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# Characterization of the non-homogenous Dirac-harmonic equation

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## Abstract

We introduce the non-homogeneous Dirac-harmonic equation for differential forms and characterize the basic properties of solutions to this new type of differential equations, including the norm estimates and the convergency of sequences of the solutions. As applications, we prove the existence and uniqueness of the solutions to a special non-homogeneous Dirac-harmonic equation and its corresponding reverse Hölder inequality.

**Keywords:** Dirac-harmonic equation; Differential forms; Norm estimates

## 1 Introduction

This is our continuous work on the Dirac-harmonic equation started in the recent paper [1] in which we studied the homogeneous Dirac-harmonic equation for differential forms and established some basic estimates, including the Caccioppoli-type inequality and the weak reverse Hölder inequality for the solutions of the homogeneous Dirac-harmonic equation. The purpose of this paper is to introduce the non-homogeneous Dirac-harmonic equation  $d^*A(x, Du) = B(x, Du)$  for differential forms and study its solvability as well as establish some essential estimates for its solutions, where  $D = d + d^*$  is the Hodge–Dirac operator,  $d$  is the exterior differential operator,  $d^*$  is the Hodge codifferential that is the formal adjoint operator of  $d$ ,  $A$  and  $B$  are operators satisfying certain conditions. In the last several decades, the  $A$ -harmonic equation  $d^*A(x, du) = 0$  and the  $p$ -harmonic equation  $d^*(du|du|^{p-2}) = 0$ , which are special cases of our new equation ( $Du = du$  if  $u$  is a function (0-form) or a co-closed form), have been very well studied [2]. These equations only involve  $du$ . However, in many situations, we need to deal with  $du$ ,  $d^*u$ , and  $Du = du + d^*u$ , such as in the case of Poisson’s equation  $\omega = D(D(u)) + H(\omega)$ , where  $\omega \in L^p(\Omega, \Lambda^l)$  is any differential form defined on the bounded domain  $M \subset \mathbb{R}^n$ ,  $n \geq 2$ ,  $u = G(\omega)$  and  $G$  is Green’s operator [2]. Hence, we introduced and studied the homogeneous Dirac-harmonic equation  $d^*A(x, Du) = 0$  for differential forms in [1].

In this paper, we extend our previous work and introduce the non-homogeneous Dirac-harmonic equation  $d^*A(x, Du) = B(x, Du)$  for differential forms. We establish some essential estimates, including the Caccioppoli-type estimate, the reverse Hölder inequality and the Poincaré–Sobolev imbedding theorems with Orlicz norm for solutions of the new

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equation. We also show that the limit of a convergent sequence of solutions for the non-homogeneous Dirac-harmonic equation is still a solution of the equation. Finally, we study the existence and uniqueness of solutions to a special non-homogeneous Dirac-harmonic equation.

Throughout this paper, let  $Q$  be a ball (or a cube) in  $M \subset \mathfrak{R}^n$ , and  $\Lambda^k = \Lambda^k(\mathfrak{R}^n)$  be the set of all differential  $k$ -forms  $u(x)$  with the expression

$$u(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \dots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$$

in  $\mathfrak{R}^n$ , where  $I = (i_1, i_2, \dots, i_k)$ ,  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . As extensions of functions, differential forms and the related equations have been very well investigated and widely used in some fields of mathematics and physics, see [3–8] for example. The space of all differential  $k$ -forms is denoted by  $D'(M, \Lambda^k)$  and the space of all differential forms in  $\mathfrak{R}^n$  is denoted by  $D'(M, \Lambda)$ . For any  $u \in D'(M, \Lambda^k)$ , the vector-valued differential form

$$\nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right)$$

consists of differential forms

$$\frac{\partial u}{\partial x_i} \in D'(M, \Lambda^k), \quad i = 1, 2, \dots, n,$$

where the partial differentiation is applied to the coefficients of  $u$ . The norm  $\|\nabla u\|_{p,M}$  is defined by

$$\|\nabla u\|_{p,M} = \left( \int_M \left( \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{p/2} dx \right)^{\frac{1}{p}}.$$

We use  $L^p(M, \Lambda^k)$  to denote the classical  $L^p$  space of differential  $k$ -forms with the norm defined by

$$\|u\|_{p,M} = \left( \int_M |u(x)|^p dx \right)^{\frac{1}{p}} = \left( \int_M \left( \sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{\frac{1}{p}}$$

Similarly,  $L^p(M, \Lambda)$  is used to denote the  $L^p$  space of all differential forms defined in  $M$ , where  $\Lambda = \Lambda(\mathfrak{R}^n) = \bigoplus_{l=0}^n \Lambda^l(\mathfrak{R}^n)$  is a graded algebra with respect to the exterior product and  $1 < p < \infty$ . For the set  $\Lambda$ , we denote the pointwise inner product by  $\langle \cdot, \cdot \rangle$  and the module by  $|\cdot|$ , then for any  $\alpha \in \Lambda$  and  $\beta \in \Lambda$ , the global inner product  $(\cdot, \cdot)$  is given by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle. \tag{1.1}$$

A non-homogeneous Dirac-harmonic equation for differential forms is of the form

$$d^* A(x, Du) = B(x, Du), \tag{1.2}$$

where  $D = d + d^*$  is the Dirac operator, operators  $A : M \times \Lambda(\mathfrak{R}^n) \rightarrow \Lambda(\mathfrak{R}^n)$  and  $B : M \times \Lambda(\mathfrak{R}^n) \rightarrow \Lambda(\mathfrak{R}^n)$  satisfy the following conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p, \quad \text{and} \quad |B(x, \xi)| \leq b|\xi|^{p-1} \tag{1.3}$$

for almost every  $x \in M$  and all  $\xi \in \Lambda^l(\mathfrak{R}^n)$ . Here,  $0 < a < 1$  and  $b > 0$  are constants and  $1 < p < \infty$  is a fixed exponent associated with (1.2). Let  $W_{loc}^{1,p}(M, \Lambda) = \bigcap W^{1,p}(M', \Lambda)$ , where the intersection is for all  $M'$  compactly contained in  $M$ . A solution to (1.2) is an element of the Sobolev space  $W_{loc}^{1,p}(M, \Lambda^{l-1})$  such that

$$\int_M \langle A(x, Du), d\varphi \rangle + \langle B(x, Du), \varphi \rangle dx = 0 \tag{1.4}$$

for all  $\varphi \in W^{1,p}(M, \Lambda^{l-1})$  with compact support. The corresponding homogeneous equation to (1.2) is of the form

$$d^*A(x, Du) = 0, \tag{1.5}$$

where  $A$  satisfies the corresponding conditions defined in (1.3).

