Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Ostrowski-type inequalities for n-polynomial $\mathscr{P}$-convex function for k-fractional Hilfer–Katugampola derivative

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Abstract
In this article, we develop a novel framework to study a new class of convex functions known as n-polynomial $\mathscr{P}$-convex functions. The purpose of this article is to establish a new generalization of Ostrowski-type integral inequalities by using a generalized k-fractional Hilfer–Katugampola derivative. We employ this technique by using the Hölder and power-mean integral inequalities. We present analogs of the Ostrowski-type integrals inequalities connected with the n-polynomial $\mathscr{P}$-convex function. Some new exceptional cases from the main results are obtained, and some known results are recaptured. In the end, an application to special means is given as well. The article seeks to create an exciting combination of a convex function and special functions in fractional calculus. It is supposed that this investigation will provide new directions in fractional calculus.

MSC: 26D10; 26D15; 90C23

Keywords: Generalized k-fractional Hilfer–Katugampola derivative; Generalized k-Riemann–Liouville fractional integral; Convex function; Hermite–Hadamard inequality; Ostrowski inequality

1 Introduction

In the last few decades, fractional calculus has been viably utilized in models over an enormous assortment of designing and applied science processes and structures. Fractional calculus in, for example, fluid mechanics, including exothermal compound responses or autocatalytic responses, characterizes the broad applications for explicit issues. It was created as a productive strategy for comprehension and demonstrating different issues in material science and applied mathematics. Fragmentary necessary conditions incorporate a derivative of any unpredictable or real requirement, which will likewise be viewed as differing conditions of an overall sort. Many explorations are considered having been proposed to upgrade demonstrating the accuracy while indicating the diffusion process, displaying various types of viscoelastic damping, unequivocally leading to the reliance on power-law frequencies, and modeling fragmentary Maxwell liquid streaming (see [3, 5, 20]).

Integral inequalities are among the best-known techniques relevant for numerous practical points: optimization, design, and innovation. In many fields of science and technology, integral inequalities in fractional strategies are dramatically more common. There has been a focus on the massive degree of the advantages of integral inequalities in taking a derivative and doing integration via convexity [13–16].

Presenting the idea of n-polynomial $\mathcal{P}$-convex functions and characterizing the Ostrowski-type inequalities for n-polynomial $\mathcal{P}$-convex functions are the fundamental subject of this paper. In the deterministic case, the vast majority of the results introduced are the refinements in the general writing of the current outcomes for new and classical convex functions.

This article is organized as follows: in Sect. 2, some basic and essential definitions and lemmas are recalled. In Sect. 3, for n-polynomial $\mathcal{P}$-convex functions, we proved an inequality of the Ostwoski type and related results. In Sect. 4, we give our concluding remarks.

2 Preliminaries

Firstly, we include some mandatory definitions and mathematical preliminaries of the fractional operators of calculus, which are further used in this article.

Definition 2.1 (See [9]) Let $[a_1, a_2]$ be a finite or infinite interval on the real axis $\mathbb{R} = (-\infty, \infty)$. By $M_q = (a_1, a_2)$, we denote the set of the complex-valued Lebesgue measurable function ψ on $[a_1, a_2]$.

$$M_q(a_1, a_2) = \left\{ \psi : \|\psi_q\| = \sqrt[q]{\int \left|\psi(z)\right|^q dz} < +\infty \right\}, \quad 1 \le q < \infty.$$

In the case if $q = 1$, we have $M_q(a_1, a_2) = M(a_1, a_2)$.

Definition 2.2 (See [4]) Diaz et al. defined the k-Gamma function as

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}}\, \mathrm{d}t. \tag{2.1}$$

Here $z, k > 0$. We have $\Gamma_k(z + k) = z\Gamma_k(z)$ and $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k})$.

Definition 2.3 (See [17]) Sarikaya et al. presented the left and right generalized k-fractional integral of order ω with $m - 1 < \omega \le m$, $m \in \mathbb{N}$, $\rho > 0$, $k > 0$, $\omega > 0$. We have

$$\left({}^\rho_k\Im^\omega_{a_1}\psi\right)(z) = \frac{\rho^{1-\frac{\omega}{k}}}{k\Gamma_k(\omega)} \int_{a_1}^z \left(z^\rho - y^\rho\right)^{\frac{\omega}{k}-1} y^{\rho-1}\psi(y)\, dy, \quad z > a_1, \tag{2.2}$$

$$\left({}^\rho_k\Im^\omega_{a_2}\psi\right)(z) = \frac{\rho^{1-\frac{\omega}{k}}}{k\Gamma_k(\omega)} \int_z^{a_2} \left(y^\rho - z^\rho\right)^{\frac{\omega}{k}-1} y^{\rho-1}\psi(y)\, dy, \quad z < a_2. \tag{2.3}$$

Definition 2.4 (See [2]) Nisar et al. presented the left and right generalized k-fractional derivative of order ω can be written as in terms of the integral defined in definition (2.3),

$${}^\rho_k\mathcal{D}^\gamma_{a_1}\psi(z) = \left(z^{1-\rho}\frac{d}{dz}\right)^m \left(k^m \cdot {}^\rho_k\Im^{km-\omega}_{a_1}\psi\right)(z), \quad z > a_1, \tag{2.4}$$

$${}^\rho_k\mathcal{D}^\gamma_{a_2}\psi(z) = \left(z^{1-\rho}\frac{d}{dz}\right)^m \left(k^m \cdot {}^\rho_k\Im^{km-\omega}_{a_2}\psi\right)(z), \quad z < a_2. \tag{2.5}$$

Definition 2.5 (See [10]) Let $m-1<\omega\leq m$, $0\leq\theta\leq 1$, $m\in\mathbb{N}$, $\rho>0$, $k>0$ and $\psi\in M_q(a,b)$, the generalized k-fractional Hilfer–Katugampola derivative (left sided and right sided) is defined as

$$\big({}^{\rho}_{k}\mathcal{D}^{\omega,\theta}_{a_1}\psi\big)(z)=\bigg({}^{\rho}_{k}\Im^{\theta(km-\omega)}_{a_1}\bigg(z^{1-\rho}\frac{d}{dz}\bigg)^{m}\big(k^{m}\,{}^{\rho}_{\cdot k}\Im^{(1-\theta)(km-\omega)}_{a_1}\psi\big)\bigg)(z), \tag{2.6}$$

$$\big({}^{\rho}_{k}\mathcal{D}^{\omega,\theta}_{a_2}\psi\big)(z)=\bigg({}^{\rho}_{k}\Im^{\theta(km-\omega)}_{a_2}\bigg(z^{1-\rho}\frac{d}{dz}\bigg)^{m}\big(k^{m}\,{}^{k}_{\cdot\rho}\Im^{(1-\theta)(km-\omega)}_{a_2}\psi\big)\bigg)(z), \tag{2.7}$$

where $\Im$ is the integral defined in definition (2.3).

Lemma 2.1 (See [11]) *Let $m-1<\omega\leq m$, $0\leq\theta\leq 1$, $m\in\mathbb{N}$, $\rho>0$, $k>0$ and $\psi\in M_q(a,b)$, then*

$$\begin{aligned}
{}^{\rho}_{k}\mathcal{D}^{\omega,\theta}_{a_1}\psi(z)&=\bigg({}^{\rho}_{k}\Im^{\theta(km-\omega)}_{a_1}\bigg(z^{1-\rho}\frac{d}{dz}\bigg)^{m}\big(k^{m}\,{}^{\rho}_{\cdot k}\Im^{(1-\theta)(km-\omega)}_{a_1}\psi\big)\bigg)(z)\\
&=\bigg({}^{\rho}_{k}\Im^{\theta(km-\omega)}_{a_1}\bigg(z^{1-\rho}\frac{d}{dz}\bigg)^{m}\big(k^{m}\,{}^{\rho}_{\cdot k}\Im^{km-\omega-\theta(km-\omega)}_{a_1}\big)\bigg)\psi(z)\\
&=\bigg({}^{\rho}_{k}\Im^{\theta(km-\omega)}_{a_1}\bigg(z^{1-\rho}\frac{d}{dz}\bigg)^{m}\big(k^{m}\,{}^{\rho}_{\cdot k}\Im^{km-\{\omega+\theta(km-\omega)\}}_{a_1}\psi\big)\bigg)(z)\\
&=\big({}^{\rho}_{k}\Im^{\theta(km-\omega)}_{a_1}\,{}^{\rho}_{k}D^{\omega+\theta(km-\omega)}_{a_1}\psi\big)(z)\quad\big(\textit{by using Eq. (2.4)}\big)\\
&=\big({}^{\rho}_{k}\Im^{\gamma-\omega}_{a_1}\,{}^{\rho}_{k}D^{\gamma}_{a_1}\psi\big)(z)\\
&=\big({}^{\rho}_{k}\Im^{\gamma-\omega}_{a_1}\psi^{(\gamma)}\big)(z)\\
&=\frac{\rho^{1-\frac{\gamma-\omega}{k}}}{k\Gamma_k(\gamma-\omega)}\int_{a_1}^{z}\big(z^{\rho}-y^{\rho}\big)^{\frac{\gamma-\omega}{k}-1}y^{\rho-1}\psi^{(\gamma)}(y)\,dy\quad\big(\textit{by using Eq. (2.2)}\big),
\end{aligned}$$

where $\gamma=\omega+\theta(km-\omega)$ and $\omega>0$ and $\psi^{(\gamma)}$ is the derivative of ψ defined in (2.4).

So the above defined generalized k-fractional Hilfer–Katugampola derivative can be written as

$$\big({}^{\rho}_{k}\mathcal{D}^{\omega,\gamma}_{a_1}\psi\big)(z)=\frac{\rho^{1-\frac{\gamma-\omega}{k}}}{k\Gamma_k(\gamma-\omega)}\int_{a_1}^{z}\big(z^{\rho}-y^{\rho}\big)^{\frac{\gamma-\omega}{k}-1}y^{\rho-1}\psi^{(\gamma)}(y)\,dy,\quad z>a_1, \tag{2.8}$$

$$\big({}^{\rho}_{k}\mathcal{D}^{\omega,\gamma}_{a_2}\psi\big)(z)=\frac{\rho^{1-\frac{\gamma-\omega}{k}}}{k\Gamma_k(\gamma-\omega)}\int_{z}^{a_2}\big(y^{\rho}-z^{\rho}\big)^{\frac{\gamma-\omega}{k}-1}y^{\rho-1}\psi^{(\gamma)}(y)\,dy,\quad z<a_2. \tag{2.9}$$

Some novel definitions and generalized fractional derivative are presented in this section. The Hermite–Hadamard inequality for a convex function $\varsigma^{(\gamma)}:I\to\mathbb{R}$ is

$$\varsigma^{(\gamma)}\bigg(\frac{a_1+a_2}{2}\bigg)\leq(a_2-a_1)^{-1}\int_{a_1}^{a_2}\varsigma^{(\gamma)}(z)\,dz\leq\frac{\varsigma^{(\gamma)}(a_1)+\varsigma^{(\gamma)}(a_2)}{2} \tag{2.10}$$

with $\forall a_1,a_2\in I$ with $a_1\neq a_2$.

Ostrowski [12] established the integral inequality in 1928 for the integral average $(a_2-a_1)^{-1}\int_{a_1}^{a_2}\varsigma^{(\gamma)}(\eta)\,d\eta$ by the value $\varsigma^{(\gamma)}(z)$ at $z\in[a_1,a_2]$.

For $\varsigma^{(\gamma)} : [a_1, a_2] \to \mathbb{R}$ a differentiable mapping on (a_1, a_2) such that $|\varsigma^{(\gamma+1)}(x)| \le \mathcal{M}$, $\forall z \in (a_1, a_2)$, the inequality

$$\left|\varsigma^{(\gamma)}(z) - (a_2 - a_1)^{-1}\int_{a_1}^{a_2}\varsigma^{(\gamma)}(\eta)d\eta\right| \le \mathcal{M}(a_2 - a_1)\left[\frac{1}{4} + \frac{(z - \frac{a_1+a_2}{2})^2}{(a_2 - a_1)^2}\right] \tag{2.11}$$

holds for $\forall z \in (a_1, a_2)$. Here the constant $\frac{1}{4}$ is the least possible value.

