, and $TAp = Ap$. By applying Lemma 2.9(ii) and the nonexpansiveness of U, we obtain

$$\begin{aligned} \|u_n - p\|^2 &= \big\|U\big(w_n + \gamma_n A^*(T-I)Aw_n\big) - p\big\|^2 \\ &\leq \big\|w_n + \gamma_n A^*(T-I)Aw_n - p\big\|^2 \qquad (4.1) \\ &= \|w_n - p\|^2 + \gamma_n^2\big\|A^*(T-I)Aw_n\big\|^2 + 2\gamma_n\big\langle w_n - p, A^*(T-I)Aw_n\big\rangle. \qquad (4.2) \end{aligned}$$

Again, applying Lemma 2.9(ii) and the nonexpansiveness of T, we have

$$\begin{aligned}
&\langle w_n - p, A^*(T-I)Aw_n\rangle \\
&= \langle Aw_n - Ap, (T-I)Aw_n\rangle \\
&= \langle TAw_n - Ap - (T-I)Aw_n, (T-I)Aw_n\rangle \\
&= \langle TAw_n - Ap, (T-I)Aw_n\rangle - \langle (T-I)Aw_n, (T-I)Aw_n\rangle \\
&= \langle TAw_n - Ap, (T-I)Aw_n\rangle - \|(T-I)Aw_n\|^2 \\
&= \frac{1}{2}\big[\|TAw_n - Ap\|^2 + \|(T-I)Aw_n\|^2 - \|TAw_n - Ap - (T-I)Aw_n\|^2\big] \\
&\quad - \|(T-I)Aw_n\|^2 \\
&= \frac{1}{2}\big[\|TAw_n - Ap\|^2 + \|(T-I)Aw_n\|^2 - \|Aw_n - Ap\|^2\big] - \|(T-I)Aw_n\|^2 \\
&= \frac{1}{2}\big[\|TAw_n - Ap\|^2 - \|Aw_n - Ap\|^2 - \|(T-I)Aw_n\|^2\big] \\
&\leq \frac{1}{2}\big[\|Aw_n - Ap\|^2 - \|Aw_n - Ap\|^2 - \|(T-I)Aw_n\|^2\big] \\
&= -\frac{1}{2}\|(T-I)Aw_n\|^2. \qquad (4.3)
\end{aligned}$$

Applying (4.3) into (4.2) and using the definition of γ_n together with the condition on τ_n, we have

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(T-I)Aw_n\|^2 - \gamma_n\|(T-I)Aw_n\|^2 \\
&= \|w_n - p\|^2 - \gamma_n\big[\|(T-I)Aw_n\|^2 - \gamma_n\|A^*(T-I)Aw_n\|^2\big] \\
&= \|w_n - p\|^2 - \gamma_n(1-\tau_n)\|(T-I)Aw_n\|^2 \qquad (4.4) \\
&\leq \|w_n - p\|^2. \qquad (4.5)
\end{aligned}$$

Next, using the triangle inequality, we obtain from *Step* 2

$$\begin{aligned}
\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\
&\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|. \qquad (4.6)
\end{aligned}$$

Since, by Remark 3.2, $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, then there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1$ for all $n \geq 1$. Consequently, from (4.6) we obtain

$$\|w_n - p\| \leq \|x_n - p\| + \alpha_n M_1. \qquad (4.7)$$

By the conditions on η and μ, and by Lemma 2.7, we know that $\mathbb{V}$ is quasinonexpansive. Consequently, by applying the triangle inequality, and using (4.5) and (4.7), from *Step* 4 we have

$$\begin{aligned}
\|v_n - p\| &= \big\|\beta_n w_n + (1-\beta_n \mathbb{V}u_n) - p\big\| \\
&\leq \beta_n\|w_n - p\| + (1-\beta_n)\|\mathbb{V}u_n - p\| \\
&\leq \beta_n\|w_n - p\| + (1-\beta_n)\|u_n - p\| \\
&\leq \beta_n\|w_n - p\| + (1-\beta_n)\|w_n - p\| \\
&= \|w_n - p\| \\
&\leq \|x_n - p\| + \alpha_n M_1. \qquad (4.8)
\end{aligned}$$

Now, by applying (4.5) and (4.8), from *Step* 5 it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \big\|\alpha_n f(w_n) + \delta_n u_n + \xi_n v_n - p\big\| \\
&= \big\|\alpha_n\big(f(w_n) - f(p)\big) + \alpha_n\big(f(p) - p\big) + \delta_n(u_n - p) + \xi_n(v_n - p)\big\| \\
&\leq \alpha_n\rho||w_n - p|| + \alpha_n||f(p) - p|| + \delta_n||u_n - p|| + \xi_n||v_n - p|| \\
&\leq \alpha_n\rho\big(\|x_n - p\| + \alpha_n M_1\big) + \alpha_n\big\|f(p) - p\big\| + \delta_n\big(\|x_n - p\| + \alpha_n M_1\big) \\
&\quad + \xi_n\big(\|x_n - p\| + \alpha_n M_1\big) \\
&= \big(\alpha_n\rho + (1-\alpha_n)\big)||x_n - p|| + \alpha_n||f(p) - p|| + \big(\alpha_n\rho + (1-\alpha_n)\big)\alpha_n M_1 \\
&= \big(1 - \alpha_n(1-\rho)\big)\|x_n - p\| + \alpha_n(1-\rho)\left\{\frac{\|f(p) - p\|}{1-\rho} + \frac{(1-\alpha_n(1-\rho))M_1}{1-\rho}\right\} \\
&\leq \big(1 - \alpha_n(1-\rho)\big)\|x_n - p\| + \alpha_n(1-\rho)M^*,
\end{aligned}$$

where $M^* := \sup_{n\in\mathbb{N}}\{\frac{\|f(p)-p\|}{1-\rho} + \frac{(1-\alpha_n(1-\rho))M_1}{1-\rho}\}$. Set $a_n := \|x_n - p\|$; $b_n := \alpha_n(1-\rho)M^*$; $c_n := 0$, and $\sigma_n := \alpha_n(1-\rho)$. By invoking Lemma 2.11(i) together with the assumptions on the control parameters, we have that $\{\|x_n - p\|\}$ is bounded and this implies that $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{u_n\}$, and $\{v_n\}$ are all bounded. □