It should be noticed that, for any differential form  $u$  in the harmonic field  $\mathfrak{H}(M, \Lambda^l)$ , we have  $Du = 0$ . Hence,  $u$  is a solution of the non-homogeneous Dirac-harmonic equation (1.2), that is, any differential form  $u \in \mathfrak{H}(M, \Lambda^l)$  is a solution of equation (1.2). Also, if  $u$  is a function (0-form) or a co-closed form, then  $d^*u = 0$  and  $Du = du$ . Thus, both the non-homogeneous Dirac-harmonic equation (1.2) and the homogeneous Dirac-harmonic equation (1.5) reduce to the corresponding non-homogeneous  $A$ -harmonic equation and the homogeneous  $A$ -harmonic equation, that is, equation (1.2) reduces to

$$d^*A(x, du) = B(x, du), \tag{1.6}$$

equation (1.5) reduces to

$$d^*A(x, du) = 0 \tag{1.7}$$

respectively. Both the non-homogeneous and the homogeneous  $A$ -harmonic equations have received much investigation in recent years, see [9–13] for example. It is easy to see that if  $u$  is a function (0-form), both the traditional  $A$ -harmonic equation (1.7) and the Dirac-harmonic equation (1.5) become the usual  $A$ -harmonic equation

$$\operatorname{div} A(x, \nabla u) = 0 \tag{1.8}$$

for functions. Let  $A : M \times \Lambda^l(\mathfrak{R}^n) \rightarrow \Lambda^l(\mathfrak{R}^n)$  be defined by  $A(x, \xi) = \xi|\xi|^{p-2}$  with  $p > 1$ . Then,  $A$  satisfies conditions (1.3) and equation (1.5) becomes the  $p$ -Dirac-harmonic equation

$$d^*(Du|Du|^{p-2}) = 0 \tag{1.9}$$

for differential forms. Similarly, if  $u$  is a function in (1.9), we obtain the usual  $p$ -harmonic equation

$$\operatorname{div}(\nabla u |\nabla u|^{p-2}) = 0 \tag{1.10}$$

which is equivalent to the following partial differential equation:

$$(p-2) \sum_{k=1}^n \sum_{i=1}^n u_{x_k} u_{x_i} u_{x_k x_i} + |\nabla u|^2 \Delta u = 0. \tag{1.11}$$

Selecting  $p = 2$  in (1.10), we have the Laplace equation  $\Delta u = 0$  for functions in  $\mathfrak{R}^n$ .

### 2 Basic inequalities

In this section, we establish some basic estimates for solutions to the non-homogeneous Dirac-harmonic equation for differential forms.

**Theorem 2.1** *Let  $u \in D'(M, \Lambda^k), k = 0, 1, \dots, n$ , be a solution of the non-homogenous Dirac-harmonic equation (1.2) in a domain  $M \subset \mathfrak{R}^n$ . Assume that  $1 < p < \infty$  is a fixed exponent associated with the non-homogeneous Dirac-harmonic equation (1.2) and  $\eta \in C_0^\infty(M), \eta > 0$ . Then there exist constants  $C_1$  and  $C_2$ , independent of  $u$  and  $Du$ , such that*

$$\left( \int_M |Du|^p |\eta|^p dx \right)^{1/p} \leq C_1 \left( \int_M |u|^p |\nabla \eta|^p dx \right)^{1/p} + C_2 \left( \int_M |u|^p |\eta|^p dx \right)^{1/p}. \tag{2.1}$$

*Proof* The proof is similar to that of Theorem 2.2 in [1]. We include the key steps that are different from [1]. We choose the test form  $\phi = -u\eta^p$ . Hence,

$$d\phi = -d(u \wedge \eta^p) = -du \wedge \eta^p + (-1)^k u \wedge d(\eta^p) = -(du)\eta^p + (-1)^k u \wedge d(\eta^p)$$

and

$$\langle A(x, Du), d\phi \rangle = \langle A(x, Du), -(du)\eta^p \rangle + \langle A(x, Du), (-1)^k u \wedge d(\eta^p) \rangle.$$

Notice that

$$|(-1)^k u \wedge d(\eta^p)| \leq p|\eta|^{p-1}|u||d\eta| \leq p|\eta|^{p-1}|u||\nabla \eta|.$$

Then

$$\int_M |\langle A(x, Du), (-1)^k u \wedge d(\eta^p) \rangle| \leq \int_M ap|Du|^{p-1}|\eta|^{p-1}|u||\nabla \eta| dx.$$

By (1.3) and (1.4), we obtain

$$\begin{aligned} & \int_M \langle A(x, Du), d\phi \rangle \\ &= \int_M \langle A(x, Du), -(Du)\eta^p \rangle + \langle A(x, Du), (d^*u)\eta^p \rangle + \langle A(x, Du), (-1)^k u \wedge d(\eta^p) \rangle \end{aligned}$$

$$\begin{aligned}
 &\leq - \int_M |Du|^p |\eta|^p dx + \int_M a |Du|^{p-1} (|d^*u| |\eta|^p + p |\eta|^{p-1} |u| |\nabla \eta|) dx \\
 &\leq - \int_M |Du|^p |\eta|^p dx + \int_M a |Du|^{p-1} |Du| |\eta|^p dx + \int_M pa |Du|^{p-1} |u| |\eta|^{p-1} |\nabla \eta| dx \\
 &= - \int_M |Du|^p |\eta|^p dx + \int_M a |Du|^p |\eta|^p dx + \int_M pa |Du|^{p-1} |u| |\eta|^{p-1} |\nabla \eta| dx \\
 &= -(1-a) \int_M |Du|^p |\eta|^p dx + \int_M pa |Du|^{p-1} |u| |\eta|^{p-1} |\nabla \eta| dx.
 \end{aligned}$$

Then

$$\begin{aligned}
 0 &= \int_M (\langle A(x, Du), d\phi \rangle + \langle B(x, Du), \varphi \rangle) \\
 &\leq -(1-a) \int_M |Du|^p |\eta|^p dx + \int_M pa |Du|^{p-1} |u| |\eta|^{p-1} |\nabla \eta| dx \\
 &\quad + \int_M b |Du|^{p-1} |u| |\eta|^p dx.
 \end{aligned}$$

Therefore, using the Hölder inequality with  $1 = (p-1)/p + 1/p$ , it follows that

$$\begin{aligned}
 &(1-a) \int_M |Du|^p |\eta|^p dx \\
 &\leq \int_M pa |Du|^{p-1} |u| |\eta|^{p-1} |\nabla \eta| dx + \int_M b |Du|^{p-1} |u| |\eta|^p dx \\
 &\leq pa \left( \int_M |Du|^p |\eta|^p dx \right)^{(p-1)/p} \left( \int_M |u|^p |\nabla \eta|^p dx \right)^{1/p} \\
 &\quad + b \left( \int_M |Du|^p |\eta|^p dx \right)^{(p-1)/p} \left( \int_M |u|^p |\eta|^p dx \right)^{1/p}.
 \end{aligned}$$

Since  $0 < a < 1$ , we have

$$\begin{aligned}
 &\left( \int_M |Du|^p |\eta|^p dx \right)^{1/p} \\
 &\leq \frac{pa}{1-a} \left( \int_M |u|^p |\nabla \eta|^p dx \right)^{1/p} + \frac{b}{1-a} \left( \int_M |u|^p |\eta|^p dx \right)^{1/p}
 \end{aligned}$$

which is (2.1) with  $C_1 \geq pa/(1-a)$  and  $C_2 \geq b/(1-a)$ . □

If we let  $Q$  be any ball with  $\sigma Q \subset M$ , where  $\sigma > 1$ . Let  $\eta \in C_0^\infty(\sigma Q)$  with  $\eta = 1$  in  $Q$  and  $|\nabla \eta| \leq C_3 |Q|^{-1/n}$ , where  $C_3 > 0$  is a constant. Then we have the following simple version of the Caccioppoli-type estimate.