Theorem 2.1 (See [7]) *For $\lambda, \vartheta > 1$ with $\lambda + \vartheta = \lambda\vartheta$, and $\varsigma_1^{(\gamma)}$ and $\varsigma_2^{(\gamma)}$ two integrable real-valued functions defined on $[a_1, a_2]$ such that $|\varsigma_1^{(\gamma)}|^\lambda$ and $|\varsigma_2^{(\gamma)}|^\vartheta$ are integrable on $[a_1, a_2]$, we have*

$$\begin{aligned}
&\int_{a_1}^{a_2}\left|\varsigma_1^{(\gamma)}(z)\varsigma_2^{(\gamma)}(z)\right| dz \\
&\quad\le \frac{1}{a_2 - a_1}\left[\left(\int_{a_1}^{a_2}(a_2 - z)\left|\varsigma_1^{(\gamma)}(z)\right|^\lambda dz\right)^{1/\lambda}\left(\int_{a_1}^{a_2}(a_2 - z)\left|\varsigma_2^{(\gamma)}(z)\right|^\vartheta dz\right)^{1/\vartheta}\right. \\
&\qquad\left. + \left(\int_{a_1}^{a_2}(z - a_1)\left|\varsigma_1^{(\gamma)}(z)\right|^\lambda dz\right)^{1/\lambda}\left(\int_{a_1}^{a_2}(z - a_1)\left|\varsigma_2^{(\gamma)}(z)\right|^\vartheta dz\right)^{1/\vartheta}\right].
\end{aligned} \tag{2.12}$$

Theorem 2.2 (See [8]) *For $\lambda, \vartheta > 1$ with $\lambda + \vartheta = \lambda\vartheta$, and $\varsigma_1^{(\gamma)}$ and $\varsigma_2^{(\gamma)}$ two integrable real-valued functions defined on $[a_1, a_2]$ such that $|\varsigma_1^{(\gamma)}|^\lambda$ and $|\varsigma_2^{(\gamma)}|^\vartheta$ are integrable on $[a_1, a_2]$*

$$\begin{aligned}
&\int_{a_1}^{a_2}\left|\varsigma_1^{(\gamma)}(z)\varsigma_2^{(\gamma)}(z)\right| dz \\
&\quad\le \frac{1}{a_2 - a_1}\left[\left(\int_{a_1}^{a_2}(a_2 - z)\left|\varsigma_1^{(\gamma)}(z)\right|^\lambda dz\right)^{1-1/\lambda}\left(\int_{a_1}^{a_2}(a_2 - z)\left|\varsigma_1^{(\gamma)}(z)\right|\left|\varsigma_2^{(\gamma)}(z)\right|^\vartheta dz\right)^{1/\vartheta}\right. \\
&\qquad + \left(\int_{a_1}^{a_2}(z - a_1)\left|\varsigma_1^{(\gamma)}(z)\right|^\lambda dz\right)^{1-1/\lambda} \\
&\qquad\left. \times\left(\int_{a_1}^{a_2}(z - a_1)\left|\varsigma_1^{(\gamma)}(z)\right|\left|\varsigma_2^{(\gamma)}(z)\right|^\vartheta dz\right)^{1/\vartheta}\right].
\end{aligned} \tag{2.13}$$

Definition 2.6 (See [21]) For $\mathcal{P} > 0$ and $\Omega \subseteq \mathbb{R}$ an interval, Ω is said to be $\mathcal{P}$-convex if with $\forall a_1, a_2 \in \Omega$ and $\eta \in [0, 1]$

$$\left(\eta a_1{}^{\mathcal{P}} + (1 - \eta)a_2{}^{\mathcal{P}}\right)^{1/\mathcal{P}} \in \Omega \tag{2.14}$$

Definition 2.7 (See [21]) For $\mathcal{P} > 0$ and $\Omega \subseteq \mathbb{R}$ a $\mathcal{P}$-convex interval, the real-valued function $\varsigma^{(\gamma)} : \Omega \to \mathbb{R}$ is said to be $\mathcal{P}$-convex if the following inequality holds with $\forall a_1, a_2 \in \Omega$ and $\eta \in [0, 1]$:

$$\varsigma^{(\gamma)}\left(\left[\eta a_1{}^{\mathcal{P}} + (1 - \eta)a_2{}^{\mathcal{P}}\right]^{1/\mathcal{P}}\right) \le \eta\varsigma^{(\gamma)}(a_1) + (1 - \eta)\varsigma^{(\gamma)}(a_2). \tag{2.15}$$

Definition 2.8 (See [6]) For $\Omega \subseteq \mathbb{R}$ an interval, a real-valued function $\varsigma^{(\gamma)} : \Omega \to \mathbb{R}$ is said to be harmonically convex if the following inequality holds for $\forall a_1, a_2 \in \Omega$ and $\eta \in [0, 1]$:

$$\varsigma^{(\gamma)}\left(\frac{a_1 a_2}{\eta a_1 + (1 - \eta)a_2}\right) \le \eta\varsigma^{(\gamma)}(a_2) + (1 - \eta)\varsigma^{(\gamma)}(a_1). \tag{2.16}$$

Definition 2.9 (See [19]) For $n \in \mathbb{N}$, the non-negative function $\varsigma^{(\gamma)} : \Omega \to [0,\infty)$ is said to be a n-polynomial convex function if the following inequality holds for $\forall a_1, a_2 \in \Omega$ and $\eta \in [0,1]$:

$$\varsigma^{(\gamma)}\big(\eta a_1 + (1-\eta)a_2\big) \le \frac{1}{n}\sum_{\theta=1}^{n}\big[1-(1-\eta)^{\theta}\big]\varsigma^{(\gamma)}(a_1) + \frac{1}{n}\sum_{\theta=1}^{n}\big[\big(1-\eta^{\theta}\big)\big]\varsigma^{(\gamma)}(a_2). \quad (2.17)$$

Definition 2.10 (See [1]) For $n \in \mathbb{N}$, the non-negative function $\varsigma^{(\gamma)} : \Omega \to [0,\infty)$ is said to be a n-polynomial harmonically convex function if the following inequality holds with $\forall a_1, a_2 \in \Omega$ and $\eta \in [0,1]$:

$$\begin{aligned}&\varsigma^{(\gamma)}\left(\frac{a_1 a_2}{\eta a_1 + (1-\eta)a_2}\right)\\ &\quad\le \frac{1}{n}\sum_{\vartheta=1}^{n}\big[1-(1-\eta)^{\vartheta}\big]\varsigma^{(\gamma)}(a_2) + \frac{1}{n}\sum_{\vartheta=1}^{n}\big[\big(1-\eta^{\vartheta}\big)\big]\varsigma^{(\gamma)}(a_1).\end{aligned} \quad (2.18)$$

For $n = 2$, we have

$$\varsigma^{(\gamma)}\left(\frac{a_1 a_2}{\eta a_1 + (1-\eta)a_2}\right) \le \frac{3\eta-\eta^2}{2}\varsigma^{(\gamma)}(a_2) + \frac{2-\eta-\eta^2}{2}\varsigma^{(\gamma)}(a_1). \quad (2.19)$$

Definition 2.11 (See [17]) Let $n \in \mathbb{N}$, $\mathscr{P} > 0$ and $\Omega \subseteq \mathbb{R}$ be a $\mathscr{P}$-convex interval. Then the non-negative real-valued function $\varsigma^{(\gamma)} : \Omega \to [0,\infty)$ is said to be a n-polynomial $\mathscr{P}$-convex function if the following inequality holds with $\forall a_1, a_2 \in \Omega$ and $\eta \in [0,1]$:

$$\begin{aligned}&\varsigma^{(\gamma)}\big(\big[\eta {a_1}^{\mathscr{P}} + (1-\eta){a_2}^{\mathscr{P}}\big]^{1/\mathscr{P}}\big)\\ &\quad\le \frac{1}{n}\sum_{\vartheta=1}^{n}\big[1-(1-\eta)^{\vartheta}\big]\varsigma^{(\gamma)}(a_1) + \frac{1}{n}\sum_{\vartheta=1}^{n}\big[\big(1-\eta^{\vartheta}\big)\big]\varsigma^{(\gamma)}(a_2).\end{aligned} \quad (2.20)$$

Remark 2.1 From Definition 2.11 we conclude:

(i) If $\mathscr{P} = -1$, then Definition 2.11 becomes Definition 2.5 for an n-polynomial harmonically convex function.

(ii) If $\mathscr{P} = 1$, then Definition 2.11 reduces to Definition (2.4) for an n-polynomial convex function.

Definition 2.12 The beta function $\mathbb{B}$ and Gaussian hypergeometric function $\mathcal{F}_1$ are defined by

$$\mathbb{B}(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} = \int_0^1 \eta^{z_1-1}(1-\eta)^{z_2-1}\,d\eta \quad (z_1,z_2>0) \quad (2.21)$$

and

$$\begin{aligned}&\mathcal{F}_1(z_1,z_2;z_3,z) = \frac{1}{\mathbb{B}(z_2,z_3-z_2)}\int_0^1 \eta^{z_2-1}(1-\eta)^{z_3-z_2-1}(1-z\eta)^{-z_1}\,d\eta\\ &\quad\big(z_3>z_2>0, |z|<1\big),\end{aligned} \quad (2.22)$$

respectively, where $\Gamma(z) = \int_0^\infty e^{-\eta}\eta^{z-1}\,d\eta$ is the Euler Gamma function.

3 Ostrowski-type inequalities for *n*-polynomial $\mathscr{P}$-convex function

In this section the Ostrowski inequality is proved for an n-polynomial $\mathscr{P}$-convex function via a generalized k-fractional Hilfer–Katugampola derivative.

Lemma 3.1 (See [18]) *For a differentiable function $\mu > 0$, $\mathscr{P} \in \mathbb{R} \setminus \{0\}$ and $\psi^{(\mu)} : \Omega \to \mathbb{R}$ on Ω° such that $a_1, a_2 \in \Omega$ with $a_1 < a_2$ and $\psi^{(\mu+1)} \in M([a_1, a_2])$, the following inequality holds*:

$$\begin{aligned} &\frac{(z^{\mathscr{P}} - {a_1}^{\mathscr{P}})^{\mu} \psi^{(\mu)}(a_1) + ({a_2}^{\mathscr{P}} - z^{\mathscr{P}})^{\mu} \psi^{(\mu)}(a_2)}{\mathscr{P}^{\mu}(a_2 - a_1)} \\ &\quad - \frac{\Gamma_k(\mu + k)}{a_2 - a_1} \bigl[\bigl({}^{\mathscr{P}}_{k}\mathcal{D}^{\mu}_{a_1^+} \psi\bigr)(z) + \bigl({}^{\mathscr{P}}_{k}\mathcal{D}^{\mu}_{a_2^-} \psi\bigr)(z) \bigr] \\ &\quad = - \frac{(z^{\mathscr{P}} - {a_1}^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2 - a_1)} \\ &\qquad \times \int_0^1 \zeta^{\mu} \bigl(\zeta {a_1}^{\mathscr{P}} + (1 - \zeta) z^{\mathscr{P}}\bigr)^{\frac{1-\mathscr{P}}{\mathscr{P}}} \bigl|\psi^{(\mu+1)}\bigl(\sqrt[\mathscr{P}]{\zeta {a_1}^{\mathscr{P}} + (1 - \zeta) z^{\mathscr{P}}}\bigr)\bigr| \, d\zeta \\ &\qquad + \frac{({a_2}^{\mathscr{P}} - z^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2 - a_1)} \\ &\qquad \times \int_0^1 \zeta^{\mu} \bigl(\zeta {a_2}^{\mathscr{P}} + (1 - \zeta) z^{\mathscr{P}}\bigr)^{\frac{1-\mathscr{P}}{\mathscr{P}}} \bigl|\psi^{(\mu+1)}\bigl(\sqrt[\mathscr{P}]{\zeta {a_2}^{\mathscr{P}} + (1 - \zeta) z^{\mathscr{P}}}\bigr)\bigr| \, d\zeta. \end{aligned} \tag{3.1}$$