Lemma 4.2 *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 2 *and* $p \in \Omega$. *Then, under conditions* (C1)–(C4) *the following inequality holds for all* $n \in \mathbb{N}$:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \left(1 - \frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\right)\|x_n - p\|^2 \\
&\quad + \frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\left\{\frac{\alpha_n}{2(1-\rho)}M + \frac{3M_2((1-\alpha_n)^2 + \alpha_n\rho)}{2(1-\rho)}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right. \\
&\quad \left. + \frac{1}{(1-\rho)}\langle f(p) - p, x_{n+1} - p\rangle\right\} \\
&\quad - \frac{\xi_n(1-\alpha_n)(1-\beta_n)}{(1-\alpha_n\rho)}\big\{\gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2 + \beta_n\|w_n - \mathbb{V}u_n\|\big\}.
\end{aligned}$$

Proof Let $p \in \Omega$. Then, by applying the Cauchy–Schwartz inequality together with Lemma 2.9(ii), we obtain

$$\|w_n - p\|^2 = \big\|x_n + \theta_n(x_n - x_{n-1}) - p\big\|^2$$

$$
\begin{aligned}
&= \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|\big(\theta_n\|x_n - x_{n-1}\| + 2\|x_n - p\|\big) \\
&\leq \|x_n - p\|^2 + 3M_2\theta_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|, \qquad (4.9)
\end{aligned}
$$

where $M_2 := \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \theta_n\|x_n - x_{n-1}\|\} > 0$.

Also, by applying Lemma 2.9(iii), (4.4) and (4.9), we have

$$
\begin{aligned}
\|v_n - p\|^2 &= \big\|\beta_n w_n + (1-\beta_n)\mathbb{V}u_n - p\big\|^2 \\
&= \beta_n\|w_n - p\|^2 + (1-\beta_n)\|\mathbb{V}u_n - p\|^2 - \beta_n(1-\beta_n)\|w_n - \mathbb{V}u_n\| \\
&\leq \beta_n\|w_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 - \beta_n(1-\beta_n)\|w_n - \mathbb{V}u_n\| \\
&\leq \beta_n\|w_n - p\|^2 + (1-\beta_n)\big\{\|w_n - p\|^2 - \gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2\big\} \\
&\quad - \beta_n(1-\beta_n)\|w_n - \mathbb{V}u_n\| \\
&= \|w_n - p\|^2 - (1-\beta_n)\gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2 \\
&\quad - \beta_n(1-\beta_n)\|w_n - \mathbb{V}u_n\| \qquad (4.10) \\
&\leq \|w_n - p\|^2. \qquad (4.11)
\end{aligned}
$$

Next, invoking Lemma 2.9(i), and applying (4.5), (4.9) and (4.10) we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\|\alpha_n f(w_n) + \delta_n u_n + \xi_n v_n - p\big\|^2 \\
&\leq \big\|\delta_n(u_n - p) + \xi_n(v_n - p)\big\|^2 + 2\alpha_n\big\langle f(w_n) - p, x_{n+1} - p\big\rangle \\
&\leq \delta_n^2\|u_n - p\|^2 + \xi_n^2\|v_n - p\|^2 + 2\delta\xi_n\|u_n - p\|\|v_n - p\| \\
&\quad + 2\alpha_n\big\langle f(w_n) - f(p), x_{n+1} - p\big\rangle \\
&\quad + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\leq \delta_n^2\|u_n - p\|^2 + \xi_n^2\|v_n - p\|^2 + \delta\xi_n\big\{\|u_n - p\|^2 + \|v_n - p\|^2\big\} \\
&\quad + 2\alpha_n\rho\|w_n - p\|\|x_{n+1} - p\| \\
&\quad + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\leq \delta_n(\delta_n + \xi_n)\|u_n - p\|^2 + \xi_n(\xi_n + \delta_n)\|v_n - p\|^2 \\
&\quad + \alpha_n\rho\big\{\|w_n - p\|^2 + \|x_{n+1} - p\|^2\big\} \\
&\quad + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\leq \delta_n(1-\alpha_n)\|w_n - p\|^2 + \xi_n(1-\alpha_n)\big\{\|w_n - p\|^2 \\
&\quad - (1-\beta_n)\gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2 \\
&\quad - \beta_n(1-\beta_n)\|w_n - \mathbb{V}u_n\|\big\} + \alpha_n\rho\big\{\|w_n - p\|^2 + \|x_{n+1} - p\|^2\big\} \\
&\quad + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \big((1-\alpha_n)^2 + \alpha_n\rho\big)\|w_n - p\|^2 \\
&\quad - \xi_n(1-\alpha_n)(1-\beta_n)\big\{\gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2 + \beta_n\|w_n - \mathbb{V}u_n\|\big\} \\
&\quad + \alpha_n\rho\|x_{n+1} - p\|^2 + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\leq \big((1-\alpha_n)^2 + \alpha_n\rho\big)\left\{\|x_n - p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right\} \\
&\quad + \alpha_n\rho\|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle - \xi_n(1-\alpha_n)(1-\beta_n)\big\{\gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2 \\
&\quad + \beta_n\|w_n - \mathbb{V}u_n\|\big\}.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \frac{(1 - 2\alpha_n + \alpha_n^2 + \alpha_n\rho)}{(1-\alpha_n\rho)}\|x_n - p\|^2 \\
&\quad + 3M_2\frac{((1-\alpha_n)^2 + \alpha_n\rho)}{(1-\alpha_n\rho)}\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\quad + \frac{2\alpha_n}{(1-\alpha_n\rho)}\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\quad - \frac{\xi_n(1-\alpha_n)(1-\beta_n)}{(1-\alpha_n\rho)}\big\{\gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2 \\
&\quad + \beta_n\|w_n - \mathbb{V}u_n\|\big\} \\
&= \frac{(1 - 2\alpha_n + \alpha_n\rho)}{(1-\alpha_n\rho)}\|x_n - p\|^2 + \frac{\alpha_n^2}{(1-\alpha_n\rho)}\|x_n - p\|^2 \\
&\quad + 3M_2\frac{((1-\alpha_n)^2 + \alpha_n\rho)}{(1-\alpha_n\rho)}\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\quad + \frac{2\alpha_n}{(1-\alpha_n\rho)}\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\quad - \frac{\xi_n(1-\alpha_n)(1-\beta_n)}{(1-\alpha_n\rho)}\big\{\gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2 \\
&\quad + \beta_n\|w_n - \mathbb{V}u_n\|\big\} \\
&\leq \left(1 - \frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\right)\|x_n - p\|^2 \\
&\quad + \frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\left\{\frac{\alpha_n}{2(1-\rho)}M + \frac{3M_2((1-\alpha_n)^2 + \alpha_n\rho)}{2(1-\rho)}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\right. \\
&\quad \left. + \frac{1}{(1-\rho)}\big\langle f(p) - p, x_{n+1} - p\big\rangle\right\} \\
&\quad - \frac{\xi_n(1-\alpha_n)(1-\beta_n)}{(1-\alpha_n\rho)}\big\{\gamma_n(1-\tau_n)\big\|(T-I)Aw_n\big\|^2 + \beta_n\|w_n - \mathbb{V}u_n\|\big\},
\end{aligned}
$$

where $M := \sup\{\|x_n - p\|^2 : n \in \mathbb{N}\}$. Hence, we have the required inequality. □