**Corollary 2.2** *Suppose that  $u$  is a solution of equation (1.2) and  $Q$  is a ball with  $\sigma Q \subset M$ , where  $\sigma > 1$ . Then there is a constant  $C$ , which is independent of  $u$  and  $Du$ , such that*

$$\left( \int_Q |Du|^p dx \right)^{1/p} \leq C |Q|^{-1/n} \left( \int_{\sigma Q} |u|^p dx \right)^{1/p}. \tag{2.2}$$

From [1], we also have

$$\left(\int_Q |Du|^p dx\right)^{1/p} \leq C|Q|^{-1/n} \left(\int_{\sigma Q} |u - c|^p dx\right)^{1/p}, \tag{2.3}$$

where  $c$  is a harmonic form.

Similar to the solutions of the homogeneous Dirac-harmonic equation [1], we also have the following weak reverse Hölder inequality for the solutions of the non-homogeneous Dirac-harmonic equation.

**Theorem 2.3** *Let  $u$  be a solution to equation (1.2) in  $M$ ,  $\sigma > 1$ , and  $0 < s, t < \infty$ . Then there exists a constant  $C$ , that is independent of  $u$ , such that*

$$\|u\|_{s,Q} \leq C|Q|^{(t-s)/ts} \|u\|_{t,\sigma Q} \tag{2.4}$$

for all cubes or balls  $Q$  with  $\sigma Q \subset M$ .

We will need the following results that can be found from [1].

**Lemma 2.4** ([1]) *Let  $u \in D'(Q, \Lambda^l)$  and  $Du \in L^p(Q, \Lambda)$ . Then  $u - u_Q$  is in  $W^{1,p}(Q, \Lambda)$  and*

$$\|u - u_Q\|_{W^{1,p}(Q,\Lambda)} \leq C \text{diam}(Q) \|Du\|_{p,Q}. \tag{2.5}$$

Sometimes, we need to estimate  $Du$ . We prove the following version of the reverse Hölder inequality for  $Du$ .

**Theorem 2.5** *Let  $u \in D'(Q, \Lambda^l)$  and  $Du \in L^p(Q, \Lambda)$  and  $0 < s, t < \infty$ . Then there exists a constant  $C$ , independent of  $u$  and  $Du$ , such that*

$$\|Du\|_{s,Q} \leq C|Q|^{(t-s)/st} \|Du\|_{t,\sigma Q} \tag{2.6}$$

for all  $Q$  with  $\sigma Q \subset M$ , here  $\sigma > 1$  is a constant.

*Proof* Note that  $|d^*u| = |d \star u|$  and  $d \star u$  is a closed form, so it is a solution of the  $A$ -harmonic equation. Hence, we can apply the weak reverse Hölder inequality [1] for solutions of the  $A$ -harmonic equation to  $d \star u$  and obtain

$$\|d^*u\|_{s,Q} = \|d \star u\|_{s,Q} \leq C_1|Q|^{1/s-1/t} \|d \star u\|_{t,Q} = C_1|Q|^{1/s-1/t} \|d^*u\|_{t,\sigma_1 Q} \tag{2.7}$$

for any constants  $0 < s, t < \infty$ , and  $\sigma_1 > 1$ . Similarly, since  $du$  is also a closed form, we have

$$\|du\|_{s,Q} \leq C_2|Q|^{1/s-1/t} \|du\|_{t,\sigma_2 Q} \tag{2.8}$$

for some constant  $\sigma_2 > 1$ . Combining (2.7) and (2.8), we derive that

$$\begin{aligned} \|Du\|_{s,Q} &= \|du + d^*u\|_{s,Q} \\ &\leq \|du\|_{s,Q} + \|d^*u\|_{s,Q} \end{aligned}$$

$$\begin{aligned}
 &\leq C_2|Q|^{1/s-1/t}\|du\|_{t,\sigma_2Q} + C_1|Q|^{1/s-1/t}\|d^*u\|_{t,\sigma_1Q} \\
 &\leq C_3|Q|^{1/s-1/t}(\|du\|_{t,\sigma_2Q} + \|d^*u\|_{t,\sigma_1Q}) \\
 &\leq C_3|Q|^{1/s-1/t}(\|Du\|_{t,\sigma_3Q} + \|Du\|_{t,\sigma_3Q}) \\
 &\leq C_4|Q|^{1/s-1/t}\|Du\|_{t,\sigma_3Q}
 \end{aligned}$$

where  $\sigma_3 = \max\{\sigma_1, \sigma_2\}$ , that is,

$$\|Du\|_{s,Q} \leq C_4|Q|^{1/s-1/t}\|Du\|_{t,\sigma Q}$$

for any  $Q$  with  $\sigma Q \subset M$  and any constants  $0 < s, t < \infty$ . The proof of Theorem 2.5 is completed.  $\square$

### 3 Imbedding theorems with Orlicz norms

In this section, we prove the Poincaré–Sobolev imbedding theorems with Orlicz norms for solutions of the non-homogeneous Dirac-harmonic equation.

We define an Orlicz function to be any continuously increasing function  $\Psi : [0, \infty) \rightarrow [0, \infty)$  with  $\Psi(0) = 0$ . A convex Orlicz function is a Young function which is finite valued and vanishes only at 0. The Orlicz space  $L^\Psi(M)$  consists of all measurable functions  $f$  on  $M$  such that  $\int_M \Psi\left(\frac{|f|}{t}\right) dx < \infty$  for some  $t = t(f) > 0$ .  $L^\Psi(M)$  is equipped with the nonlinear Luxemburg norm  $\|\cdot\|_{L^\Psi(M)}$  by

$$\|f\|_{L^\Psi(M)} = \inf\left\{t > 0 : \int_M \Psi\left(\frac{|f|}{t}\right) dx \leq 1\right\}. \tag{3.1}$$

**Definition 3.1** ([14]) We say that a Young function  $\Psi$  lies in the class  $G(p, q, C)$ ,  $1 \leq p \leq q \leq \infty$ ,  $C \geq 1$ , if (i)  $1/C \leq \Psi(t^{1/p})/g(t) \leq C$  and (ii)  $1/C \leq \Psi(t^{1/q})/h(t) \leq C$  for all  $t > 0$ , where  $g$  is a convex increasing function and  $h$  is a concave increasing function on  $[0, \infty)$ .