Theorem 3.1 *For a differentiable function $n \in \mathbb{N}$, $\mu > 0$, $a_1, a_2 \in \Omega$ with $a_1 < a_2$, and $\psi^{(\mu)} : \Omega \subset (0, \infty) \to \mathbb{R}$ on Ω° such that $\psi^{(\mu+1)} \in M([a_1, a_2])$ and $|\psi^{(\mu+1)}|$ a n-polynomial $\mathscr{P}$-convex function satisfying $|\psi^{(\mu+1)}(z)| \le \mathcal{Q}$, $\forall z \in [a_1, a_2]$, the following inequality holds for all $z \in (a_1, a_2)$ and $\mathscr{P} \in (1, \infty)$*:

$$\begin{aligned} &\biggl| \frac{(z^{\mathscr{P}} - {a_1}^{\mathscr{P}})^{\mu} \psi^{(\mu)}(a_1) + ({a_2}^{\mathscr{P}} - z^{\mathscr{P}})^{\mu} \psi^{(\mu)}(a_2)}{\mathscr{P}^{\mu}(a_2 - a_1)} \\ &\quad - \frac{\Gamma_k(\mu + k)}{a_2 - a_1} \bigl[\bigl({}^{\mathscr{P}}_{k}\mathcal{D}^{\mu}_{a_1^+} \psi\bigr)(z) + \bigl({}^{\mathscr{P}}_{k}\mathcal{D}^{\mu}_{a_2^-} \psi\bigr)(z) \bigr] \biggr| \\ &\quad \le \frac{{a_1}^{1-\mathscr{P}} \mathcal{Q}}{\mathscr{P}^{1+\mu}} \biggl[\frac{(z^{\mathscr{P}} - {a_1}^{\mathscr{P}})^{\mu+1} + ({a_2}^{\mathscr{P}} - z^{\mathscr{P}})^{\mu+1}}{(a_2 - a_1)} \biggr] \\ &\qquad \times \frac{1}{n} \sum_{\theta=1}^{n} \biggl[\frac{\mu + 2\theta + 1}{(\mu + 1)(\mu + \theta + 1)} - \mathbb{B}(\mu - 1, \theta - 1) \biggr], \end{aligned} \tag{3.2}$$

and the following inequality holds with $\forall z \in (a_1, a_2)$ and $\mathscr{P} \in (-\infty, 0) \cup (0, 1)$:

$$\begin{aligned} &\biggl| \frac{(z^{\mathscr{P}} - {a_1}^{\mathscr{P}})^{\mu} \psi^{(\mu)}(a_1) + ({a_2}^{\mathscr{P}} - z^{\mathscr{P}})^{\mu} \psi^{(\mu)}(a_2)}{\mathscr{P}^{\mu}(a_2 - a_1)} \\ &\quad - \frac{\Gamma_k(\mu + k)}{a_2 - a_1} \bigl[\bigl({}^{\mathscr{P}}_{k}\mathcal{D}^{\mu}_{a_1^+} \psi\bigr)(z) + \bigl({}^{\mathscr{P}}_{k}\mathcal{D}^{\mu}_{a_2^-} \psi\bigr)(z) \bigr] \biggr| \\ &\quad \le \frac{{a_2}^{1-\mathscr{P}} \mathcal{Q}}{\mathscr{P}^{1+\mu}} \biggl[\frac{(z^{\mathscr{P}} - {a_1}^{\mathscr{P}})^{\mu+1} + ({a_2}^{\mathscr{P}} - z^{\mathscr{P}})^{\mu+1}}{(a_2 - a_1)} \biggr] \\ &\qquad \times \frac{1}{n} \sum_{\theta=1}^{n} \biggl[\frac{\mu + 2\theta + 1}{(\mu + 1)(\mu + \theta + 1)} - \mathbb{B}(\mu - 1, \theta - 1) \biggr]. \end{aligned} \tag{3.3}$$

Proof We use Lemma 3.1 to prove the inequality (3.2) for the n-polynomial $\mathcal{P}$-convexity of $|\psi^{(\mu+1)}|$ to yield

$$\begin{aligned}
&\left|\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_1)+(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2-a_1)}\right.\\
&\qquad\left.-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\left[\left({}_k^{\mathcal{P}}\mathcal{D}^{\mu}_{a_1^+}\psi\right)(z)+\left({}_k^{\mathcal{P}}\mathcal{D}^{\mu}_{a_2^-}\psi\right)(z)\right]\right|\\
&\quad\le\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\int_0^1\zeta^{\mu}\left(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|d\zeta\\
&\qquad+\frac{(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\\
&\qquad\times\int_0^1\zeta^{\mu}\left(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|d\zeta\\
&\quad\le\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\int_0^1\zeta^{\mu}\left(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\\
&\qquad\times\left[\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]\left|\psi^{(\mu+1)}(a_1)\right|+\frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\left|\psi^{(\mu+1)}(z)\right|\right]d\zeta\\
&\qquad+\frac{(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\int_0^1\zeta^{\mu}\left(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\\
&\qquad\times\left[\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]\left|\psi^{(\mu+1)}(a_2)\right|+\frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\left|\psi^{(\mu+1)}(z)\right|\right]d\zeta.
\end{aligned}\tag{3.4}$$

As $\mathcal{P}\in(1,\infty)$, we can infer that

$$\left(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\le\left(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\le a_1{}^{1-\mathcal{P}}.\tag{3.5}$$

We proceed by simplifying

$$\begin{aligned}
&\int_0^1\zeta^{\mu}\left[\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]+\frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\right]d\zeta\\
&\quad=\frac{1}{n}\sum_{\theta=1}^{n}\left[\frac{\mu+2\theta+1}{(\mu+1)(\mu+\theta+1)}-\mathbb{B}(\mu-1,\theta-1)\right].
\end{aligned}\tag{3.6}$$

The first inequality of Theorem 3.1 is proved. Now for the second part, we let $\mathcal{P}\in(-\infty,0)\cup(0,1)$ to yield

$$\left(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\le\left(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\le a_2{}^{1-\mathcal{P}}.\tag{3.7}$$

The above inequality completes the proof of the second part of Theorem 3.1. □

Theorem 3.2 *For a differentiable function $n\in\mathbb{N}$, $\lambda,\vartheta>1$ with $\lambda^{-1}+\vartheta^{-1}=1$, $a_1,a_2\in\Omega$ with $a_1<a_2$, and $\psi^{(\mu)}:\Omega\subset(0,\infty)\to\mathbb{R}$ on Ω° such that $\psi^{(\mu+1)}\in M([a_1,a_2])$ and $|\psi^{(\mu+1)}|^{\vartheta}$ an n-polynomial convex function satisfying $|\psi^{(\mu+1)}(z)|\le\mathcal{Q}$, $\forall z\in[a_1,a_2]$, the following in-*

equality holds for all $z \in (a_1, a_2)$ and $\mathcal{P} \in (1, \infty)$:

$$\begin{aligned} &\left| \frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu} \psi^{(\mu)}(a_1) + ({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu} \psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2 - a_1)} \right. \\ &\quad \left. - \frac{\Gamma_k(\mu + k)}{a_2 - a_1} \left[\left({}^{\mathcal{P}}_{k} \mathcal{D}^{\mu}_{a_1^+} \psi \right)(z) + \left({}^{\mathcal{P}}_{k} \mathcal{D}^{\mu}_{a_2^-} \psi \right)(z) \right] \right| \\ &\quad \leq \frac{{a_1}^{1-\mathcal{P}} \mathcal{Q}}{\mathcal{P}^{1+\mu}(1 + \lambda\mu)^{1/\lambda}} \left[\frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu+1} + ({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{(a_2 - a_1)} \right] \left(\frac{1}{n} \sum_{\theta=1}^{n} \frac{2\theta}{\theta + 1} \right)^{1/\vartheta}, \end{aligned} \tag{3.8}$$

and the following inequality holds with $\forall z \in (a_1, a_2)$ and $\mathcal{P} \in (-\infty, 0) \cup (0, 1)$:

$$\begin{aligned} &\left| \frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu} \psi^{(\mu)}(a_1) + ({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu} \psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2 - a_1)} \right. \\ &\quad \left. - \frac{\Gamma_k(\mu + k)}{a_2 - a_1} \left[\left({}^{\mathcal{P}}_{k} \mathcal{D}^{\mu}_{a_1^+} \psi \right)(z) + \left({}^{\mathcal{P}}_{k} \mathcal{D}^{\mu}_{a_2^-} \psi \right)(z) \right] \right| \\ &\quad \leq \frac{{a_2}^{1-\mathcal{P}} \mathcal{Q}}{\mathcal{P}^{1+\mu}(1 + \lambda\mu)^{1/\lambda}} \left[\frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu+1} + ({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{(a_2 - a_1)} \right] \left(\frac{1}{n} \sum_{\theta=1}^{n} \frac{2\theta}{\theta + 1} \right)^{1/\vartheta}. \end{aligned} \tag{3.9}$$

Proof We use Lemma 3.1, (3.5) and the Hölder inequality to prove the first inequality of Theorem 3.3,

$$\begin{aligned} &\left| \frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu} \psi^{(\mu)}(a_1) + ({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu} \psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2 - a_1)} \right. \\ &\quad \left. - \frac{\Gamma_k(\mu + k)}{a_2 - a_1} \left[\left({}^{\mathcal{P}}_{k} \mathcal{D}^{\mu}_{a_1^+} \psi \right)(z) + \left({}^{\mathcal{P}}_{k} \mathcal{D}^{\mu}_{a_2^-} \psi \right)(z) \right] \right| \\ &\quad \leq \frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)} \int_0^1 \zeta^{\mu} \left(\zeta {a_1}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}} \right)^{\frac{1-\mathcal{P}}{\mathcal{P}}} \left| \psi^{(\mu+1)} \left(\sqrt[\mathcal{P}]{\zeta {a_1}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}}} \right) \right| d\zeta \\ &\quad + \frac{({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)} \\ &\quad \times \int_0^1 \zeta^{\mu} \left(\zeta {a_2}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}} \right)^{\frac{1-\mathcal{P}}{\mathcal{P}}} \left| \psi^{(\mu+1)} \left(\sqrt[\mathcal{P}]{\zeta {a_2}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}}} \right) \right| d\zeta \\ &\quad \leq \frac{{a_1}^{1-\mathcal{P}} (z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)} \\ &\quad \times \int_0^1 \zeta^{\mu} \left(\zeta {a_1}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}} \right)^{\frac{1-\mathcal{P}}{\mathcal{P}}} \left| \psi^{(\mu+1)} \left(\sqrt[\mathcal{P}]{\zeta {a_1}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}}} \right) \right| d\zeta \\ &\quad + \frac{{a_2}^{1-\mathcal{P}} ({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)} \\ &\quad \times \int_0^1 \zeta^{\mu} \left(\zeta {a_2}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}} \right)^{\frac{1-\mathcal{P}}{\mathcal{P}}} \left| \psi^{(\mu+1)} \left(\sqrt[\mathcal{P}]{\zeta {a_2}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}}} \right) \right| d\zeta \\ &\quad \leq \frac{{a_1}^{1-\mathcal{P}} (z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)} \\ &\quad \times \left(\int_0^1 \zeta^{\lambda\mu} \, d\zeta \right)^{1/\lambda} \left(\int_0^1 \left| \psi^{(\mu+1)} \left(\sqrt[\mathcal{P}]{\zeta {a_1}^{\mathcal{P}} + (1 - \zeta) z^{\mathcal{P}}} \right) \right|^{\vartheta} d\zeta \right)^{1/\vartheta} \end{aligned} \tag{3.10}$$

$$+\frac{{a_2}^{1-\mathcal{P}}({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)} \times\left(\int_0^1\zeta^{\lambda\mu}\,d\zeta\right)^{1/\lambda}\left(\int_0^1\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta {a_2}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta}d\zeta\right)^{1/\vartheta}.$$

As $|\psi^{(\mu+1)}|^{\vartheta}$ is n-polynomial $\mathcal{P}$-convex and $|\psi^{(\mu+1)}(z)|\leq\mathcal{Q}\forall z\in[a_1,a_2]$, we have

$$\begin{aligned}&\int_0^1\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta {a_1}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta}d\zeta\\&\leq\int_0^1\left[\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]\left|\psi^{(\mu+1)}(a_1)\right|^{\vartheta}+\frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\left|\psi^{(\mu+1)}(z)\right|^{\vartheta}\right]d\zeta\\&\leq\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\int_0^1\left[2-(1-\zeta)^{\theta}-\zeta^{\theta}\right]d\zeta\\&\leq\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{2\theta}{\theta+1}\end{aligned}\tag{3.11}$$

and

$$\int_0^1\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta {a_2}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta}d\zeta\leq\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{2\theta}{\theta+1}.\tag{3.12}$$

Since $\int_0^1\zeta^{\lambda\mu}\,d\zeta=\frac{1}{\lambda\mu+1}$. We get the first inequality of Theorem 3.3 by combining all above inequalities. Continuing in the same way the second inequality of Theorem 3.3 can be proved. □