Lemma 4.3 *Suppose $\{x_n\}$ is a sequence generated by Algorithm 2 such that conditions* (C1)–(C4) *are satisfied. Then, the following inequality holds for all $p \in \Omega$ and $n \in \mathbb{N}$:*

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\left\{\left\|f(w_n)-p\right\|^2 + 3M_2(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\right\} - \delta\xi_n\|u_n-v_n\|^2.$$

Proof Let $p \in \Omega$. By applying Lemma 2.9(iii) together with (4.5), (4.9) and (4.11) we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\alpha_n f(w_n) + \delta_n u_n + \xi_n v_n - p\right\|^2 \\
&\le \alpha_n\left\|f(w_n)-p\right\|^2 + \delta\|u_n-p\|^2 + \xi_n\|v_n-p\|^2 - \delta\xi_n\|u_n-v_n\|^2 \\
&\le \alpha_n\left\|f(w_n)-p\right\|^2 + \delta\|w_n-p\|^2 + \xi_n\|w_n-p\|^2 - \delta\xi_n\|u_n-v_n\|^2 \\
&= \alpha_n\left\|f(w_n)-p\right\|^2 + (1-\alpha_n)\|w_n-p\|^2 - \delta\xi_n\|u_n-v_n\|^2 \\
&\le \alpha_n\left\|f(w_n)-p\right\|^2 + (1-\alpha_n)\left\{\|x_n-p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\right\} \\
&\quad - \delta\xi_n\|u_n-v_n\|^2 \\
&= (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\left\{\left\|f(w_n)-p\right\|^2 + 3M_2(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\right\} \\
&\quad - \delta\xi_n\|u_n-v_n\|^2,
\end{aligned}$$

which is the required inequality. □

Now, we are in a position to state and prove the strong-convergence theorem for the proposed algorithm.

Theorem 4.4 *Let H_1 and H_2 be two real Hilbert spaces and let $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in (0,1)$. Suppose $\{x_n\}$ is a sequence generated by Algorithm 2 such that conditions* (C1)–(C4) *hold. Then, the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega$, where $\hat{x} = P_\Omega \circ f(\hat{x})$.*

Proof Let $\hat{x} = P_\Omega \circ f(\hat{x})$. From Lemma 4.2, we obtain

$$\begin{aligned}
\|x_{n+1}-\hat{x}\|^2 &\le \left(1-\frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\right)\|x_n-\hat{x}\|^2 \\
&\quad + \frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\left\{\frac{\alpha_n}{2(1-\rho)}M + \frac{3M_2((1-\alpha_n)^2+\alpha_n\rho)}{2(1-\rho)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\right. \\
&\quad \left. + \frac{1}{(1-\rho)}\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle\right\}.
\end{aligned} \tag{4.12}$$

Next, we claim that the sequence $\{\|x_n-\hat{x}\|\}$ converges to zero. In order to establish this, by Lemma 2.10, it suffices to show that $\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x}, x_{n_k+1}-\hat{x}\rangle \le 0$ for every subsequence $\{\|x_{n_k}-\hat{x}\|\}$ of $\{\|x_n-\hat{x}\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-\hat{x}\| - \|x_{n_k}-\hat{x}\|\right) \ge 0.$$

Suppose that $\{\|x_{n_k} - \hat{x}\|\}$ is a subsequence of $\{\|x_n - \hat{x}\|\}$ such that

$$\liminf_{k\to\infty}\big(\|x_{n_k+1} - \hat{x}\| - \|x_{n_k} - \hat{x}\|\big) \geq 0. \tag{4.13}$$

Again, from Lemma 4.2 we obtain

$$\begin{aligned}
&\frac{\xi_{n_k}(1-\alpha_{n_k})(1-\beta_{n_k})}{(1-\alpha_{n_k}\rho)}\gamma_{n_k}(1-\tau_{n_k})\big\|(T-I)Aw_{n_k}\big\|^2 \\
&\quad\leq \left(1 - \frac{2\alpha_{n_k}(1-\rho)}{(1-\alpha_{n_k}\rho)}\right)\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 + \frac{2\alpha_{n_k}(1-\rho)}{(1-\alpha_{n_k}\rho)}\left\{\frac{\alpha_{n_k}}{2(1-\rho)}M\right. \\
&\qquad + \frac{3M_2((1-\alpha_{n_k})^2 + \alpha_{n_k}\rho)}{2(1-\rho)}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| \\
&\qquad \left. + \frac{1}{(1-\rho)}\langle f(p) - p, x_{n_k+1} - p\rangle\right\}.
\end{aligned}$$

By applying (4.13) together with condition (C2) and the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we have

$$\gamma_{n_k}(1-\tau_{n_k})\big\|(T-I)Aw_{n_k}\big\|^2 \to 0, \quad k\to\infty.$$

By the definition of γ_n, we obtain

$$\tau_{n_k}(1-\tau_{n_k})\frac{\|(T-I)Aw_{n_k}\|^4}{\|A^*(T-I)Aw_{n_k}\|^2} \to 0, \quad k\to\infty.$$