From [14], each of  $\Psi$ ,  $g$ , and  $h$  in the above definition is doubling in the sense that its values at  $t$  and  $2t$  are uniformly comparable for all  $t > 0$ , and the consequent fact that

$$\begin{aligned}
 C_1t^q &\leq h^{-1}(\Psi(t)) \leq C_2t^q, \\
 C_1t^p &\leq g^{-1}(\Psi(t)) \leq C_2t^p,
 \end{aligned} \tag{3.2}$$

where  $C_1$  and  $C_2$  are constants. Also, for all  $1 \leq p_1 \leq p \leq p_2$  and  $\alpha \in \mathfrak{R}$ , the function  $\Psi(t) = t^p \log_+^\alpha t$  belongs to  $G(p_1, p_2, C)$  for some constant  $C = C(p, \alpha, p_1, p_2)$ . Here  $\log_+(t)$  is defined by  $\log_+(t) = 1$  for  $t \leq e$ ; and  $\log_+(t) = \log(t)$  for  $t > e$ . Particularly, if  $\alpha = 0$ , we see that  $\Psi(t) = t^p$  lies in  $G(p_1, p_2, C)$ ,  $1 \leq p_1 \leq p \leq p_2$ .

For any subset  $E \subset \mathfrak{R}^n$ , we use  $W^{1,\Psi}(E, \Lambda)$  to denote the Orlicz–Sobolev space of  $l$ -forms which equals  $L^\Psi(E, \Lambda) \cap L_1^\Psi(E, \Lambda)$  with the norm

$$\|u\|_{W^{1,\Psi}(E)} = \|u\|_{W^{1,\Psi}(E,\Lambda)} = \text{diam}(E)^{-1}\|u\|_{L^\Psi(E)} + \|\nabla u\|_{L^\Psi(E)}.$$

If we choose  $\Psi(t) = t^p$ ,  $p > 1$ , we obtain the norm for  $W^{1,p}(E, \Lambda)$  defined by

$$\|u\|_{W^{1,p}(E)} = \|u\|_{W^{1,p}(E,\Lambda)} = \text{diam}(E)^{-1}\|u\|_{p,E} + \|\nabla u\|_{p,E}.$$

**Lemma 3.2** ([1]) *Suppose that  $u \in L^p_{\text{loc}}(M, \Lambda^l)$  is such that  $Du \in L^p_{\text{loc}}(M, \Lambda^{l+1})$ ,  $1 \leq p < \infty$ ,  $l = 1, \dots, n$ , and  $T$  is the homotopy operator defined on differential forms. Then*

$$\|u - u_B\|_{W^{1,p}(Q)} \leq A_p(n)|B|\|Du\|_{p,Q}, \tag{3.3}$$

$$\|T du\|_{W^{1,p}(Q)} \leq A_p(n)|Q|\|Du\|_{p,Q}, \tag{3.4}$$

$$\|T du\|_{p,Q} \leq C|Q|\text{diam}(Q)\|Du\|_{p,Q}, \tag{3.5}$$

where  $T : L^p(M, \Lambda^l) \rightarrow L^p(M, \Lambda^{l-1})$  is the homotopy operator defined in [15].

Similar to the proof of Theorem 2.3 in [6], by using Theorem 2.5, we have the following  $L^\Psi$  norm estimate.

**Lemma 3.3** *Let  $\Psi$  be a Young function in the class  $G(p, q, C)$ ,  $1 \leq p < q < \infty$ ,  $C \geq 1$ .  $M$  be a bounded and convex domain, and  $T$  be the homotopy operator. Assume that  $\Psi(|Du|) \in L^1_{\text{loc}}(M)$  and  $u$  is a differential form with  $Du \in L^p_{\text{loc}}(M, \Lambda^l)$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|T(Du)\|_{L^\Psi(Q)} \leq C\|Du\|_{L^\Psi(\sigma Q)} \tag{3.6}$$

for all balls  $Q$  with  $\sigma Q \subset M$ , where  $\sigma > 1$  is a constant.

*Proof* We give the proof here for the purpose of completeness. By (3.5), for any  $q > 1$ , we have

$$\|T(Du)\|_{q,Q} \leq C_1|Q|^{1+1/n}\|Du\|_{q,Q} \tag{3.7}$$

for all balls  $Q$  with  $\sigma Q \subset M$ . From the reverse Hölder inequality, for any positive numbers  $p$  and  $q$ , we have

$$\left(\int_Q |Du|^q dx\right)^{1/q} \leq C_2|Q|^{(p-q)/pq} \left(\int_{\sigma Q} |Du|^p dx\right)^{1/p}, \tag{3.8}$$

where  $\sigma > 1$  is a constant. Using Jensen’s inequality for  $h^{-1}$ , (3.2), (3.7), and (3.8), (i) in Definition 3.1, the fact that  $\Psi$  and  $h$  are doubling, and  $\Psi$  is an increasing function, we have

$$\begin{aligned} & \int_Q \Psi(|T(Du)|) dx \\ &= h\left(h^{-1}\left(\int_Q \Psi(|T(Du)|) dx\right)\right) \\ &\leq h\left(\int_Q h^{-1}(\Psi(|T(Du)|)) dx\right) \\ &\leq h\left(C_3 \int_Q |T(Du)|^q dx\right) \\ &\leq C_4 \Psi\left(\left(C_3 \int_Q |T(Du)|^q dx\right)^{1/q}\right) \end{aligned}$$



$$\begin{aligned}
 &\leq C_4 \Psi \left( C_5 |Q|^{1+1/n} \left( \int_Q |Du|^q dx \right)^{1/q} \right) \\
 &\leq C_4 \Psi \left( C_6 |Q|^{1+1/n+(p-q)/pq} \left( \int_{\sigma Q} |Du|^p dx \right)^{1/p} \right) \\
 &\leq C_4 \Psi \left( C_6^p |Q|^{p+1/n+(p-q)/q} \left( \int_{\sigma Q} |Du|^p dx \right)^{1/p} \right) \\
 &\leq C_7 g \left( \int_{\sigma Q} C_6^p |Q|^{p+1/n+(p-q)/q} \int_{\sigma Q} |Du|^p dx \right) \\
 &\leq C_7 \int_{\sigma Q} g \left( C_6^p |Q|^{p+1/n+(p-q)/q} \int_{\sigma Q} |Du|^p \right) dx \\
 &\leq C_8 \int_{\sigma Q} \Psi (C_6 |Q|^{1+1/n+(p-q)/pq} |Du|) dx,
 \end{aligned} \tag{3.9}$$

where  $1 + \frac{1}{n} + \frac{(p-q)}{pq} = \frac{1}{n} + \frac{p(q+1)-q}{pq} > 0$  by  $p \geq 1$ , so that

$$|Q|^{1+\frac{1}{n}+\frac{(p-q)}{pq}} \leq |M|^{1+\frac{1}{n}+\frac{(p-q)}{pq}} \leq C_9.$$

We know that  $\Psi$  is doubling, so that

$$\Psi (C_6 |Q|^{1+1/n+(p-q)/pq} |Du|) \leq C_{10} \Psi (|Du|).$$

Therefore, combining with (3.9), we have

$$\int_Q \Psi (|T(Du)|) dx \leq C_{11} \int_{\sigma Q} \Psi (|Du|). \tag{3.10}$$

Again  $\Psi$ ,  $g$ , and  $h$  are all doubling, from (3.10) we have

$$\int_Q \Psi \left( \frac{|T(Du)|}{\lambda} \right) dx \leq C \int_{\sigma Q} \Psi \left( \frac{|Du|}{\lambda} \right) \tag{3.11}$$

for all balls  $Q$  with  $\sigma Q \subset M$  and any constant  $\lambda > 0$ . Thus, with the Luxemburg norm, we have

$$\|T(Du)\|_{L^\Psi(Q)} \leq C \|Du\|_{L^\Psi(\sigma Q)}. \tag{3.12}$$

Lemma 3.3 is proved. □

From the proof of Lemma 3.3, noticing that  $\Psi$  is doubling and  $\|du\|_{p,Q} \leq \|Du\|_{p,Q}$ , we could also get

$$\|T(du)\|_{L^\Psi(Q)} \leq C \text{diam}(Q) \|Du\|_{L^\Psi(\sigma Q)}. \tag{3.13}$$

Since  $\Psi$  is an increasing function, it is also obvious that

$$\|du\|_{L^\Psi(Q)} \leq \|Du\|_{L^\Psi(Q)}. \tag{3.14}$$

Then, similar to Theorem 2.5 in [6], we have the following local Poincaré–Sobolev imbedding theorem.