Theorem 3.3 *For a differentiable function $n\in\mathbb{N}$, $\lambda,\vartheta>1$ with $\lambda^{-1}+\vartheta^{-1}=1$, $a_1,a_2\in\Omega$ with $a_1<a_2$, and $\psi^{(\mu)}:\Omega\subset(0,\infty)\to\mathbb{R}$ on Ω° such that $\psi^{(\mu+1)}\in M([a_1,a_2])$ and $|\psi^{(\mu+1)}|^{\vartheta}$ an n-polynomial $\mathcal{P}$-convex function satisfying $|\psi^{(\mu+1)}(z)|\leq\mathcal{Q}$, $\forall z\in[a_1,a_2]$, the following inequality holds for all $z\in(a_1,a_2)$ and $\mathcal{P}\in(1,\infty)$:*

$$\begin{aligned}&\left|\frac{(z^{\mathcal{P}}-{a_1}^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_1)+({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2-a_1)}\right.\\&\qquad\left.-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\left[\left({}_k^{\mathcal{P}}\mathcal{D}_{a_1^+}^{\mu}\psi\right)(z)+\left({}_k^{\mathcal{P}}\mathcal{D}_{a_2^-}^{\mu}\psi\right)(z)\right]\right|\\&\leq\frac{{a_1}^{1-\mathcal{P}}\mathcal{Q}}{\mathcal{P}^{1+\mu}(1+\lambda\mu)^{1/\lambda}}\left[\frac{(z^{\mathcal{P}}-{a_1}^{\mathcal{P}})^{\mu+1}+({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{(a_2-a_1)}\right]\\&\qquad\times\left(\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\left[\frac{\vartheta\mu+2\theta+1}{(\mu\vartheta+1)(\mu\vartheta+\theta+1)}-\mathbb{B}(\theta+1,\vartheta\mu+1)\right]\right)^{1/\vartheta}\end{aligned}\tag{3.13}$$

and the following inequality holds for all $z\in(a_1,a_2)$ and $\mathcal{P}\in(-\infty,0)\cup(0,1)$:

$$\begin{aligned}&\left|\frac{(z^{\mathcal{P}}-{a_1}^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_1)+({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2-a_1)}\right.\\&\qquad\left.-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\left[\left({}_k^{\mathcal{P}}\mathcal{D}_{a_1^+}^{\mu}\psi\right)(z)+\left({}_k^{\mathcal{P}}\mathcal{D}_{a_2^-}^{\mu}\psi\right)(z)\right]\right|\end{aligned}\tag{3.14}$$

$$\leq \frac{{a_2}^{1-\mathcal{P}}\mathcal{Q}}{\mathcal{P}^{1+\mu}(1+\lambda\mu)^{1/\lambda}}\left[\frac{(z^{\mathcal{P}}-{a_1}^{\mathcal{P}})^{\mu+1}+({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{(a_2-a_1)}\right]$$
$$\times\left(\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\left[\frac{\vartheta\mu+2\theta+1}{(\mu\vartheta+1)(\mu\vartheta+\theta+1)}-\mathbb{B}(\theta+1,\vartheta\mu+1)\right]\right)^{1/\vartheta}.$$

Proof We use the Lemma 3.1, to prove Theorem 3.3 and the power-mean inequality to yield

$$\begin{aligned}
&\left|\frac{(z^{\mathcal{P}}-{a_1}^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_1)+({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2-a_1)}\right.\\
&\quad\left.-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\left[\left({}^{\mathcal{P}}_{k}\mathcal{D}^{\mu}_{a_1^+}\psi\right)(z)+\left({}^{\mathcal{P}}_{k}\mathcal{D}^{\mu}_{a_2^-}\psi\right)(z)\right]\right|\\
&\quad\leq\frac{(z^{\mathcal{P}}-{a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\int_0^1\zeta^{\mu}\left(\zeta{a_1}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta{a_1}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|d\zeta\\
&\quad\quad+\frac{({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\int_0^1\zeta^{\mu}\left(\zeta{a_2}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta{a_2}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|d\zeta\\
&\quad\leq\frac{{a_1}^{1-\mathcal{P}}(z^{\mathcal{P}}-{a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\int_0^1\zeta^{\mu}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta{a_1}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|d\zeta\\
&\quad\quad+\frac{{a_2}^{1-\mathcal{P}}({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\int_0^1\zeta^{\mu}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta{a_2}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|d\zeta\\
&\quad\leq\frac{{a_1}^{1-\mathcal{P}}(z^{\mathcal{P}}-{a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\left(\int_0^1\zeta^{\vartheta\mu}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta{a_1}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta}d\zeta\right)^{1/\vartheta}\\
&\quad\quad+\frac{{a_2}^{1-\mathcal{P}}({a_2}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\left(\int_0^1\zeta^{\vartheta\mu}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta{a_2}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta}d\zeta\right)^{1/\vartheta}.
\end{aligned}\tag{3.15}$$

As $|\psi^{(\mu)}|^{\vartheta}$ is n-polynomial $\mathcal{P}$-convex and $|\psi^{(\mu+1)}(z)|\leq\mathcal{Q}$, $\forall z\in[a_1,a_2]$, we obtain

$$\begin{aligned}
&\int_0^1\zeta^{\vartheta\mu}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta{a_1}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta}d\zeta\\
&\quad\leq\int_0^1\zeta^{\vartheta\mu}\left[\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]\left|\psi^{(\mu+1)}(a_1)\right|^{\vartheta}+\frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\left|\psi^{(\mu+1)}(z)\right|^{\vartheta}\right]d\zeta\\
&\quad=\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\int_0^1\left[2\zeta^{\mu\vartheta}-\zeta^{\mu\vartheta}(1-\zeta)^{\theta}+\zeta^{\vartheta\mu}\left(1-\zeta^{\theta}\right)\right]d\zeta\\
&\quad\leq\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\left[\frac{\vartheta\mu+2\theta+1}{(\mu\vartheta+1)(\mu\vartheta+\theta+1)}-\mathbb{B}(\theta+1,\vartheta\mu+1)\right].
\end{aligned}\tag{3.16}$$

Similarly,

$$\begin{aligned}
&\int_0^1\zeta^{\vartheta\mu}\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta{a_2}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta}d\zeta\\
&\quad\leq\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\left[\frac{\vartheta\mu+2\theta+1}{(\mu\vartheta+1)(\mu\vartheta+\theta+1)}-\mathbb{B}(\theta+1,\vartheta\mu+1)\right].
\end{aligned}\tag{3.17}$$

We arrive at the first inequality of Theorem 3.3 by combining all above inequalities. For the second part continuing in the same fashion, we find the required result. □

Theorem 3.4 *For a differentiable function $n \in \mathbb{N}$, $\lambda, \vartheta > 1$ with $\lambda^{-1} + \vartheta^{-1} = 1$, $a_1, a_2 \in \Omega$ with $a_1 < a_2$, and $\psi^{(\mu)} : \Omega \subset (0,\infty) \to \mathbb{R}$ on Ω° such that $\psi^{(\mu+1)} \in M([a_1,a_2])$ and $|\psi^{(\mu+1)}|^\vartheta$ be an n-polynomial $\mathcal{P}$-convex function satisfying $|\psi^{(\mu+1)}(z)| \leq \mathcal{Q}$, $\forall z \in [a_1,a_2]$, the following inequality holds for all $z \in (a_1,a_2)$ and $\mathcal{P} \in (1,\infty)$:*

$$\begin{aligned}
&\bigg|\frac{(z^{\mathcal{P}} - a_1{}^{\mathcal{P}})^\mu \psi^{(\mu)}(a_1) + (a_2{}^{\mathcal{P}} - z^{\mathcal{P}})^\mu \psi^{(\mu)}(a_2)}{\mathcal{P}^\mu (a_2 - a_1)} \\
&\quad - \frac{\Gamma_k(\mu+k)}{a_2 - a_1}\big[\big({}^{\mathcal{P}}_k\mathcal{D}^\mu_{a_1^+}\psi\big)(z) + \big({}^{\mathcal{P}}_k\mathcal{D}^\mu_{a_2^-}\psi\big)(z)\big]\bigg| \\
&\quad \leq \frac{(z^{\mathcal{P}} - a_1{}^{\mathcal{P}})^{\mu+1} + (a_2{}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)}\left[\frac{(a_1{}^{\lambda(1-\mathcal{P})})}{\lambda(\lambda\mu+1)} + \frac{1}{\vartheta}\left(\frac{\mathcal{Q}}{n}\sum_{\theta=1}^{n}\frac{2\theta}{\theta+1}\right)^\vartheta\right]
\end{aligned} \tag{3.18}$$

and the following inequality holds for all $z \in (a_1,a_2)$ and $\mathcal{P} \in (-\infty,0) \cup (0,1)$:

$$\begin{aligned}
&\bigg|\frac{(z^{\mathcal{P}} - a_1{}^{\mathcal{P}})^\mu \psi^{(\mu)}(a_1) + (a_2{}^{\mathcal{P}} - z^{\mathcal{P}})^\mu \psi^{(\mu)}(a_2)}{\mathcal{P}^\mu (a_2 - a_1)} \\
&\quad - \frac{\Gamma_k(\mu+k)}{a_2 - a_1}\big[\big({}^{\mathcal{P}}_k\mathcal{D}^\mu_{a_1^+}\psi\big)(z) + \big({}^{\mathcal{P}}_k\mathcal{D}^\mu_{a_2^-}\psi\big)(z)\big]\bigg| \\
&\quad \leq \frac{(z^{\mathcal{P}} - a_1{}^{\mathcal{P}})^{\mu+1} + (a_2{}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)}\left[\frac{(a_2{}^{\lambda(1-\mathcal{P})})}{\lambda(\lambda\mu+1)} + \frac{1}{\vartheta}\left(\frac{\mathcal{Q}}{n}\sum_{\theta=1}^{n}\frac{2\theta}{\theta+1}\right)^\vartheta\right].
\end{aligned} \tag{3.19}$$

Proof The Young inequality is $cd \leq \frac{1}{\lambda}c^\lambda + \frac{1}{\vartheta}d^\lambda$, $c,d \geq 0$, $\lambda, \vartheta > 1$, $\lambda^{-1} + \vartheta^{-1} = 1$. We use Lemma 3.1 to prove the first part of Theorem 3.3, and using the n-polynomial $\mathcal{P}$-convexity of $|\psi^{(\mu+1)}|^\vartheta$ we find

$$\begin{aligned}
&\bigg|\frac{(z^{\mathcal{P}} - a_1{}^{\mathcal{P}})^\mu \psi^{(\mu)}(a_1) + (a_2{}^{\mathcal{P}} - z^{\mathcal{P}})^\mu \psi^{(\mu)}(a_2)}{\mathcal{P}^\mu (a_2 - a_1)} \\
&\quad - \frac{\Gamma_k(\mu+k)}{a_2 - a_1}\big[\big({}^{\mathcal{P}}_k\mathcal{D}^\mu_{a_1^+}\psi\big)(z) + \big({}^{\mathcal{P}}_k\mathcal{D}^\mu_{a_2^-}\psi\big)(z)\big]\bigg| \\
&\quad \leq \frac{(z^{\mathcal{P}} - a_1{}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)} \\
&\quad \times \int_0^1 \left(\frac{1}{\lambda}\big|\zeta^\mu\big(\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\big|^\lambda + \frac{1}{\vartheta}\big|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}}\big)\big|^\vartheta\right) d\zeta \\
&\quad + \frac{(a_2{}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)} \\
&\quad \times \int_0^1 \left(\frac{1}{\lambda}\big|\zeta^\mu\big(\zeta a_2{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\big|^\lambda + \frac{1}{\vartheta}\big|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}}\big)\big|^\vartheta\right) d\zeta
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\quad \leq \frac{(z^{\mathcal{P}} - a_1{}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)}\left(\frac{\zeta^{\lambda\mu}}{\lambda}\big|\big(\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\big|^\lambda\right. \\
&\quad \left. + \frac{1}{\vartheta}\left|\frac{1}{n}\sum_{\theta=1}^{n}\big[1-(1-\zeta)^\theta\big]\big|\psi^{(\mu+1)}(a_1)\big| + \frac{1}{n}\sum_{\theta=1}^{n}\big[1-\zeta^\theta\big]\right|\psi^{(\mu+1)}(z)||^\vartheta\right) d\zeta
\end{aligned}$$

$$+\frac{(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}\int_0^1\left(\frac{\zeta^{\lambda\mu}}{\lambda}\left|\left(\zeta a_2{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\right)^{\frac{1-\mathscr{P}}{\mathscr{P}}}\right|^{\lambda}\right.$$
$$\left.+\frac{1}{\vartheta}\left|\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]\left|\psi^{(\mu+1)}(a_2)\right|+\frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\right|\psi^{(\mu+1)}(z)||^{\vartheta}\right)d\zeta$$
$$\leq\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu+1}+(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}\left[\frac{(a_1{}^{\lambda(1-\mathscr{P})})}{\lambda(\lambda\mu+1)}+\frac{1}{\vartheta}\left(\frac{\mathcal{Q}}{n}\sum_{\theta=1}^{n}\frac{2\theta}{\theta+1}\right)^{\vartheta}\right].$$