Consequently, we have

$$\frac{\|(T-I)Aw_{n_k}\|^2}{\|A^*(T-I)Aw_{n_k}\|} \to 0, \quad k\to\infty.$$

Since $\|A^*(T-I)Aw_{n_k}\|$ is bounded, it follows that

$$\big\|(T-I)Aw_{n_k}\big\| \to 0, \quad k\to\infty. \tag{4.14}$$

Consequently, we obtain

$$\begin{aligned}
\big\|A^*(T-I)Aw_{n_k}\big\| &\leq \big\|A^*\big\|\big\|(T-I)Aw_{n_k}\big\| \\
&= \|A\|\big\|(T-I)Aw_{n_k}\big\| \to 0, \quad k\to\infty.
\end{aligned} \tag{4.15}$$

Following a similar argument, from Lemma 4.2 we obtain

$$\beta_{n_k}\|w_{n_k} - \mathbb{V}u_{n_k}\| \to 0, \quad k\to\infty.$$

By condition (C2), it follows that

$$\|w_{n_k} - \mathbb{V}u_{n_k}\| \to 0, \quad k\to\infty. \tag{4.16}$$

Next, from Lemma 4.3 we obtain

$$\delta\xi_{n_k}\|u_{n_k}-v_{n_k}\|^2 \le (1-\alpha_{n_k})\|x_{n_k}-\hat{x}\|^2 - \|x_{n_{k}k+1}-\hat{x}\|^2 + \alpha_{n_k}\Big\{\big\|f(w_{n_k})-\hat{x}\big\|^2 + 3M_2(1-\alpha_{n_k})\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\Big\}.$$

By (4.13), Remark 3.2, and the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we obtain

$$\delta\xi_{n_k}\|u_{n_k}-v_{n_k}\|^2 \to 0, \quad k\to\infty. \tag{4.17}$$

Consequently, we have

$$\|u_{n_k}-v_{n_k}\| \to 0, \quad k\to\infty. \tag{4.18}$$

By Remark 3.2, we have

$$\|w_{n_k}-x_{n_k}\| = \theta_{n_k}\|x_{n_k}-x_{n_k-1}\| \to 0, \quad k\to\infty. \tag{4.19}$$

By applying (4.16), from *Step* 4 we have

$$\|v_{n_k}-w_{n_k}\| \le \beta_{n_k}\|w_{n_k}-w_{n_k}\| + (1-\beta_{n_k})\|\mathbb{V}u_{n_k}-w_{n_k}\| \to 0, \quad k\to\infty. \tag{4.20}$$

Next, by applying (4.16), (4.18), (4.19) and (4.20) we obtain

$$\begin{aligned} &\|w_{n_k}-u_{n_k}\| \to 0, \quad k\to\infty; \\ &\|x_{n_k}-\mathbb{V}u_{n_k}\| \to 0, \quad k\to\infty; \\ &\|x_{n_k}-v_{n_k}\| \to 0, \quad k\to\infty, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} &\|x_{n_k}-u_{n_k}\| \to 0, \quad k\to\infty; \\ &\|u_{n_k}-\mathbb{V}u_{n_k}\| \to 0, \quad k\to\infty. \end{aligned} \tag{4.22}$$

Now, applying (4.21) and (4.22) together with the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we obtain

$$\begin{aligned} \|x_{n_k+1}-x_{n_k}\| &= \big\|\alpha_{n_k}f(w_{n_k}) + \delta_{n_k}u_{n_k} + \xi_{n_k}v_{n_k} - x_{n_k}\big\| \\ &\le \alpha_{n_k}\big\|f(w_{n_k})-x_{n_k}\big\| + \delta_{n_k}\|u_{n_k}-x_{n_k}\| \\ &\quad + \xi_{n_k}\|v_{n_k}-x_{n_k}\| \to 0, \quad k\to\infty. \end{aligned} \tag{4.23}$$

To complete the proof, we need to show that $w_\omega(x_n)\subset\Omega$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^*\in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$. From (4.22), we have that $u_{n_k}\rightharpoonup x^*$ as $k\to\infty$. Since $I-\mathbb{V}$ is demiclosed at zero, then it follows from (4.22) and Lemma 2.7 that $x^*\in F(\mathbb{V})=F(S)$. That is, $w_\omega(x_n)\subset F(S)$.

Next, we show that $w_\omega(x_n) \subset \Gamma$. From *Step* 3 and by applying (4.21), we have

$$\lim_{k\to\infty} \big\| U\big(I + \gamma_n A^*(T - I)A\big)w_{n_k} - w_{n_k}\big\| = \lim_{k\to\infty} \|u_{n_k} - w_{n_k}\| = 0. \tag{4.24}$$

Since the operators U and $I + \gamma_n A^*(T - I)A$ are averaged, it follows from Lemma 2.12(ii) that the composition $U(I + \gamma_n A^*(T - I)A)$ is also average and consequently nonexpansive. By the Demiclosedness Principle for nonexpansive mappings, and by applying (4.19) and (4.24) we obtain $U(I + \gamma_n A^*(T - I)A)x^* = x^*$. Since $\Omega \neq \emptyset$, then by Lemma 2.12(iii) we have $Ux^* = x^*$ and $(I + \gamma_n A^*(T - I)A)x^* = x^*$. It then follows from Lemma 2.12(iii) and Lemma 2.14(I) that

$$0 \in \bigcap_{i=1}^{m} (h_i + B_i)x^*. \tag{4.25}$$

Sine T is nonexpansive, then by the Demiclosedness Principle for nonexpansive mappings, and by applying (4.14) and (4.19) we have $TAx^* = Ax^*$. It then follows from Lemma 2.12(iii) and Lemma 2.14(I) that

$$0 \in \bigcap_{j=1}^{k} (g_i + D_i)Ax^*. \tag{4.26}$$

From (4.25) and (4.26), we obtain $w_\omega(x_n) \subset \Gamma$. Consequently, we have that $w_\omega(x_n) \subset \Omega$.