**Theorem 3.4** *Let  $\Psi$  be a Young function in the class  $G(p, q, C)$ ,  $1 \leq p < q < \infty$ ,  $C \geq 1$ ,  $M$  be a bounded and convex domain. Assume that  $\Psi(|Du|) \in L^1_{loc}(M, \Lambda)$  and  $u$  is differential form with  $Du \in L^p_{loc}(M, \Lambda^l)$ . Then, there exists a constant  $C$ , independent of  $u$ , such that*

$$\|u - u_Q\|_{W^{1,\Psi}(Q)} \leq C \|Du\|_{L^\Psi(\sigma Q)} \tag{3.15}$$

for all balls  $Q$  with  $\sigma Q \subset M$ .

*Proof* First we notice that the following  $L^\Psi$  norm inequality holds for any differential forms, see [6]:

$$\|\nabla(T(du))\|_{L^\Psi(Q)} \leq C|Q|\|du\|_{L^\Psi(\sigma Q)} \leq C|Q|\|Du\|_{L^\Psi(\sigma Q)}. \tag{3.16}$$

Then, by (3.13), (3.14), and (3.16), we have

$$\begin{aligned} \|u - u_Q\|_{W^{1,\Psi}(Q,\Lambda^l)} &= \|T(du)\|_{W^{1,\Psi}(Q,\Lambda^l)} \\ &= (\text{diam}(Q))^{-1} \|T(du)\|_{L^\Psi(Q)} + \|\nabla(T(du))\|_{L^\Psi(Q)} \\ &\leq C_1 \|Du\|_{L^\Psi(\sigma_1 Q)} + C_2 \|Du\|_{L^\Psi(\sigma_2 Q)} \\ &\leq C_3 \|Du\|_{L^\Psi(\sigma Q)}, \end{aligned}$$

where  $\sigma_1 > 1$ ,  $\sigma_2 > 1$  and  $\sigma = \max\{\sigma_1, \sigma_2\}$  and  $\sigma Q \subset M$ . □

**Lemma 3.5** ([12]) *Each domain  $M$  has a modified Whitney cover of cubes  $\mathcal{V} = \{Q_i\}$  such that*

$$\bigcup_i Q_i = M, \quad \sum_{Q_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \leq N \chi_M$$

and some  $N > 1$ , and if  $Q_i \cap Q_j \neq \emptyset$ , then there exists a cube  $R$  (this cube need not be a member of  $\mathcal{V}$ ) in  $Q_i \cap Q_j$  such that  $Q_i \cup Q_j \subset NR$ . Moreover, if  $M$  is  $\delta$ -John, then there is a distinguished cube  $Q_0 \in \mathcal{V}$  which can be connected with every cube  $Q \in \mathcal{V}$  by a chain of cubes  $Q_0, Q_1, \dots, Q_k = Q$  from  $\mathcal{V}$  and such that  $Q \subset \rho Q_i$ ,  $i = 0, 1, 2, \dots, k$ , for some  $\rho = \rho(n, \delta)$ .

Finally, we have the following global Poincaré–Sobolev imbedding theorem.

**Theorem 3.6** *Let  $\Psi$  be a Young function in the class  $G(p, q, C)$ ,  $1 \leq p < q < \infty$ ,  $K \geq 1$ ,  $M$  be a bounded domain. Assume that  $\Psi(|Du|) \in L^1(M, \Lambda)$  and  $u$  is a differential form with  $Du \in L^p_{loc}(M, \Lambda^l)$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|u - u_M\|_{W^{1,\Psi}(M)} \leq C \|Du\|_{L^\Psi(M)} \tag{3.17}$$

for any bounded domain  $M \subset \mathbb{R}^n$ .

*Proof* Using (3.13) and Lemma 3.5, we have the global estimate

$$\|T du\|_{L^\Psi(M)} \leq C_1 \text{diam}(M) \|Du\|_{L^\Psi(M)}. \tag{3.18}$$

Note that, for any differential form  $u$  and the constant  $p > 1$ , we have

$$\|\nabla T u\|_{p,Q} \leq C_2 |Q| \|u\|_{p,Q} \tag{3.19}$$

for all balls  $Q \subset \mathfrak{R}^n$ . Hence

$$\|\nabla T(du)\|_{p,Q} \leq C_2 |Q| \|du\|_{p,Q} \leq C_3 |Q| \|Du\|_{p,Q}. \tag{3.20}$$

Starting from (3.20), using Theorem 2.5 and the same skills developed in the proof of Lemma 3.3, we obtain

$$\|\nabla T du\|_{L^\Psi(Q)} \leq C_4 |Q| \|Du\|_{L^\Psi(\sigma Q)}, \tag{3.21}$$

where  $\sigma > 1$  is a constant. From (3.21) and Lemma 3.5, it follows that

$$\|\nabla T du\|_{L^\Psi(M)} \leq C_5 \|Du\|_{L^\Psi(M)}. \tag{3.22}$$

Thus,

$$\begin{aligned} \|u - u_M\|_{W^{1,\Psi}(M)} &= \|T du\|_{W^{1,\Psi}(M)} \\ &= (\text{diam}(M))^{-1} \|T du\|_{L^\Psi(M)} + \|\nabla T du\|_{L^\Psi(M)} \\ &\leq (\text{diam}(M))^{-1} (C_1 \text{diam}(M) \|Du\|_{L^\Psi(M)}) + C_5 \|Du\|_{L^\Psi(M)} \\ &\leq C_6 \|Du\|_{L^\Psi(M)} \end{aligned}$$

We have completed the proof of Theorem 3.6. □

*Remark 1* If we choose  $\Psi(t)$  to be some special function in  $G(p, q, C)$ , we will obtain some special versions of the imbedding theorem. For example, if we select  $\Psi(t) = t^p \log^\alpha(e + t)$  with  $p \geq 1, \alpha > 0$  or  $\Psi(t) = t^p, p \geq 1$ , we will have  $L^p(\log L)^\alpha$ -norm or  $L^p$ -norm imbedding theorem, respectively.