Continuing in the same fashion, we can prove the second part. □

Theorem 3.5 *For a differentiable function $n\in\mathbb{N}$, $\lambda,\vartheta>1$ with $\lambda^{-1}+\vartheta^{-1}=1$, $a_1,a_2\in\Omega$ with $a_1<a_2$, and $\psi^{(\mu)}:\Omega\subset(0,\infty)\to\mathbb{R}$ on Ω° such that $\psi^{(\mu+1)}\in M([a_1,a_2])$ and $|\psi^{(\mu+1)}|^{\vartheta}$ a n-polynomial $\mathscr{P}$-convex function satisfying $|\psi^{(\mu+1)}(z)|\leq\mathcal{Q}$, $\forall z\in[a_1,a_2]$, the following inequality holds for all $z\in(a_1,a_2)$ and $\mathscr{P}\in(1,\infty)$:*

$$\left|\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu}\psi^{(\mu)}(a_1)+(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathscr{P}^{\mu}(a_2-a_1)}\right.$$
$$\left.-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\left[\left({}_k^{\mathscr{P}}\mathcal{D}^{\mu}_{a_1^+}\psi\right)(z)+\left({}_k^{\mathscr{P}}\mathcal{D}^{\mu}_{a_2^-}\psi\right)(z)\right]\right| \tag{3.21}$$
$$\leq\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu+1}+(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}\left[\frac{\lambda a_1{}^{1-\mathscr{P}}}{(\mu+1)}+\frac{\vartheta\mathcal{Q}}{n}\sum_{\theta=1}^{n}\frac{2\theta}{\theta+1}\right]$$

and the following inequality holds for all $z\in(a_1,a_2)$ and $\mathscr{P}\in(-\infty,0)\cup(0,1)$,

$$\left|\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu}\psi^{(\mu)}(a_1)+(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathscr{P}^{\mu}(a_2-a_1)}\right.$$
$$\left.-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\left[\left({}_k^{\mathscr{P}}\mathcal{D}^{\mu}_{a_1^+}\psi\right)(z)+\left({}_k^{\mathscr{P}}\mathcal{D}^{\mu}_{a_2^-}\psi\right)(z)\right]\right| \tag{3.22}$$
$$\leq\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu+1}+(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}\left[\frac{\lambda a_2{}^{1-\mathscr{P}}}{(\mu+1)}+\frac{\vartheta\mathcal{Q}}{n}\sum_{\theta=1}^{n}\frac{2\theta}{\theta+1}\right].$$

Proof Use the weighted $\mathcal{AM}-\mathcal{GM}$ inequality

$$c^{\lambda}d^{\vartheta}\leq\lambda c+\vartheta d,\quad c,d\geq0,\lambda,\vartheta>0,\lambda+\vartheta=1. \tag{3.23}$$

Using Lemma 3.1 and by using the n-polynomial $\mathscr{P}$-convexity of $|\psi^{(\mu+1)}|^{\vartheta}$ we find

$$\left|\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu}\psi^{(\mu)}(a_1)+(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathscr{P}^{\mu}(a_2-a_1)}\right.$$
$$\left.-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\left[\left({}_k^{\mathscr{P}}\mathcal{D}^{\mu}_{a_1^+}\psi\right)(z)+\left({}_k^{\mathscr{P}}\mathcal{D}^{\mu}_{a_2^-}\psi\right)(z)\right]\right|$$
$$\leq\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}$$
$$\times\int_0^1\left[\zeta^{\mu}\left(\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\right)^{\frac{1-\mathscr{P}}{\mathscr{P}}}\right]^{\lambda}\left[\left|\psi^{(\mu+1)}\left(\sqrt[\mathscr{P}]{\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}}\right)\right|\right]^{\vartheta}d\zeta$$

$$
\begin{aligned}
&+\frac{(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\\
&\times\int_0^1\zeta^{\mu}\bigl(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\bigr)^{\frac{1-\mathcal{P}}{\mathcal{P}}}]^{\lambda}\bigl[\bigl|\psi^{(\mu+1)}\bigl(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\bigr)\bigr|\bigr]^{\vartheta}\,d\zeta\\
\le{}&\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\\
&\times\biggl[\int_0^1\lambda\zeta^{\mu}\bigl|\bigl(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\bigr)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\bigr|\,d\zeta\\
&+\int_0^1\vartheta\bigl|\psi^{(\mu+1)}\bigl(\sqrt[\mathcal{P}]{\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\bigr)\bigr|\,d\zeta\biggr]\\
&+\frac{(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)} \qquad (3.24)\\
&\times\biggl[\int_0^1\lambda\zeta^{\mu}\bigl|\bigl(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\bigr)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\bigr|\,d\zeta\\
&+\int_0^1\vartheta\bigl|\psi^{(\mu+1)}\bigl(\sqrt[\mathcal{P}]{\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\bigr)\bigr|\,d\zeta\biggr]\\
\le{}&\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\\
&\times\biggl[\int_0^1 a_1{}^{1-\mathcal{P}}\lambda\zeta^{\mu}\,d\zeta+\int_0^1\vartheta\biggl|\frac{1}{n}\sum_{\theta=1}^{n}\bigl[1-(1-\zeta)^{\theta}\bigr]\bigl|\psi^{(\mu+1)}(a_1)\bigr|\\
&+\frac{1}{n}\sum_{\theta=1}^{n}\bigl[1-\zeta^{\theta}\bigr]\bigl|\psi^{(\mu+1)}(z)\bigr|\biggr|\,d\zeta\biggr]\\
&+\frac{(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\\
&\times\biggl[\int_0^1 a_1{}^{1-\mathcal{P}}\lambda\zeta^{\mu}\,d\zeta+\int_0^1\vartheta\biggl|\frac{1}{n}\sum_{\theta=1}^{n}\bigl[1-(1-\zeta)^{\theta}\bigr]\bigl|\psi^{(\mu+1)}(a_2)\bigr|\\
&+\frac{1}{n}\sum_{\theta=1}^{n}\bigl[1-\zeta^{\theta}\bigr]\bigl|\psi^{(\mu+1)}(z)\bigr|\biggr|\,d\zeta\biggr]\\
\le{}&\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu+1}+(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\biggl[\frac{\lambda a_1{}^{1-\mathcal{P}}}{(\mu+1)}+\frac{\vartheta\mathcal{Q}}{n}\sum_{\theta=1}^{n}\frac{2\theta}{\theta+1}\biggr].
\end{aligned}
$$

Continuing in the same fashion, we can prove the second part. □

Theorem 3.6 *For a differentiable function* $n\in\mathbb{N}$, $\lambda,\vartheta>1$ *with* $\lambda^{-1}+\vartheta^{-1}=1$, $a_1,a_2\in\Omega$ *with* $a_1<a_2$, *and* $\psi^{(\mu)}:\Omega\subset(0,\infty)\to\mathbb{R}$ *on* Ω° *such that* $\psi^{(\mu+1)}\in M([a_1,a_2])$ *and* $|\psi^{(\mu+1)}|^{\vartheta}$ *a n-polynomial* $\mathcal{P}$*-convex function satisfying* $|\psi^{(\mu+1)}(z)|\le\mathcal{Q}$, $\forall z\in[a_1,a_2]$, *the following inequality holds for all* $z\in(a_1,a_2)$ *and* $\mathcal{P}\in(1,\infty)$:

$$
\begin{aligned}
&\biggl|\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_1)+(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2-a_1)}\\
&\quad-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\bigl[\bigl({}^{\mathcal{P}}_{k}\mathcal{D}^{\mu}_{a_1{}^{+}}\psi\bigr)(z)+\bigl({}^{\mathcal{P}}_{k}\mathcal{D}^{\mu}_{a_2{}^{-}}\psi\bigr)(z)\bigr]\biggr| \qquad (3.25)
\end{aligned}
$$

$$\leq \frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)}\Bigg[\big(\Lambda_1(a_1, z; \mathcal{P})\big)^{1/\lambda}\left(\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{\theta}{\theta+1}\right)^{1/\vartheta} + \big(\Lambda_2(a_1, z; \mathcal{P})\big)^{1/\lambda}\left(\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{\theta^2+2\theta-1}{(\theta+2)(\theta+1)}\right)^{1/\vartheta}\Bigg]$$

and the following inequality holds for all $z \in (a_1, a_2)$ *and* $\mathcal{P} \in (-\infty, 0) \cup (0, 1)$:

$$\begin{aligned} &\Bigg|\frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_1) + ({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2 - a_1)} - \frac{\Gamma_k(\mu+k)}{a_2 - a_1}\big[\big({}^{\mathcal{P}}_{k}\mathcal{D}^{\mu}_{a_1^+}\psi\big)(z) + \big({}^{\mathcal{P}}_{k}\mathcal{D}^{\mu}_{a_2^-}\psi\big)(z)\big]\Bigg| \\ &\quad\leq \frac{({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)}\Bigg[\big(\Lambda_3(a_2, z; \mathcal{P})\big)^{1/\lambda}\left(\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{\theta}{\theta+1}\right)^{1/\vartheta}, \\ &\qquad + \big(\Lambda_4(a_2, z; \mathcal{P})\big)^{1/\lambda}\left(\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{\theta^2+2\theta-1}{(\theta+2)(\theta+1)}\right)^{1/\vartheta}\Bigg]. \end{aligned} \tag{3.26}$$

Here

$$\begin{aligned} \Lambda_1(a_1, z; \mathcal{P}) &= \begin{cases} \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}), \lambda\mu+1, \lambda\mu+3, 1-(a_1/z)^{\mathcal{P}})]}{z^{\lambda(\mathcal{P}-1)}(\lambda\mu+1)(\lambda\mu+2)}, & \mathcal{P} \in (-\infty, 0) \cup (0, 1), \\ \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}), \lambda\mu+1, \lambda\mu+3, 1-(z/a_1)^{\mathcal{P}})]}{{a_1}^{\lambda(\mathcal{P}-1)}(\lambda\mu+1)(\lambda\mu+2)}, & \mathcal{P} \in (1, \infty), \end{cases} \\ \Lambda_2(a_1, z; \mathcal{P}) &= \begin{cases} \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}), \lambda\mu+2, \lambda\mu+3, 1-(a_1/z)^{\mathcal{P}})]}{z^{\lambda(\mathcal{P}-1)}(\lambda\mu+2)}, & \mathcal{P} \in (-\infty, 0) \cup (0, 1), \\ \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}), \lambda\mu+2, \lambda\mu+3, 1-(z/a_1)^{\mathcal{P}})]}{{a_1}^{\lambda(\mathcal{P}-1)}(\lambda\mu+2)}, & \mathcal{P} \in (1, \infty), \end{cases} \\ \Lambda_3(a_2, z; \mathcal{P}) &= \begin{cases} \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}), \lambda\mu+1, \lambda\mu+3, 1-(a_2/z)^{\mathcal{P}}]}{z^{\lambda(\mathcal{P}-1)}(\lambda\mu+1)(\lambda\mu+2)}, & \mathcal{P} \in (-\infty, 0) \cup (0, 1), \\ \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}), \lambda\mu+1, \lambda\mu+3, 1-(z/a_2)^{\mathcal{P}}]}{{a_2}^{\lambda(\mathcal{P}-1)}(\lambda\mu+1)(\lambda\mu+2)}, & \mathcal{P} \in (1, \infty), \end{cases} \end{aligned} \tag{3.27}$$

and

$$\Lambda_4(a_2, z; \mathcal{P}) = \begin{cases} \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}), \lambda\mu+2, \lambda\mu+3, 1-(a_2/z)^{\mathcal{P}})]}{z^{\lambda(\mathcal{P}-1)}(\lambda\mu+2)}, & \mathcal{P} \in (-\infty, 0) \cup (0, 1), \\ \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}), \lambda\mu+2, \lambda\mu+3, 1-(z/a_2)^{\mathcal{P}})]}{{a_2}^{\lambda(\mathcal{P}-1)}(\lambda\mu+2)}, & \mathcal{P} \in (1, \infty). \end{cases} \tag{3.28}$$