Next, from (4.21) we have that $w_\omega\{v_{n_k}\} = w_\omega\{x_{n_k}\}$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\begin{aligned}
\lim_{j\to\infty}\big\langle f(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\big\rangle &= \limsup_{k\to\infty}\big\langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\big\rangle \\
&= \limsup_{k\to\infty}\big\langle f(\hat{x}) - \hat{x}, v_{n_k} - \hat{x}\big\rangle.
\end{aligned}$$

Since $\hat{x} = P_\Omega \circ f(\hat{x})$, it follows that

$$\begin{aligned}
\limsup_{k\to\infty}\big\langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\big\rangle &= \lim_{j\to\infty}\big\langle f(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\big\rangle \\
&= \big\langle f(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\big\rangle \le 0.
\end{aligned}\tag{4.27}$$

Now, from (4.23) and (4.27), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\big\langle f(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x}\big\rangle &= \limsup_{k\to\infty}\big\langle f(\hat{x}) - \hat{x}, x_{n_k+1} - x_{n_k}\big\rangle \\
&\quad + \limsup_{k\to\infty}\big\langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\big\rangle \\
&= \big\langle f(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\big\rangle \le 0.
\end{aligned}\tag{4.28}$$

Applying Lemma 2.10 to (4.12), and using (4.28) together with the fact that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty} \alpha_n = 0$, we deduce that $\lim_{n\to\infty} \|x_n - \hat{x}\| = 0$ as required. □

Remark 4.5 The results of this paper improve the results of Yao et al. [48] and Chang et al. [16] in the following ways:

(i) Our result extends the result of Yao et al. [48] and the result of Chang et al. [16] from SMVIP (1.4) and (1.5) and a system of monotone variational inclusion problems (1.11), respectively, to the problem of finding a common solution of the system of monotone variational inclusion problems (1.11) and the fixed-point problem of quasipseudocontractions.

(ii) While Yao et al. [48] were only able to prove a weak-convergence result, in this paper we established a strong-convergence result for our proposed algorithm.

(iii) The proposed method of Chang et al. [16] requires knowledge of the operator norm for its implementation, while our proposed method is independent of the operator norm.

(iv) Our method employs a very efficient inertial technique that does not require stringent conditions, like one has in condition (iii) of Algorithm 1 of Yao et al. [48] and condition (iv) of Algorithm (1.13) of Chang et al. [16].

(v) The viscosity technique we employed accommodates a larger class of contractions than the one employed by Chang et al. [16].

Remark 4.6 Since the class of quasipseudocontractions contains several other classes of nonlinear mappings such as the pseudocontractions, the demicontractive operators, the quasinonexpansive operators, the directed operators, and the strictly pseudocontractive mappings with fixed points as special cases, our results present a unified framework for studying these classes of operators.

5 Applications

In this section we consider some applications of our results to approximating solutions of related optimization problems in the framework of Hilbert spaces.

5.1 System of equilibrium problems with fixed-point constraint

Let C be a nonempty closed convex subset of a real Hilbert space H, and let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* (EP) for the bifunction F on C is to find a point $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall y \in C. \tag{5.1}$$

We denote the solution of the EP (5.1) by EP(F). The EP serves as a unifying framework for several mathematical problems, such as variational inequality problems, minimization problems, complementarity problems, saddle-point problems, mathematical programming problems, Nash-equilibrium problems in noncooperative games, and others; see [2, 23, 28, 29, 35] and the references therein. Several problems in economics, physics, and optimization can be formulated as finding a solution of EP (5.1).

In solving the EP (5.1), we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) F is upper hemicontinuous, that is, for all $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.
The following theorem is required in establishing our next result.

Theorem 5.1 ([43]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Define a multivalued mapping $A_F : H \to 2^H$ by*

$$A_F(x) = \begin{cases} \{y \in H : F(x,z) \geq \langle z - x, y\rangle, \forall z \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then, the following hold:

(i) *A_F is maximal monotone*;

(ii) $\mathrm{EP}(F) = A_F^{-1}(0)$;

(iii) *$T_r^F = (I + rA_F)^{-1}$ for $r > 0$, where T_r^F is the resolvent of A_F and is given by*

$$T_r^F(x) = \left\{ y \in C : F(y,z) + \frac{1}{r}\langle z - y, y - x\rangle \geq 0, \forall z \in C \right\}.$$

Here, we consider the following system of equilibrium problems (SEPs) with fixed-point constraint:

$$\begin{cases} \text{Find } x^* \in F(S) \text{ such that } F_i(x^*,x) \geq 0, \quad \forall x \in C, i = 1,2,\dots,m; \quad \text{and} \\ y^* = Ax^* \quad \text{solves} \quad G_j(y^*,y) \geq 0, \quad \forall y \in Q, j = 1,2,\dots,k, \end{cases} \tag{5.2}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $S : H_1 \to H_1$ is a quasipseudocontractive mapping, for each $i = 1,2,\dots,m$ and $j = 1,2,\dots,k$, F_i and G_j are bifunctions satisfying conditions (A1)–(A4) above, and $A : H_1 \to H_2$ is a bounded linear operator. We denote the solution set of problem (5.2) by $\Gamma_{\mathrm{SEP}} = \bigcap_{i=1}^m EP(F_i) \cap A^{-1}(\bigcap_{j=1}^k (EP(G_j)))$.

Now, taking $B_i = A_{F_i}$, $i = 1,2,\dots,m$ and $D_j = H_{G_j}$, $j = 1,2,\dots,k$ and setting $h_i = g_j = 0$ in Theorem 4.4, we obtain the following result for approximating solutions of problem (5.2) in Hilbert spaces.

Theorem 5.2 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*, and for each $i = 1,2,\dots,m$ and $j = 1,2,\dots,k$ let $F_i : C \times C \to \mathbb{R}$ and $G_j : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions (A1)–(A4). Let $S : H_1 \to H_1$ be a K-Lipschitz continuous quasipseudocontractive mapping, which is demiclosed at zero and with $K \geq 1$, and $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in (0,1)$. Suppose that the solution set $\Omega = \Gamma_{\mathrm{SEP}} \cap F(S) \neq \emptyset$, and conditions (C1)–(C4) are satisfied. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to a point $\hat{x} \in \Omega$, where $\hat{x} = P_\Omega \circ f(\hat{x})$.*

5.2 System of convex minimization problems with fixed-point constraint

Suppose that $F : H \to \mathbb{R}$ is a convex and differentiable function, and $M : H \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function. It is known that if ∇F is $\frac{1}{\mu}$-Lipschitz

Algorithm 3

Step 0. Let $x_0, x_1 \in H_1$ be two arbitrary initial points and set $n = 1$.