#### 4 Limits of convergent sequences

In this section, we consider the limits of convergent sequences of differential  $l$ -forms  $u_n(x)$  defined in a bounded domain  $M \subset \mathfrak{R}^n$ . We say an  $l$ -form  $u_n(x)$  converges uniformly in  $M$  if all its coefficient functions under the base  $\{dx_{i_1}, dx_{i_2}, \dots, dx_{i_l}\}$  converge uniformly in  $M$ . For example, we say the sequence

$$u_n(x) = \sum_I u_I^n(x) dx_I = \sum u_{i_1 i_2 \dots i_k}^n(x) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$$

converges uniformly in  $M$  if all its coefficient functions  $u_{i_1 i_2 \dots i_k}^n(x)$  converge uniformly in  $M$  as  $n$  goes to infinity. For example, for  $x \in M \subset \mathfrak{R}^3$ , let

$$u_n(x) = P_n(x) dx_1 \wedge dx_2 + Q_n(x) dx_1 \wedge dx_3 + R_n(x) dx_2 \wedge dx_3.$$

We say that  $u_n(x)$  converges uniformly in  $M$  as  $n \rightarrow \infty$  if its all coefficient functions  $P_n(x)$ ,  $Q_n(x)$ , and  $R_n(x)$  converge uniformly in  $M$  as  $n \rightarrow \infty$ .

In addition to condition (1.3), we also assume that the operators  $A$  and  $B$  are Lipschitz continuous with respect to  $\xi$  and satisfy

$$\begin{aligned}
 |A(x, \xi) - A(x, \eta)| &\leq L_1(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|, \\
 |B(x, \xi) - B(x, \eta)| &\leq L_2(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|
 \end{aligned}
 \tag{4.1}$$

for all  $x \in M$  and all  $\xi, \eta \in \wedge^l$ . Here  $L_1$  and  $L_2$  are positive constants. See [16] for Lipschitz continuous condition and other conditions that the operators  $A$  and  $B$  could satisfy. From (2.1) in [16], we know that  $A(x, \xi)$  and  $B(x, \xi)$  have a polynomial growth with respect to the variable  $\xi$ . Specifically, for any  $x \in M$  and  $\xi \in \wedge^l$ , we have

$$\begin{aligned}
 m_1|\xi|^{p-1} &\leq |A(x, \xi)| \leq L_1|\xi|^{p-1}, \\
 m_2|\xi|^{p-1} &\leq |B(x, \xi)| \leq L_2|\xi|^{p-1},
 \end{aligned}
 \tag{4.2}$$

where  $m_1, m_2, L_1$ , and  $L_2$  are positive constants. Also, a simple example of this kind of operators is the  $p$ -Laplace system  $A(x, \xi) = B(x, \xi) = |\xi|^{p-2}\xi$ .

**Theorem 4.1** *Let  $u_n(x)$  be a solution of the non-homogeneous Dirac-harmonic equation (1.2) with conditions (1.3) and (4.1) such that  $Du_n(x)$  converges uniformly to  $Du(x)$  in  $M$  and  $u_n(x_0)$  converges for some  $x_0 \in M$ . Then  $u(x)$  is also a solution of the non-homogeneous Dirac-harmonic equation (1.2).*

*Proof* Assume that  $u_n(x)$  is a solution of the non-homogeneous Dirac-harmonic equation (1.2), that is,

$$\int_M \langle A(x, Du_n), d\varphi \rangle + \langle B(x, Du_n), \varphi \rangle = 0.
 \tag{4.3}$$

Using the basic properties of the inner product and (4.1), we obtain

$$\begin{aligned}
 &\left| \left( \int_M \langle A(x, Du_n), d\varphi \rangle + \langle B(x, Du_n), \varphi \rangle \right) - \left( \int_M \langle A(x, Du), d\varphi \rangle + \langle B(x, Du), \varphi \rangle \right) \right| \\
 &\leq \int_M |\langle A(x, Du_n), d\varphi \rangle - \langle A(x, Du), d\varphi \rangle| + \int_M |\langle B(x, Du_n), \varphi \rangle - \langle B(x, Du), \varphi \rangle| \\
 &= \int_M |\langle A(x, Du_n) - A(x, Du), d\varphi \rangle| + \int_M |\langle B(x, Du_n) - B(x, Du), \varphi \rangle| \\
 &\leq \int_M |A(x, Du_n) - A(x, Du)| |d\varphi| dx + \int_M |B(x, Du_n) - B(x, Du)| |\varphi| dx \\
 &\leq \int_M L_1(|Du_n|^2 + |Du|^2)^{\frac{p-2}{2}} |Du_n - Du| |d\varphi| dx \\
 &\quad + \int_M L_2(|Du_n|^2 + |Du|^2)^{\frac{p-2}{2}} |Du_n - Du| |\varphi| dx.
 \end{aligned}$$

Letting  $n \rightarrow \infty$  in the above inequality and noticing that  $Du_n(x)$  converges uniformly to  $Du(x)$  in  $M$ , we can switch the limit operation with the integral operation and obtain

$$\lim_{n \rightarrow \infty} \left| \left( \int_M \langle A(x, Du_n), d\varphi \rangle + \langle B(x, Du_n), \varphi \rangle \right) - \left( \int_M \langle A(x, Du), d\varphi \rangle + \langle B(x, Du), \varphi \rangle \right) \right| = 0.$$

Hence, it follows that

$$\begin{aligned} & \int_M \langle A(x, Du), d\varphi \rangle + \langle B(x, Du), \varphi \rangle \\ &= \lim_{n \rightarrow \infty} \int_M \langle A(x, D_n u), d\varphi \rangle + \langle B(x, D_n u), \varphi \rangle \\ &= 0 \end{aligned}$$

by (4.3), which indicates that  $u(x)$  is also a solution of the non-homogeneous Dirac-harmonic equation (1.2). We have completed the proof of Theorem 4.1.  $\square$

### 5 Existence and uniqueness of solutions

As mentioned in Sect. 1, there exist many solutions to equation (1.2) in general if the operators  $A$  and  $B$  only satisfy condition (1.3). However, if we require that the operators  $A$  and  $B$  satisfy some more conditions or one of these operators in (1.2) is replaced with certain type of differential form, we need to study the existence and uniqueness of solutions to equation (1.2). For example, we consider the following type of the non-homogenous Dirac-harmonic equation for differential forms:

$$d^*A(x, Du) = d^*f(x), \tag{5.1}$$

where the natural space we consider in (5.1) is the Sobolev space  $W^{1,q}(M, \Lambda)$ ,  $D = d + d^*$  is the Dirac operator;  $f \in W^{1,p}(M, \Lambda^l)$  is a differential form, and the operator  $A : M \times \Lambda(M) \rightarrow \Lambda(M)$  satisfies the following conditions:

- (i) The mapping  $x \rightarrow A(x, \xi)$  is measurable for all  $\xi \in \Lambda(M)$ ;
- (ii) The Lipschitz type inequality  $|A(x, \xi) - A(x, \eta)| \leq L_1(|\xi|^2 + |\eta|^2)^{\frac{q-2}{2}}|\xi - \eta|$ .
- (iii) The monotonicity inequality  $|\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle| \geq L_2(|\xi|^2 + |\eta|^2)^{\frac{q-2}{2}}|\xi - \eta|^2$ .
- (iv)  $A(x, 0) \in L^p(M, \Lambda)$ .