Proof We use Lemma 3.1 to prove the first part of the inequality and we use the Hölder–İşcan inequality to find

$$\begin{aligned} &\Bigg|\frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_1) + ({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2 - a_1)} - \frac{\Gamma_k(\mu+k)}{a_2 - a_1}\big[\big({}^{\mathcal{P}}_{k}\mathcal{D}^{\mu}_{a_1^+}\psi\big)(z) + \big({}^{\mathcal{P}}_{k}\mathcal{D}^{\mu}_{a_2^-}\psi\big)(z)\big]\Bigg| \\ &\quad\leq \frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)}\int_0^1 \zeta^{\mu}\big(\zeta {a_1}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\Big|\psi^{(\mu+1)}\Big(\sqrt[\mathcal{P}]{\zeta {a_1}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}}\Big)\Big|\,d\zeta \\ &\qquad + \frac{({a_2}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)}\int_0^1 \zeta^{\mu}\big(\zeta {a_2}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\Big|\psi^{(\mu+1)}\Big(\sqrt[\mathcal{P}]{\zeta {a_2}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}}\Big)\Big|\,d\zeta \\ &\quad\leq \frac{(z^{\mathcal{P}} - {a_1}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)}\Bigg[\left(\int_0^1 \zeta^{\lambda\mu}(1-\zeta)\big(\zeta {a_1}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})}\,d\zeta\right)^{1/\lambda} \end{aligned}$$

$$\begin{aligned}
&\times \left(\int_0^1 (1-\zeta)\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_1^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta\right)^{1/\vartheta} \\
&+ \left(\int_0^1 \zeta^{\lambda\mu+1}\left(\zeta a_1^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})} d\zeta\right)^{1/\lambda} \\
&\times \left(\int_0^1 \zeta\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_1^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta\right)^{1/\vartheta}\Bigg] \\
&+ \frac{(a_2^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\Bigg[\left(\left(\int_0^1 \zeta^{\lambda\mu}(1-\zeta)\left(\zeta a_2^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})} d\zeta\right)^{1/\lambda}\right. \\
&\times \left(\int_0^1 (1-\zeta)\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_2^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta\right)^{1/\vartheta} \\
&+ \left(\int_0^1 \zeta^{\lambda\mu+1}\left(\zeta a_2^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\right)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})} d\zeta\right)^{1/\lambda} \\
&\times \left(\int_0^1 \zeta\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_2^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta\right)^{1/\vartheta}\Bigg] \\
&\leq \frac{(z^{\mathcal{P}}-a_1^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)} \\
&\times \Bigg[\left(\Lambda_1(a_1,z;\mathcal{P})\right)^{1/\lambda}\left(\int_0^1 (1-\zeta)\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_1^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta\right)^{1/\vartheta} \\
&+ \left(\Lambda_2(a_1,z;\mathcal{P})\right)^{1/\lambda}\left(\int_0^1 \zeta\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_1^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta\right)^{1/\vartheta}\Bigg] \\
&+ \frac{(a_2^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)} \\
&\times \Bigg[\left(\Lambda_3(a_2,z;\mathcal{P})\right)^{1/\lambda}\left(\int_0^1 (1-\zeta)\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_2^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta\right)^{1/\vartheta} \\
&+ \left(\Lambda_4(a_2,z;\mathcal{P})\right)^{1/\lambda}\left(\int_0^1 \zeta\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_2^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta\right)^{1/\vartheta}\Bigg].
\end{aligned} \tag{3.29}$$

As $|\psi^{(\mu+1)}|^{\vartheta}$ is n-polynomial $\mathcal{P}$-convex and $|\psi^{(\mu+1)}(z)| \leq \mathcal{Q}$, $\forall z \in [a_1, a_2]$, we find

$$\begin{aligned}
&\int_0^1 (1-\zeta)\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_1^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta \\
&\leq \int_0^1 (1-\zeta)\left[\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]\left|\psi^{(\mu+1)}(a_1)\right|^{\vartheta} + \frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\left|\psi^{(\mu+1)}(z)\right|^{\vartheta}\right] d\zeta \\
&\leq \frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\int_0^1\left[2(1-\zeta)-(1-\zeta)^{\theta+1}-\zeta^{\theta}(1-\zeta)\right] d\zeta \\
&= \frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{\theta}{\theta+1},
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&\int_0^1 \zeta\left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_1^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta \\
&\leq \int_0^1 \zeta\left[\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]\left|\psi^{(\mu+1)}(a_1)\right|^{\vartheta} + \frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\left|\psi^{(\mu+1)}(z)\right|^{\vartheta}\right] d\zeta
\end{aligned}$$

$$= \frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\int_0^1 \big[2\zeta - \zeta(1-\zeta)^{\theta} - \zeta^{\theta+1}\big]\,d\zeta$$
$$\le \frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{\theta^2+2\theta-1}{(\theta+2)(\theta+1)}.$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^1 (1-\zeta)\big|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\big|^{\vartheta}\,d\zeta \le \frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{\theta}{\theta+1},\\
&\int_0^1 \zeta\big|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\big|^{\vartheta}\,d\zeta \le \frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\frac{\theta^2+2\theta-1}{(\theta+2)(\theta+1)}.
\end{aligned} \tag{3.31}$$

We have the result

$$\begin{aligned}
\Lambda_1(a_1,z;\mathcal{P}) &:= \int_0^1 \zeta^{\lambda\mu}(1-\zeta)\big(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})}\,d\zeta\\
&= \begin{cases} \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}),\lambda\mu+1,\lambda\mu+3,1-(a_1/z)^{\mathcal{P}}]}{z^{\lambda(\mathcal{P}-1)}(\lambda\mu+1)(\lambda\mu+2)}, & \mathcal{P}\in(-\infty,0)\cup(0,1),\\ \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}),\lambda\mu+1,\lambda\mu+3,1-(z/a_1)^{\mathcal{P}}]}{a_1{}^{\lambda(\mathcal{P}-1)}(\lambda\mu+1)(\lambda\mu+2)}, & \mathcal{P}\in(1,\infty),\end{cases}
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\Lambda_2(a_1,z;\mathcal{P}) &:= \int_0^1 \zeta^{\lambda\mu+1}\big(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})}\,d\zeta\\
&= \begin{cases} \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}),\lambda\mu+2,\lambda\mu+3,1-(a_1/z)^{\mathcal{P}})]}{z^{\lambda(\mathcal{P}-1)}(\lambda\mu+2)}, & \mathcal{P}\in(-\infty,0)\cup(0,1),\\ \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}),\lambda\mu+2,\lambda\mu+3,1-(z/a_1)^{\mathcal{P}})]}{a_1{}^{\lambda(\mathcal{P}-1)}(\lambda\mu+2)}, & \mathcal{P}\in(1,\infty),\end{cases}
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
\Lambda_3(a_2,z;\mathcal{P}) &:= \int_0^1 \zeta^{\lambda\mu}(1-\zeta)\big(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})}\,d\zeta\\
&= \begin{cases} \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}),\lambda\mu+1,\lambda\mu+3,1-(a_2/z)^{\mathcal{P}}]}{z^{\lambda(\mathcal{P}-1)}(\lambda\mu+1)(\lambda\mu+2)}, & \mathcal{P}\in(-\infty,0)\cup(0,1),\\ \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}),\lambda\mu+1,\lambda\mu+3,1-(z/a_2)^{\mathcal{P}}]}{a_2{}^{\lambda(\mathcal{P}-1)}(\lambda\mu+1)(\lambda\mu+2)}, & \mathcal{P}\in(1,\infty),\end{cases}
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
\Lambda_4(a_2,z;\mathcal{P}) &:= \int_0^1 \zeta^{\lambda\mu+1}\big(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})}\,d\zeta\\
&= \begin{cases} \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}),\lambda\mu+2,\lambda\mu+3,1-(a_2/z)^{\mathcal{P}})]}{z^{\lambda(\mathcal{P}-1)}(\lambda\mu+2)}, & \mathcal{P}\in(-\infty,0)\cup(0,1),\\ \frac{[{}_2\mathcal{F}_1(\lambda(1-1/\mathcal{P}),\lambda\mu+2,\lambda\mu+3,1-(z/a_2)^{\mathcal{P}})]}{a_2{}^{\lambda(\mathcal{P}-1)}(\lambda\mu+2)}, & \mathcal{P}\in(1,\infty).\end{cases}
\end{aligned} \tag{3.35}$$

□

Theorem 3.7 *For a differentiable function $n \in \mathbb{N}$, $\lambda, \vartheta > 1$ with $\lambda^{-1}+\vartheta^{-1}=1$, $a_1, a_2 \in \Omega$ with $a_1 < a_2$, and $\psi^{(\mu)} : \Omega \subset (0,\infty) \to \mathbb{R}$ on Ω° such that $\psi^{(\mu+1)} \in M([a_1,a_2])$ and $|\psi^{(\mu+1)}|^{\vartheta}$ an n-polynomial $\mathcal{P}$-convex function satisfying $|\psi^{(\mu+1)}(z)| \le \mathcal{Q}$, $\forall z \in [a_1,a_2]$, the following inequality holds for all $z \in (a_1,a_2)$ and $\mathcal{P} \in (1,\infty)$:*

$$\begin{aligned}
&\bigg|\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_1)+(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2-a_1)}\\
&\quad - \frac{\Gamma_k(\mu+k)}{a_2-a_1}\big[\big({}_k^{\mathcal{P}}\mathcal{D}^{\mu}_{a_1^+}\psi\big)(z)+\big({}_k^{\mathcal{P}}\mathcal{D}^{\mu}_{a_2^-}\psi\big)(z)\big]\bigg|
\end{aligned} \tag{3.36}$$
$$\le \frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\Bigg[\big(\Lambda_1^*(a_1,z;\mathcal{P})\big)^{1/\lambda}\Bigg(\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\mathrm{T}_1(a_1,z;\mathcal{P})\Bigg)^{1/\vartheta}$$

$$+ \left(\Lambda_2^*(a_1, z; \mathcal{P})\right)^{1/\lambda} \left(\frac{\mathcal{Q}^{\vartheta}}{n} \sum_{\theta=1}^{n} \mathrm{T}_2(a_1, z; \mathcal{P}) \right)^{1/\vartheta} \Bigg]$$

and the following inequality holds for all $z \in (a_1, a_2)$ and $\mathcal{P} \in (-\infty, 0) \cup (0, 1)$:

$$\begin{aligned} &\left| \frac{(z^{\mathcal{P}} - a_1{}^{\mathcal{P}})^{\mu} \psi^{(\mu)}(a_1) + (a_2{}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu} \psi^{(\mu)}(a_2)}{\mathcal{P}^{\mu}(a_2 - a_1)} \right. \\ &\quad \left. - \frac{\Gamma_k(\mu + k)}{a_2 - a_1} \left[\left({}_k^{\mathcal{P}}\mathcal{D}_{a_1^+}^{\mu} \psi\right)(z) + \left({}_k^{\mathcal{P}}\mathcal{D}_{a_2^-}^{\mu} \psi\right)(z) \right] \right| \\ &\quad \le \frac{(a_2{}^{\mathcal{P}} - z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2 - a_1)} \left[\left(\Lambda_3^*(a_2, z; \mathcal{P})\right)^{1/\lambda} \left(\frac{\mathcal{Q}^{\vartheta}}{n} \sum_{\theta=1}^{n} \mathrm{T}_3(a_2, z; \mathcal{P}) \right)^{1/\vartheta} \right. \\ &\quad \left. + \left(\Lambda_4^*(a_2, z; \mathcal{P})\right)^{1/\lambda} \left(\frac{\mathcal{Q}^{\vartheta}}{n} \sum_{\theta=1}^{n} \mathrm{T}_4(a_2, z; \mathcal{P}) \right)^{1/\vartheta} \right]. \end{aligned} \tag{3.37}$$