Step 1. Given the $(n-1)$th and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{5.3}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$u_n = \hat{U}\big(w_n + \gamma_n A^*(\hat{T} - I)Aw_n\big),$$

where

$$\begin{cases} \hat{U} := T_\lambda^{F_1} \circ T_\lambda^{F_2} \circ \dots T_\lambda^{F_m}, \\ \hat{T} := T_\lambda^{G_1} \circ T_\lambda^{G_2} \circ \dots T_\lambda^{G_k}, \end{cases} \tag{5.4}$$

and

$$\gamma_n := \begin{cases} \tau_n \frac{\|(\hat{T}-I)Aw_n\|^2}{\|A^*(\hat{T}-I)Aw_n\|^2}, & \text{if } Aw_n \neq \hat{T}Aw_n, \\ \gamma, & \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases} \tag{5.5}$$

Step 4. Compute

$$v_n = \beta_n w_n + (1-\beta_n)\mathbb{V}u_n,$$

where

$$\mathbb{V} = (1-\eta)I + \eta S\big((1-\mu)I + \mu S\big).$$

Step 5. Compute

$$x_{n+1} = \alpha_n f(w_n) + \delta_n u_n + \xi_n v_n.$$

Set $n := n + 1$ and return to *Step* 1.

continuous, then it is μ-inverse strongly monotone, where ∇F is the gradient of F. Also, it is known that the subdifferential ∂M of M is maximal monotone (see [39]). Moreover,

$$F\big(x^*\big) + M\big(x^*\big) = \min_{x \in H}\big\{F(x) + M(x)\big\} \quad \Longleftrightarrow \quad 0 \in \nabla F\big(x^*\big) + \partial M\big(x^*\big).$$

We consider the following system of convex minimization problems (SCMP) with fixed-point constraint: Find

$$x^* \in F(S) \text{ such that } \quad F_i(x^*) + M_i(x^*) = \min_{x\in F(S)}\{F_i(x) + M_i(x)\}, \quad i = 1,2,\dots,m, \tag{5.6}$$

and such that $y^* = Ax^* \in H_2$, solves

$$G_j(x^*) + N_j(x^*) = \min_{x\in H_2}\{G_j(x) + N_j(x)\}, \quad j = 1,2,\dots,k, \tag{5.7}$$

where $S : H_1 \to H_1$ is a quasipseudocontractive mapping, for each $i = 1,2,\dots,m$ and $j = 1,2,\dots,k$, $F_i : H_1 \to \mathbb{R}$ and $G_j : H_2 \to \mathbb{R}$ are convex and differentiable functions, and $M_i : H_1 \to (-\infty,+\infty]$ and $N_j : H_2 \to (-\infty,+\infty]$ are proper convex and lower semicontinuous functions. We denote the solution set of problem (5.6) and (5.7) by Γ_{SCMP}.

Now, for each $i = 1,2,\dots,m$ and $j = 1,2,\dots,k$, set $B_i = \partial M_i$, $D_j = \partial N_j$, $h_i = \nabla F_i$ and $g_j = \nabla G_j$ in Theorem 4.4, we obtain the following result for approximating solutions of problem (5.6) and (5.7) in Hilbert spaces.

Theorem 5.3 *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. For each $i = 1,2,\dots,m$ and $j = 1,2,\dots,k$, let $M_i : H_1 \to (-\infty,+\infty]$ and $N_j : H_2 \to (-\infty,+\infty]$ be proper convex and lower semicontinuous functions, $F_i : H_1 \to \mathbb{R}$, $G_j : H_2 \to \mathbb{R}$ be convex and differentiable functions such that ∇F_i, ∇G_j are $\frac{1}{\mu}$-Lipschitz continuous. Let $S : H_1 \to H_1$ be a K-Lipschitz continuous quasipseudocontractive mapping, which is demiclosed at zero and with $K \geq 1$, and $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in (0,1)$. Suppose that the solution set $\Omega = \Gamma_{\text{SCMP}} \cap F(S) \neq \emptyset$, and conditions (C1)–(C4) are satisfied. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to a point $\hat{x} \in \Omega$, where $\hat{x} = P_\Omega \circ f(\hat{x})$.*

6 Numerical examples

Here, we present some numerical experiments both in finite-dimensional and infinite-dimensional Hilbert spaces to illustrate the performance of our proposed method Algorithm 2 in comparison with Algorithm (1.13). Moreover, we experiment on the dependency of the key parameters on the performance of our method. All numerical computations were carried out using Matlab version R2019(b).

In our computations, we choose for each $n \in \mathbb{N}$ $\alpha_n = \frac{1}{2n+1}$, $\epsilon_n = \frac{1}{(2n+1)^3}$, $\delta_n = \xi_n = \frac{n}{2n+1}$, $\beta_n = \frac{2n}{3n+2}$, $\lambda = 0.5$. Let $f(x) = \frac{1}{6}x$, then $\rho = \frac{1}{6}$ is the Lipschitz constant for f. It can easily be verified that the conditions of Theorem 4.4 are satisfied. We take $\theta_n = \frac{1}{3n^2}$ and $\gamma = 0.05$ in Algorithm (1.13).

Example 6.1 Let $H_1 = H_2 = \mathbb{R}$ with the inner product defined by $\langle x,y\rangle = xy$, for all $x,y \in \mathbb{R}$, and the induced usual norm $|\cdot|$. For $i = j = 1,2,\dots,5$, we define the mappings $h_i, g_j : \mathbb{R} \to \mathbb{R}$ by $h_i(x) = ix + 6\ \forall x \in H_1$ and $g_j(y) = 2jx - 1\ \forall y \in H_2$, then we take $\lambda = 0.18$. Let $B_i, D_j : \mathbb{R} \to \mathbb{R}$ be defined by $B_i(x) = 3ix - 2\ \forall x \in H_1$, $D_j(y) = 3jy\ \forall y \in H_2$, and we define $A : H_1 \to H_2$ by $A(x) = -\frac{5}{3}x$ for all $x \in H_1$, then $A^*(y) = -\frac{5}{3}y$ for all $y \in H_2$. Define $S : \mathbb{R} \to \mathbb{R}$ by $S(x) = -2x$. Then, S is 2-Lipschitzian quasipseudocontractive. We choose $\eta = 0.23$ and $\mu = 0.28$.

Using MATLAB 2019(b), we compare the performance of Algorithm 2 with Algorithm (1.13). The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-3}$. We plot the

Algorithm 4

Step 0. Let $x_0, x_1 \in H_1$ be two arbitrary initial points and set $n = 1$.