Here,  $L_1 > 0$  and  $L_2 > 0$  are two constants, and  $1 < p, q < \infty$  are the conjugate exponents with  $1/p + 1/q = 1$  determined by conditions (ii)–(iv).

It should be noticed that we do not require that the operator  $A$  appearing in (5.1) satisfies condition (1.3). Before the upcoming argument, we first give the following definition.

**Definition 5.1** Given the formal joint operator  $d^* = (-1)^{n+l+1} \star d \star$  defined on  $D'(M, \Lambda^{l+1})$  with the values in  $D'(M, \Lambda^l)$ ,  $n \geq 1$  and  $l = 0, 1, \dots, n$ , the forms in the image of  $d^*$  are called the co-exact  $l$ -forms, and the forms in the kernel of  $d^*$  are called the co-closed  $l$ -forms.

Indeed, we should point out that the construction of equation (5.1) is applicable and reasonable. To be precise, if the differential form  $u$  is a function (0-form) defined in  $M$ , then equation (5.1) reduces to a divergence  $A$ -harmonic equation

$$\operatorname{div}(A, \nabla u) = \operatorname{div} f. \tag{5.2}$$

The properties of equation (5.2), including its solvability, have been very well studied in [17]. Equation (5.1) could be viewed as a generalization of the divergence  $A$ -harmonic equations (5.2). If the differential form  $u$  is a co-closed form, equation (5.1) is actually corresponding to the non-homogenous  $A$ -harmonic equation

$$d^*A(x, du) = d^*f.$$

For more descriptions and details, we refer the readers to [18] and [19]. From the other perspective, according to the non-homogenous Dirac-harmonic equation in Sect. 1, one may see that every element in the image of the operator  $B(x, \xi)$  in (1.3) is of the class  $D'(M, \Lambda^l)$ ,  $l = 0, 1, \dots, n$ . Specifically, assumed that  $B(x, \xi)$  is a co-exact form, by Definition 5.1 of the co-exact form, there exists a differential form  $f \in D'(M, \Lambda^{l+1})$  such that  $B(x, \xi) = d^*f$ . Thus, we are inspired to introduce the non-homogenous Dirac-harmonic equation (5.1). It is worth noting that this equation is different from equation (1.2) where a differential form  $u$  with  $Du = 0$  is always a solution of equation (1.2). The differential form  $u$  here with  $Du = 0$  is not a solution of equation (5.1) (unless  $d^*f = 0$  and  $A(x, 0) = 0$ ), since we cannot derive that  $A(x, Du) = d^*f$  from  $Du = 0$ . Therefore, our focus in this section is to explore the technique for the solvability of the non-homogenous Dirac-harmonic equation (5.1).

To facilitate the latter assertion of Theorem 5.3, we begin with the following lemma 5.2 given by Minty and Browder in [20].

**Lemma 5.2** *Let  $X$  be the real and reflexive Banach space and  $X^*$  be the dual space of  $X$ . Suppose that  $T : X \rightarrow X^*$  is hemicontinuous operator on  $X$  such that, for every  $v_1, v_2 \in X$  and  $v_1 \neq v_2$ ,*

$$(Tv_1 - Tv_2, v_1 - v_2) > 0 \tag{5.3}$$

and

$$\lim_{\|v\| \rightarrow \infty} \frac{(Tv, v)}{\|v\|} = \infty. \tag{5.4}$$

Then, for any  $b \in X^*$ , the equation  $Tx = b$  has a unique solution on  $X$ .

With this monotone operator theory, we can establish Theorem 5.3 as follows.

**Theorem 5.3** *Let the operator  $A$  satisfy conditions (i)–(iv). Then the non-homogenous Dirac-harmonic equation (5.1) has a solution in the Sobolev space  $W^{1,p}(M, \Lambda)$  for  $p > 1$  and  $l = 0, 1, \dots, n$ . Moreover, the solution to equation (5.1) is unique except for a harmonic form  $c$  satisfying  $dc = d^*c = 0$ .*

Before giving the rigorous proof, we need to make a brief analysis first for this theorem. According to  $L^p$ -Hodge decomposition, for any differential form  $u \in L^p(M, \Lambda^l)$ , there are  $\alpha \in dW^{1,p}(M, \Lambda^{l-1})$ ,  $\beta \in d^*W^{1,p}(M, \Lambda^{l+1})$ , and  $h \in \mathfrak{H}_p(M, \Lambda^l)$  such that

$$u = d\alpha + d^*\beta + h(u) \tag{5.5}$$

for  $1 < p < \infty, l = 1, 2, \dots, n$ , where  $h$  is the harmonic projection in  $L^p$ .

We should point out that there exist other two Hodge decompositions of  $L^p$ -space, which are equivalent to (5.5), see [21] for more descriptions. Without loss of generality, we only apply (5.5) to the proof of Theorem 5.3. In addition, it should be noticed that  $dW^{1,s}(M, \Lambda^l)$  and  $d^*W^{1,s}(M, \Lambda^l)$  are both Banach subspaces of  $L^s(M, \Lambda)$ ,  $s > 1$ . For simplicity, since  $d + d^* = D$ , denote

$$\begin{aligned} dW^{1,p}(M, \Lambda^l) \oplus d^*W^{1,p}(M, \Lambda^l) &= DW^{1,p}(M, \Lambda^l), \\ dW^{1,q}(M, \Lambda^l) \oplus d^*W^{1,q}(M, \Lambda^l) &= DW^{1,q}(M, \Lambda^l). \end{aligned} \tag{5.6}$$

It is obvious to see that  $DW^{1,p}(M, \Lambda^l)$  is the dual space of  $DW^{1,q}(M, \Lambda^l)$ . We can define a projection operator  $K : L^p(M, \Lambda^l) \rightarrow L^p(M, \Lambda^l)$  such that

$$Ku = d\alpha + d^*\beta. \tag{5.7}$$

By some simple observation, one may readily see that the projection operator  $K$  is a bounded linear operator. Due to (5.5) and the boundedness of the harmonic projection  $h$ , we have

$$\begin{aligned} \|Ku\|_p &= \|d\alpha + d^*\beta\|_p \\ &= \|d\alpha + d^*\beta + h(u) - h(u)\|_p \\ &\leq \|u\|_p + \|h(u)\|_p \\ &\leq \|u\|_p + C_1\|u\|_p \\ &\leq C_2\|u\|_p. \end{aligned} \tag{5.8}$$

Furthermore, given  $u$  satisfying  $KA(x, Du) = Kf$ , according to definition (5.7) of the operator  $K$ , we have  $A(x, Du) - f$  is of the class  $\mathfrak{H}_p(M, \Lambda)$  of the harmonic field, which implies that  $(A(x, Du) - f, d\omega) = 0$  for any  $\omega \in W^{1,p}(M, \Lambda)$ . Thus,  $u$  is a solution of Dirac-harmonic equation (5.1). With these facts in hand, for every fixed  $x \in M$ , we find that the key point to prove the existence in Theorem 5.3 is equivalent to showing that

$$KA(x, Du) = Kf \tag{5.9}$$

with respect to the differential form  $u$ . Namely, denote  $\mathfrak{F}(v) = KA(x, v)$ , in which  $\mathfrak{F}$  is a nonlinear mapping defined on  $DW^{1,q}(M, \Lambda^l)$  with values in  $DW^{1,p}(M, \Lambda^l)$ .