Here

$$\mathrm{T}_1(a_1 z; \mathcal{P}) = \begin{cases} \frac{1}{z^{(\mathcal{P}-1)}} \left[\frac{2}{(\mu+1)(\mu+2)} {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + 1, \mu + 3, 1 - (a_1/z)^{\mathcal{P}}) \right. \\ \quad - \mathbb{B}(\mu + 1, \theta + 2) {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + 1, \theta + \mu + 3, 1 - (a_1/z)^{\mathcal{P}})] \\ \quad - \mathbb{B}(\mu + \theta + 1, 2) {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + \theta + 1, \theta + \mu + 3, 1 - (a_1/z)^{\mathcal{P}})], \\ \quad \mathcal{P} \in (-\infty, 0) \cup (0, 1), \\ \frac{1}{a_1{}^{(\mathcal{P}-1)}} \left[\frac{2}{(\mu+1)(\mu+2)} \mathcal{F}_1(1 - 1/\mathcal{P}, \mu + 1, \mu + 3, 1 - (z/a_1)^{\mathcal{P}}) \right. \\ \quad - \mathbb{B}(\mu + 1, \theta + 2) {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + 1, \theta + \mu + 3, 1 - (z/a_1)^{\mathcal{P}})] \\ \quad - \mathbb{B}(\mu + \theta + 1, 2) {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + \theta + 1, \theta + \mu + 3, 1 - (z/a_1)^{\mathcal{P}})], \\ \quad \mathcal{P} \in (1, \infty), \end{cases}$$

$$\mathrm{T}_2(a_1 z; \mathcal{P}) = \begin{cases} \frac{1}{z^{(\mathcal{P}-1)}} \left[\frac{2}{(\mu+2)} \mathcal{F}_1(1 - 1/\mathcal{P}, \mu + 2, \mu + 3, 1 - (a_1/z)^{\mathcal{P}}) \right. \\ \quad - \mathbb{B}(\mu + 2, \theta + 2) {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + 2, \theta + \mu + 3, 1 - (a_1/z)^{\mathcal{P}})] \\ \quad - \frac{1}{\mu+\theta+2} {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + \theta + 2, \theta + \mu + 3, 1 - (a_1/z)^{\mathcal{P}})], \\ \quad \mathcal{P} \in (-\infty, 0) \cup (0, 1), \\ \frac{1}{a_1{}^{(\mathcal{P}-1)}} \left[\frac{2}{(\mu+2)} \mathcal{F}_1(1 - 1/\mathcal{P}, \mu + 2, \mu + 3, 1 - (z/a_1)^{\mathcal{P}}) \right. \\ \quad - \mathbb{B}(\mu + 2, \theta + 2) {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + 2, \theta + \mu + 3, 1 - (z/a_1)^{\mathcal{P}})] \\ \quad - \frac{1}{\mu+\theta+2} {}_2\mathcal{F}_1(1 - 1/\mathcal{P}, \mu + \theta + 2, \theta + \mu + 3, 1 - (z/a_1)^{\mathcal{P}})], \\ \quad \mathcal{P} \in (1, \infty), \end{cases}$$

$$\Lambda_1^*(a_1, z; \mathcal{P}) = \begin{cases} \frac{1}{z^{\mathcal{P}-1}(\mu+1)(\mu+2)} \mathcal{F}_1((1 - 1/\mathcal{P}), \mu + 1, \mu + 3, 1 - (a_1/z)^{\mathcal{P}}), \\ \quad \mathcal{P} \in (-\infty, 0) \cup (0, 1), \\ \frac{1}{a_1{}^{\mathcal{P}-1}(\mu+1)(\mu+2)} {}_2\mathcal{F}_1((1 - 1/\mathcal{P}), \mu + 1, \mu + 3, 1 - (z/a_1)^{\mathcal{P}}), \\ \quad \mathcal{P} \in (1, \infty), \end{cases} \tag{3.38}$$

$$\Lambda_2^*(a_1,z;\mathscr{P})=\begin{cases}\frac{1}{z^{\mathscr{P}-1}(\mu+2)}[{}_2\mathcal{F}_1((1-1/\mathscr{P}),\mu+2,a\mu+3,1-(a_1/z)^{\mathscr{P}})],\\ \quad \mathscr{P}\in(-\infty,0)\cup(0,1),\\ \frac{1}{a_1{}^{\mathscr{P}-1}(\mu+2)}[{}_2\mathcal{F}_1((1-1/\mathscr{P}),\mu+2,a\mu+3,1-(z/a_1)^{\mathscr{P}})],\\ \quad \mathscr{P}\in(1,\infty),\end{cases}$$

$$\Lambda_3^*(a_2,z;\mathscr{P})=\begin{cases}\frac{1}{z^{\mathscr{P}-1}(\mu+1)(\mu+2)}\mathcal{F}_1((1-1/\mathscr{P}),\mu+1,\mu+3,1-(a_2/z)^{\mathscr{P}}),\\ \quad \mathscr{P}\in(-\infty,0)\cup(0,1),\\ \frac{1}{a_2{}^{\mathscr{P}-1}(\mu+1)(\mu+2)}{}_2\mathcal{F}_1((1-1/\mathscr{P}),\mu+1,\mu+3,1-(z/a_2)^{\mathscr{P}}),\\ \quad \mathscr{P}\in(1,\infty),\end{cases}$$

and

$$\Lambda_4^*(a_2,z;\mathscr{P})=\begin{cases}\frac{1}{z^{\mathscr{P}-1}(\mu+2)}[{}_2\mathcal{F}_1((1-1/\mathscr{P}),\mu+2,a\mu+3,1-(a_2/z)^{\mathscr{P}})],\\ \quad \mathscr{P}\in(-\infty,0)\cup(0,1),\\ \frac{1}{a_2{}^{\mathscr{P}-1}(\mu+2)}[{}_2\mathcal{F}_1((1-1/\mathscr{P}),\mu+2,a\mu+3,1-(z/a_2)^{\mathscr{P}})],\\ \quad \mathscr{P}\in(1,\infty).\end{cases} \tag{3.39}$$

Proof By using Lemma 3.1 and the improved power-mean inequality

$$\begin{aligned}
&\bigg|\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu}\psi^{(\mu)}(a_1)+(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu}\psi^{(\mu)}(a_2)}{\mathscr{P}^{\mu}(a_2-a_1)}\\
&\quad-\frac{\Gamma_k(\mu+k)}{a_2-a_1}\big[\big({}_k^{\mathscr{P}}\mathcal{D}^{\mu}_{a_1{}^+}\psi\big)(z)+\big({}_k^{\mathscr{P}}\mathcal{D}^{\mu}_{a_2{}^-}\psi\big)(z)\big]\bigg|\\
&\quad\le\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}\int_0^1\zeta^{\mu}\big(\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\big)^{\frac{1-\mathscr{P}}{\mathscr{P}}}\big|\psi^{(\mu+1)}\big(\sqrt[\mathscr{P}]{\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}}\big)\big|\,d\zeta\\
&\quad\quad+\frac{(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}\\
&\quad\quad\times\int_0^1\zeta^{\mu}\big(\zeta a_2{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\big)^{\frac{1-\mathscr{P}}{\mathscr{P}}}\big|\psi^{(\mu+1)}\big(\sqrt[\mathscr{P}]{\zeta a_2{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}}\big)\big|\,d\zeta\\
&\quad\le\frac{(z^{\mathscr{P}}-a_1{}^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}\bigg[\bigg(\int_0^1\zeta^{\mu}(1-\zeta)\big(\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\big)^{(\frac{1-\mathscr{P}}{\mathscr{P}})}\,d\zeta\bigg)^{1-1/\lambda}\\
&\quad\quad\times\bigg(\int_0^1\zeta^{\mu}(1-\zeta)\big(\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\big)^{(\frac{1-\mathscr{P}}{\mathscr{P}})}\\
&\quad\quad\times\big|\psi^{(\mu+1)}\big(\sqrt[\mathscr{P}]{\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}}\big)\big|^{\vartheta}\,d\zeta\bigg)^{1/\vartheta}\\
&\quad\quad+\bigg(\int_0^1\zeta^{\mu+1}\big(\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\big)^{(\frac{1-\mathscr{P}}{\mathscr{P}})}\,d\zeta\bigg)^{1-1/\lambda}\\
&\quad\quad\times\bigg(\int_0^1\zeta^{\mu+1}\big(\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\big)^{(\frac{1-\mathscr{P}}{\mathscr{P}})}\big|\psi^{(\mu+1)}\big(\sqrt[\mathscr{P}]{\zeta a_1{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}}\big)\big|^{\vartheta}\,d\zeta\bigg)^{1/\vartheta}\bigg]\\
&\quad\quad+\frac{(a_2{}^{\mathscr{P}}-z^{\mathscr{P}})^{\mu+1}}{\mathscr{P}^{1+\mu}(a_2-a_1)}\bigg[\bigg(\int_0^1\zeta^{\mu}(1-\zeta)\big(\zeta a_2{}^{\mathscr{P}}+(1-\zeta)z^{\mathscr{P}}\big)^{(\frac{1-\mathscr{P}}{\mathscr{P}})}\,d\zeta\bigg)^{1-1/\lambda}
\end{aligned}$$

$$\begin{aligned}
&\times\left(\int_0^1 \zeta^{\mu}(1-\zeta)\big(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\right.\\
&\left.\times\left|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\right|^{\vartheta}d\zeta\right)^{1/\vartheta} \qquad (3.40)\\
&+\left(\int_0^1 \zeta^{\mu+1}\big(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})}d\zeta\right)^{1-1/\lambda}\\
&\left.\times\left(\int_0^1 \zeta^{\mu+1}\big(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})}\left|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\right|^{\vartheta}d\zeta\right)^{1/\vartheta}\right]\\
\leq{}&\frac{(z^{\mathcal{P}}-a_1{}^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\left[\big(\Lambda_1^*(a_1,z;\mathcal{P})\big)^{1-1/\lambda}\right.\\
&\times\left(\int_0^1 \zeta^{\mu}(1-\zeta)\big(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\right.\\
&\left.\times\left|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\right|^{\vartheta}d\zeta\right)^{1/\vartheta}\\
&+\big(\Lambda_2^*(a_1,z;\mathcal{P})\big)^{1-1/\lambda}\\
&\left.\times\left(\int_0^1 \zeta^{\mu+1}\big(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\left|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\right|^{\vartheta}d\zeta\right)^{1/\vartheta}\right]\\
&+\frac{(a_2{}^{\mathcal{P}}-z^{\mathcal{P}})^{\mu+1}}{\mathcal{P}^{1+\mu}(a_2-a_1)}\left[\big(\Lambda_3^*(a_2,z;\mathcal{P})\big)^{1-1/\lambda}\right.\\
&\times\left(\int_0^1 \zeta^{\mu}(1-\zeta)\big(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\right.\\
&\left.\times\left|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\right|^{\vartheta}d\zeta\right)^{1/\vartheta}\\
&+\big(\Lambda_4^*(a_2,z;\mathcal{P})\big)^{1-1/\lambda}\\
&\left.\times\left(\int_0^1 \zeta^{\mu+1}\big(\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{\lambda(\frac{1-\mathcal{P}}{\mathcal{P}})}\left|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_2{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\right|^{\vartheta}d\zeta\right)^{1/\vartheta}\right].
\end{aligned}$$