Step 1. Given the $(n-1)$th and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{5.8}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$u_n = \hat{U}\big(w_n + \gamma_n A^*(\hat{T} - I)Aw_n\big),$$

where

$$\begin{cases} \hat{U} := J_\lambda^{\partial M_1}(I - \lambda \nabla F_1) \circ J_\lambda^{\partial M_2}(I - \lambda \nabla F_2) \circ \ldots J_\lambda^{\partial M_m}(I - \lambda \nabla F_m), \\ \hat{T} := J_\lambda^{\partial N_1}(I - \lambda \nabla G_1) \circ J_\lambda^{\partial N_2}(I - \lambda \nabla G_2) \circ \ldots J_\lambda^{\partial N_k}(I - \lambda \nabla G_k) \end{cases} \tag{5.9}$$

and

$$\gamma_n := \begin{cases} \tau_n \frac{\|(\hat{T} - I)Aw_n\|^2}{\|A^*(\hat{T} - I)Aw_n\|^2}, & \text{if } Aw_n \neq \hat{T}Aw_n, \\ \gamma, \quad \text{otherwise } (\gamma \text{ being any nonnegative real number}). \end{cases} \tag{5.10}$$

Step 4. Compute

$$v_n = \beta_n w_n + (1 - \beta_n)\mathbb{V}u_n,$$

where

$$\mathbb{V} = (1 - \eta)I + \eta S\big((1 - \mu)I + \mu S\big).$$

Step 5. Compute

$$x_{n+1} = \alpha_n f(w_n) + \delta_n u_n + \xi_n v_n.$$

Set $n := n + 1$ and return to *Step* 1.

graphs of errors against the number of iterations in each case. The numerical results are reported in Fig. 1 and Table 1.

Example 6.2 Let $H_1 = (\ell_2(\mathbb{R}), \|\cdot\|_2) = H_2$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^\infty |x_j|^2 < \infty\}$, $\|x\|_2 = (\sum_{j=1}^\infty |x_j|^2)^{\frac{1}{2}}$ for all $x \in \ell_2(\mathbb{R})$. For $i = j = 1, 2, \ldots, 5$, we define the

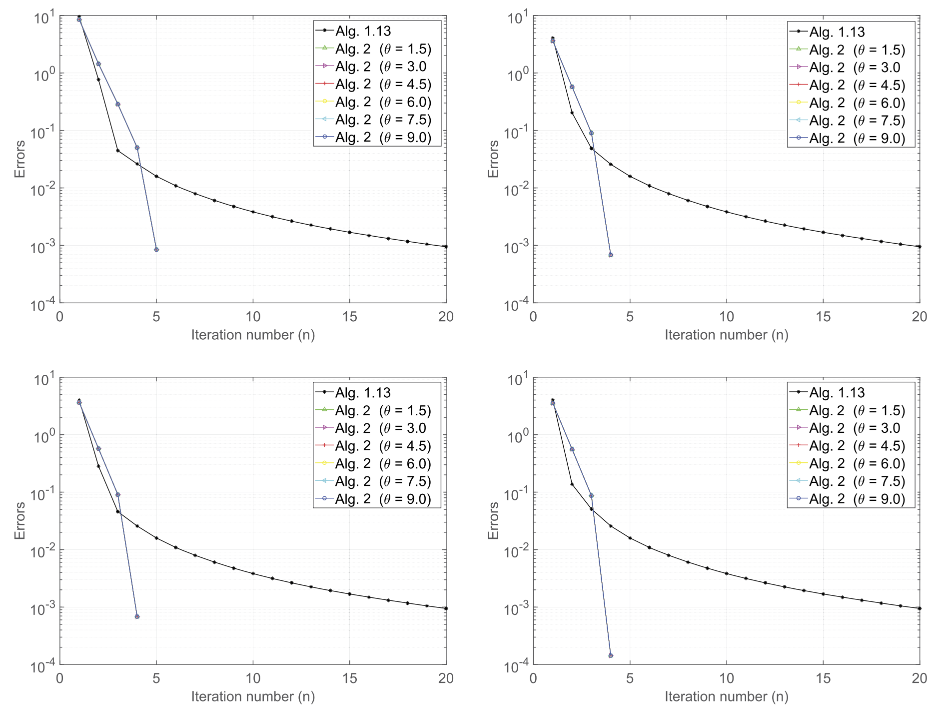

Figure 1 Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV

Table 1 Numerical results for Example 6.1 (Experiment 1)

Cases		Alg. (1.13)	Alg. 2 $\theta = 1.5$	Alg. 2 $\theta = 3.0$	Alg. 2 $\theta = 4.5$	Alg. 2 $\theta = 6.0$	Alg. 2 $\theta = 7.5$	Alg. 2 $\theta = 9.0$
I	CPU time (s)	0.0101	0.0075	0.0134	0.0096	0.0108	0.0119	0.0111
	No. of Iter.	20	5	5	5	5	5	5
II	CPU time (s)	0.0114	0.0059	0.0040	0.0039	0.0077	0.0065	0.0056
	No. of Iter.	20	4	4	4	4	4	4
III	CPU time (s)	0.0104	0.0046	0.0133	0.0081	0.0039	0.0038	0.0065
	No. of Iter.	20	4	4	4	4	4	4
IV	CPU time (s)	0.0097	0.0050	0.0075	0.0077	0.0046	0.0050	0.0044
	No. of Iter.	20	4	4	4	4	4	4

mappings $h_i, g_j : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by $h_i(x) = 2ix - 1$ $\forall x \in H_1$ and $g_j(y) = jx + 2$ $\forall y \in H_2$, then we take $\lambda = 0.15$. Let $B_i, D_j : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ be defined by $B_i(x) = \frac{7}{3i}x$ $\forall x \in H_1$, $D_j(y) = \frac{5}{3j}y \forall y \in H_2$, and we define $A : H_1 \to H_2$ by $A(x) = \frac{x}{3}x$ for all $x \in H_1$, then $A^*(y) = \frac{x}{3}y$ for all $y \in H_2$. Define $S : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by $S(x) = -\frac{5}{4}x$. Then, S is $\frac{5}{4}$-Lipschitzian quasipseudocontractive. We choose $\eta = 0.29$ and $\mu = 0.34$ and $\lambda = 0.01$ in Algorithm (1.13).

Using MATLAB 2019(b), we compare the performance of Algorithm 2 with Algorithm (1.13). The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-3}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Fig. 2 and Table 2.