Next, our primary work is to deal with the continuity, monotonicity, and coercivity of the operator  $\mathfrak{F}$ .

*Proof* For any fixed point  $x \in M$ , the expression of operator  $K$  shows that  $A(x, Du) = KA(x, Du) + h(A(x, Du))$ , where  $h(A(x, Du)) \in \mathfrak{S}_p(M, \Lambda)$ . Notice that  $(h, D\eta) = 0$  for any  $\eta \in L^q(M, \Lambda)$ . Thus, for all  $u \in W^{1,q}(M, \Lambda)$ , we have

$$\begin{aligned} (A(x, Du), D\eta) &= (KA(x, Du) + h(A(x, Du)), D\eta) \\ &= (KA(x, Du), D\eta). \end{aligned} \tag{5.10}$$

To prove the continuity, the Lipschitz inequality (ii) and bounded property (5.8) ensure that  $\mathfrak{F}$  is continuous with respect to  $v$ .

For the monotonicity, in accord to condition (iii) and (5.10), we derive that

$$\begin{aligned} (\mathfrak{F}(Du) - \mathfrak{F}(D\eta), Du - D\eta) &= (KA(x, Du) - KA(x, D\eta), Du - D\eta) \\ &= (A(x, Du) - A(x, D\eta), Du - D\eta) \\ &= \int_M \langle A(x, Du) - A(x, D\eta), Du - D\eta \rangle \\ &\geq L_2 \int_M (|Du|^2 + |D\eta|^2)^{\frac{q-2}{2}} |Du - D\eta|^2 d\mu \\ &\geq 0. \end{aligned}$$

On the other hand, using condition (iii) again gives that

$$\begin{aligned} \frac{(\mathfrak{F}(Du), Du)}{\|Du\|_q} - \frac{(\mathfrak{F}(0), Du)}{\|Du\|_q} &= \frac{(KA(x, Du), Du)}{\|Du\|_q} - \frac{(KA(x, 0), Du)}{\|Du\|_q} \\ &= \frac{(A(x, Du) - A(x, 0), Du)}{\|Du\|_q} \\ &= \frac{\int_M \langle A(x, Du) - A(x, 0), Du - 0 \rangle}{\|Du\|_q} \\ &\geq \frac{L_2 \|Du\|_q^q}{\|Du\|_q} \\ &= L_2 \|Du\|_q^{q-1}. \end{aligned} \tag{5.11}$$

Hence, it follows that

$$\frac{(\mathfrak{F}(Du), Du)}{\|Du\|_q} - \frac{(\mathfrak{F}(0), Du)}{\|Du\|_q} \rightarrow \infty \tag{5.12}$$

as  $\|Du\|_q \rightarrow \infty$ . By applying Hölder’s inequality and condition (iv), we notice that

$$\begin{aligned} \frac{|(\mathfrak{F}(0), Du)|}{\|Du\|_q} &= \frac{|(KA(x, 0), Du)|}{\|Du\|_q} \\ &= \frac{|(A(x, 0), Du)|}{\|Du\|_q} \\ &\leq \frac{\|A(x, 0)\|_p \|Du\|_q}{\|Du\|_q} \\ &= \|A(x, 0)\|_p < \infty. \end{aligned} \tag{5.13}$$



Then substituting (5.13) and (5.12) into (5.11) yields

$$\frac{(\mathfrak{F}(Du), Du)}{\|Du\|_q} \rightarrow \infty, \tag{5.14}$$

which shows that the operator  $\mathfrak{F}$  is monotonic. By applying Lemma 5.2, we find that, for any  $g \in DW^{1,p}(M, \Lambda^l)$ , there exists unique  $v \in DW^{1,q}(M, \Lambda^l)$  such that  $\mathfrak{F}(v) = g$ , in particular, for  $g = Kf$ , in view of definition (5.6), there exists unique  $u \in W^{1,q}(M, \Lambda^l)$  with  $Du \in DW^{1,q}(M, \Lambda^l)$  such that  $\mathfrak{F}(Du) = Kf$ , that is,  $KA(x, Du) = Kf$ . Thus, we derive that the solution of equation (5.1) in  $W^{1,q}(M, \Lambda)$  exists. Moreover, by the monotonic result, one may see that, except for the harmonic form  $c$ , the solution to equation (5.1) is unique. Therefore, the desired result Theorem 5.3 holds.  $\square$

Now, with the above existence theorem in mind, we can derive the following local result as an application for Theorem 2.5.

*Example 5.4* Let  $u_0 \in$  be the solution of the non-homogenous equation (5.1) and  $0 < s, t < \infty$ . Then, according to the definition of the weak solution to the non-homogenous equation, we know that  $u_0 \in D'(M)$  and  $D(u_0) \in L^p(M)$ . Thus, by applying Theorem 2.5, we know that the reverse Hölder inequality of  $Du$  holds. That is, there exists a constant  $C > 0$ , independent of  $u$  and  $Du$ , such that

$$\|Du\|_{s,Q} \leq C|Q|\|Du\|_{t,Q} \tag{5.15}$$

holds for any ball (or cube)  $Q$  with  $\sigma Q \subset M$ , where  $\sigma > 1$  is a constant.

It should be pointed out that the reverse Hölder inequality of  $Du$  is a key tool in some sense for the study on the non-homogenous equations driven by the term  $Du$ , especially for the norm inequalities, such as Poincaré–Sobolev imbedding inequalities, which play an important role in the characterization of the continuity and regularity of the solutions.

### 6 Conclusion

In this paper, we introduce a new Dirac-harmonic equation (1.2) and present an exhaustive study on the norm estimates of the solution for this equation. Precisely, in Sect. 2, using some new techniques and the methods previously developed by others, we obtain the essential inequalities, including Caccioppoli inequalities and reverse Hölder inequalities. In Sect. 3, by using the basic inequalities, we derive the Poincaré–Sobolev imbedding inequalities in terms of Orlicz norm. In Sect. 4, with these norm estimates in hand, we get the convergency of solution sequences for this equation under certain structure assumptions. In the last section, we assert that there exists a unique nontrivial solution for a concrete non-homogenous Dirac-harmonic equation.

In general, non-homogenous equation (1.2) is an extension of the  $p$ -Laplacian equation for differential forms. In fact, it is quite applicable to many related fields such as geometry analysis and elasticity theory. For example, the elasticity results involving the determinants could be understood better if they can be formulated by the equation for differential forms, such as that every conformal mapping  $f$  corresponds to a solution of a special harmonic equation for differential forms.

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### Availability of data and materials

Not applicable.

### Declarations

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All results and investigations of this article were the joint efforts of all authors. Specifically, GS mainly worked on the proofs and drafted the article. SD and BL proposed the initial idea of this article and improved the final version. All authors read and approved the final article.

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