We test Examples 6.1 and 6.2 under the following experiments:

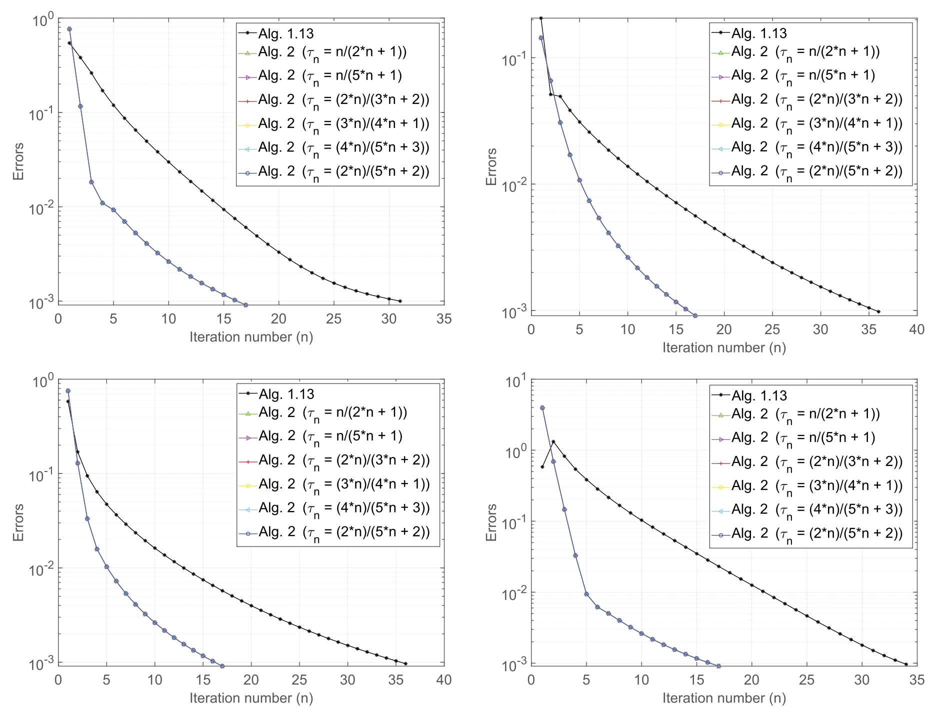

Figure 2 Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV

Table 2 Numerical results for Example 6.2 (Experiment 2)

Cases		Alg. (1.13)	Alg. 2 $\tau_n = \frac{n}{2n+1}$	Alg. 2 $\tau_n = \frac{n}{5n+1}$	Alg. 2 $\tau_n = \frac{2n}{3n+2}$	Alg. 2 $\tau_n = \frac{3n}{4n+1}$	Alg. 2 $\tau_n = \frac{4n}{5n+3}$	Alg. 2 $\tau_n = \frac{2n}{5n+2}$
I	CPU time (s)	0.0200	0.0165	0.0176	0.0187	0.0167	0.0181	0.0184
	No. of Iter.	31	17	17	17	17	17	17
II	CPU time (s)	0.0844	0.0593	0.0343	0.0346	0.0312	0.0324	0.0465
	No. of Iter.	36	17	17	17	17	17	17
III	CPU time (s)	0.0209	0.0162	0.0196	0.0185	0.0177	0.0203	0.0240
	No. of Iter.	36	17	17	17	17	17	17
IV	CPU time (s)	0.0162	0.0180	0.0194	0.0201	0.0184	0.0199	0.0177
	No. of Iter.	34	17	17	17	17	17	17

Experiment 1 In this experiment, we check the behavior of our method by fixing the other parameters and varying θ. We do this to check the effect of the parameter θ on our method.

For Example 6.1, we choose different initial values as follows:

Case I: $x_0 = -10, x_1 = 23;$

Case II: $x_0 = \frac{17}{25}, x_1 = -32;$

Case III: $x_0 = 29, x_1 = -100.23;$

Case IV: $x_0 = 29, x_1 = -100.23;$

Also, we consider $\theta \in \{1.5, 3.0, 4.5, 6.0, 7.5, 9.0\}$, which satisfies Assumption (C4). We use Algorithm (1.13) and Algorithm 2 for the experiment and report the numerical results in Table 1 and Fig. 1.

Experiment 2 In this experiment, we check the behavior of our method by fixing the other parameters and varying τ_n. We do this to check the effect of the parameter τ_n on our method.

For Example 6.2, we choose different initial values as follows:

Case I: $x_0 = (-3, 1, -\frac{1}{3}, \ldots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$;

Case II: $x_0 = (1, \frac{1}{7}, \frac{1}{49}, \ldots)$, $x_1 = (0.1, 0.01, 0.001, \ldots)$;

Case III: $x_0 = (2, \frac{4}{5}, \frac{8}{25}, \ldots)$, $x_1 = (1, -\frac{1}{6}, \frac{1}{36}, \ldots)$;

Case IV: $x_0 = (-2, \frac{4}{3}, -\frac{8}{9}, \ldots)$, $x_1 = (5, -0.5, 0.05, \ldots)$.

Also, we consider $\tau_n \in \{\frac{n}{2n+1}, \frac{n}{5n+1}, \frac{2n}{3n+2}, \frac{3n}{4n+1}, \frac{4n}{5n+1}, \frac{2n}{5n+2}, \}$, which satisfies Assumption (C4). We use Algorithm (1.13) and Algorithm 2 for the experiment and report the numerical results in Table 2 and Fig. 2.

7 Conclusion

We studied the problem of finding the solution of a system of monotone variational inclusion problems with the constraint of a fixed-point set of quasipseudocontractive mappings. We proposed a new iterative method that employs an inertial technique with a self-adaptive step size for approximating the solution of the problem in Hilbert spaces and proved a strong-convergence result for the proposed method under some mild conditions. We further applied our results to study related optimization problems and presented some numerical experiments with graphical illustration to demonstrate the efficiency and applicability of our proposed method. In Examples 6.1 and 6.2, we checked the dependency of key parameters for each starting point in order to determine if their choices affect the performance of our method. We can see from the tables and graphs that the number of iterations and CPU times for our proposed method remain consistent and well behaved for different choices of these key parameters and that our method is more efficient and outperforms a related method.

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Availability of data and materials

Not applicable.

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Conceptualization of the article was carried out by TO, OT, and VA, methodology by TO and VA, formal analysis, investigation and writing the original draft preparation by TO and VA, software and validation by OT and JC, writing, reviewing and editing by TO, OT, and JC, and project administration by OT and JC. All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

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