As $|\psi^{(\mu+1)}|^{\vartheta}$ is n-polynomial $\mathcal{P}$-convex and $|\psi^{(\mu+1)}(z)|\leq\mathcal{Q}$, $\forall z\in[a_1,a_2]$, we have

$$\begin{aligned}
&\int_0^1 \zeta^{\mu}(1-\zeta)\big(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\left|\psi^{(\mu+1)}\big(\sqrt[\mathcal{P}]{\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}}\big)\right|^{\vartheta}d\zeta\\
&\quad\leq\int_0^1 \zeta^{\mu}(1-\zeta)\big(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\\
&\qquad\times\left[\frac{1}{n}\sum_{\theta=1}^{n}\big[1-(1-\zeta)^{\theta}\big]\left|\psi^{(\mu+1)}(a_1)\right|^{\vartheta}+\frac{1}{n}\sum_{\theta=1}^{n}\big[1-\zeta^{\theta}\big]\left|\psi^{(\mu+1)}(z)\right|^{\vartheta}\right]d\zeta \qquad (3.41)\\
&\quad=\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\int_0^1\big(\zeta a_1{}^{\mathcal{P}}+(1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\big[2\zeta^{\mu}(1-\zeta)-\zeta^{\mu}(1-\zeta)^{\theta+1}-\zeta^{\mu+\theta}(1-\zeta)\big]d\zeta\\
&\quad=\frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\mathrm{T}_1(a_1z;\mathcal{P}),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{T}_1(a_1z;\mathcal{P}) &:= \int_0^1 \left(\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\right)^{(\frac{1-\mathcal{P}}{\mathcal{P}})} \\
&\quad \times \left[2\zeta^{\mu}(1-\zeta) - \zeta^{\mu}(1-\zeta)^{\theta+1} - \zeta^{\mu+\theta}(1-\zeta)\right] d\zeta \qquad (3.42) \\
&= \begin{cases}
\frac{1}{z^{(\mathcal{P}-1)}}\left[\frac{2}{(\mu+1)(\mu+2)}{}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+1,\mu+3,1-(a_1/z)^{\mathcal{P}})\right. \\
\quad - \mathbb{B}(\mu+1,\theta+2){}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+1,\theta+\mu+3,1-(a_1/z)^{\mathcal{P}})] \\
\quad - \mathbb{B}(\mu+\theta+1,2){}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+\theta+1,\theta+\mu+3,1-(a_1/z)^{\mathcal{P}})], \\
\quad \mathcal{P} \in (-\infty,0)\cup(0,1), \\
\frac{1}{a_1{}^{(\mathcal{P}-1)}}\left[\frac{2}{(\mu+1)(\mu+2)}\mathcal{F}_1(1-1/\mathcal{P},\mu+1,\mu+3,1-(z/a_1)^{\mathcal{P}})\right. \\
\quad - \mathbb{B}(\mu+1,\theta+2){}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+1,\theta+\mu+3,1-(z/a_1)^{\mathcal{P}})] \\
\quad - \mathbb{B}(\mu+\theta+1,2){}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+\theta+1,\theta+\mu+3,1-(z/a_1)^{\mathcal{P}})], \\
\quad \mathcal{P} \in (1,\infty).
\end{cases}
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^1 \zeta^{\mu+1}\left(\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\right)^{(\frac{1-\mathcal{P}}{\mathcal{P}})} \left|\psi^{(\mu+1)}\left(\sqrt[\mathcal{P}]{\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}}\right)\right|^{\vartheta} d\zeta \\
&\quad \le \int_0^1 \zeta^{\mu+1}\left(\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\right)^{(\frac{1-\mathcal{P}}{\mathcal{P}})} \\
&\qquad \times \left[\frac{1}{n}\sum_{\theta=1}^{n}\left[1-(1-\zeta)^{\theta}\right]\left|\psi^{(\mu+1)}(a_1)\right|^{\vartheta} + \frac{1}{n}\sum_{\theta=1}^{n}\left[1-\zeta^{\theta}\right]\left|\psi^{(\mu+1)}(z)\right|^{\vartheta}\right] d\zeta \qquad (3.43) \\
&\quad = \frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\int_0^1 \left(\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\right)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\left[2\zeta^{\mu+1} - \zeta^{\mu+1}(1-\zeta)^{\theta} - \zeta^{\mu+\theta+1}\right] d\zeta \\
&\quad = \frac{\mathcal{Q}^{\vartheta}}{n}\sum_{\theta=1}^{n}\mathrm{T}_2(a_1z;\mathcal{P}),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{T}_2(a_1z;\mathcal{P}) &:= \int_0^1 \left(\zeta a_1{}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\right)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\left[2\zeta^{\mu+1} - \zeta^{\mu+1}(1-\zeta)^{\theta} - \zeta^{\mu+\theta+1}\right] d\zeta \\
&= \begin{cases}
\frac{1}{z^{(\mathcal{P}-1)}}\left[\frac{2}{(\mu+2)}\mathcal{F}_1(1-1/\mathcal{P},\mu+2,\mu+3,1-(a_1/z)^{\mathcal{P}})\right. \\
\quad - \mathbb{B}(\mu+2,\theta+2){}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+2,\theta+\mu+3,1-(a_1/z)^{\mathcal{P}})] \\
\quad - \frac{1}{\mu+\theta+2}{}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+\theta+2,\theta+\mu+3,1-(a_1/z)^{\mathcal{P}})], \\
\quad \mathcal{P} \in (-\infty,0)\cup(0,1), \\
\frac{1}{a_1{}^{(\mathcal{P}-1)}}\left[\frac{2}{(\mu+2)}\mathcal{F}_1(1-1/\mathcal{P},\mu+2,\mu+3,1-(z/a_1)^{\mathcal{P}})\right. \\
\quad - \mathbb{B}(\mu+2,\theta+2){}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+2,\theta+\mu+3,1-(z/a_1)^{\mathcal{P}})] \\
\quad - \frac{1}{\mu+\theta+2}{}_2\mathcal{F}_1(1-1/\mathcal{P},\mu+\theta+2,\theta+\mu+3,1-(z/a_1)^{\mathcal{P}})], \\
\quad \mathcal{P} \in (1,\infty).
\end{cases} \qquad (3.44)
\end{aligned}$$

We obtain $\mathrm{T}_3(a_2, z; \mathcal{P})$ and $\mathrm{T}_4(a_2, z; \mathcal{P})$ by replacing a_1 into a_2 in the above results and using the facts

$$\Lambda_1^*(a_1, z; \mathcal{P}) := \int_0^1 \zeta^{\mu}(1-\zeta)\big(\zeta {a_1}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\, d\zeta$$

$$= \begin{cases} \frac{1}{z^{\mathcal{P}-1}(\mu+1)(\mu+2)} \mathcal{F}_1((1-1/\mathcal{P}), \mu+1, \mu+3, 1-(a_1/z)^{\mathcal{P}}), \\ \quad \mathcal{P} \in (-\infty, 0) \cup (0,1), \\ \frac{1}{{a_1}^{\mathcal{P}-1}(\mu+1)(\mu+2)} {}_2\mathcal{F}_1((1-1/\mathcal{P}), \mu+1, \mu+3, 1-(z/a_1)^{\mathcal{P}}), \\ \quad \mathcal{P} \in (1, \infty), \end{cases} \tag{3.45}$$

$$\Lambda_2^*(a_1, z; \mathcal{P}) := \int_0^1 \zeta^{\mu+1}\big(\zeta {a_1}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\, d\zeta$$

$$= \begin{cases} \frac{1}{z^{\mathcal{P}-1}(\mu+2)} [{}_2\mathcal{F}_1((1-1/\mathcal{P}), \mu+2, a\mu+3, 1-(a_1/z)^{\mathcal{P}})], \\ \quad \mathcal{P} \in (-\infty, 0) \cup (0,1), \\ \frac{1}{{a_1}^{\mathcal{P}-1}(\mu+2)} [{}_2\mathcal{F}_1((1-1/\mathcal{P}), \mu+2, a\mu+3, 1-(z/a_1)^{\mathcal{P}})], \\ \quad \mathcal{P} \in (1, \infty), \end{cases} \tag{3.46}$$

$$\Lambda_3^*(a_2, z; \mathcal{P}) := \int_0^1 \zeta^{\mu}(1-\zeta)\big(\zeta {a_2}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{(\frac{1-\mathcal{P}}{\mathcal{P}})}\, d\zeta$$

$$= \begin{cases} \frac{1}{z^{\mathcal{P}-1}(\mu+1)(\mu+2)} \mathcal{F}_1((1-1/\mathcal{P}), \mu+1, \mu+3, 1-(a_2/z)^{\mathcal{P}}), \\ \quad \mathcal{P} \in (-\infty, 0) \cup (0,1), \\ \frac{1}{{a_2}^{\mathcal{P}-1}(\mu+1)(\mu+2)} {}_2\mathcal{F}_1((1-1/\mathcal{P}), \mu+1, \mu+3, 1-(z/a_2)^{\mathcal{P}}), \\ \quad \mathcal{P} \in (1, \infty), \end{cases} \tag{3.47}$$

and

$$\Lambda_4^*(a_2, z; \mathcal{P}) := \int_0^1 \zeta^{\mu+1}\big(\zeta {a_2}^{\mathcal{P}} + (1-\zeta)z^{\mathcal{P}}\big)^{\frac{1-\mathcal{P}}{\mathcal{P}}}\, d\zeta$$

$$= \begin{cases} \frac{1}{z^{\mathcal{P}-1}(\mu+2)} [{}_2\mathcal{F}_1((1-1/\mathcal{P}), \mu+2, a\mu+3, 1-(a_2/z)^{\mathcal{P}})], \\ \quad \mathcal{P} \in (-\infty, 0) \cup (0,1), \\ \frac{1}{{a_2}^{\mathcal{P}-1}(\mu+2)} [{}_2\mathcal{F}_1((1-1/\mathcal{P}), \mu+2, a\mu+3, 1-(z/a_2)^{\mathcal{P}})], \\ \quad \mathcal{P} \in (1, \infty), \end{cases} \tag{3.48}$$

which completes the proof. □

4 Special bi-variate means

Let $\lambda_1, \lambda_2, w_1, w_2 > 0$. Then the arithmetic mean $\mathcal{A}(\lambda_1, \lambda_2)$, harmonic mean $\mathcal{H}(\lambda_1, \lambda_2)$ and weighted arithmetic mean $\mathcal{B}(\lambda_1, \lambda_2; w_1; w_2)$ are defined by $\mathcal{A}(\lambda_1, \lambda_2) = \frac{\lambda_1+\lambda_2}{2}$, $\mathcal{H}(\lambda_1, \lambda_2) = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}$ and $\mathcal{B}(\lambda_1, \lambda_2; w_1; w_2) = \frac{\lambda_1 w_1+\lambda_2 w_2}{w_1+w_2}$. The given propositions can be obtained by making some proper substitutions in Theorem 3.1.

Proposition 4.1 *The inequality*

$$\left|\frac{2}{3(\lambda_2-\lambda_1)}\left[\left(2\mathcal{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\lambda_1\right)^3-\left(2\mathcal{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\lambda_2\right)^3\right]\right.$$
$$\left.-\left[2\mathcal{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\mathcal{B}(\lambda_1,\lambda_2)\right]^2\right|$$

holds for all $\lambda_2 > \lambda_1 > 0$.

Proposition 4.2 *The inequality*

$$\left|\frac{2}{(m+1)(\lambda_2-\lambda_1)}\left[\left(2\mathscr{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\lambda_1\right)^{m+1}\right.\right.$$
$$\left.\left.-\left(2\mathscr{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\lambda_2\right)^{m+1}\right]-\left[2\mathscr{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\mathscr{B}(\lambda_1,\lambda_2)\right]^m\right|$$
$$\leq \frac{3m(\lambda_2-\lambda_1)}{4}\mathscr{B}\left(\left(\mathscr{B}\left(\lambda_1^{\frac{m-1}{2}},d^{\frac{m-2}{2^2}}\right),\mathscr{B}\left(\lambda_2^{m-2},\lambda_2^{m-1}\right)\right)\right.$$

holds for all $\lambda_2 > \lambda_1 > 0$ *and* $m \in \mathbb{N}$ *with* $m \geq 2$.

Proposition 4.3 *The inequality*

$$\left|\frac{2}{(m+1)(\lambda_2-\lambda_1)}\left[\left(2\mathscr{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\lambda_1\right)^{m+1}\right.\right.$$
$$\left.\left.-\left(2\mathscr{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\lambda_2\right)^{m+1}\right]-\left[2\mathscr{B}(\sqrt{\lambda_1},\sqrt{\lambda_2})-\mathscr{B}(\lambda_1,\lambda_2)\right]^m\right|$$
$$\leq \frac{3m(\lambda_2-\lambda_1)}{4}\mathscr{B}\left(\left(\mathscr{B}\left(\lambda_1^{\frac{m-1}{2}},\lambda_2^{\frac{m-1}{2}}\right),\mathscr{B}\left(\lambda_2^{m-2},\lambda_2^{m-1}\right)\right)\right.$$

holds for all $\lambda_2 > \lambda_1 > 0$ *and* $m \in \mathbb{N}$ *with* $m \geq 2$.

5 Concluding remarks

In this article, we establish novel Ostrowski-type inequalities for n-polynomial $\mathscr{P}$-convex functions. To the best of our knowledge, these results are new in the literature. Since convex functions have immense applications in many mathematical areas, we hope that our new developments can be applied to special functions, and in convex analysis, quantum analysis, post-quantum analysis, related optimization theory, mathematical inequalities and that they may stimulate further research in various areas of pure and applied sciences. In the end, we have given some applications.

Acknowledgements
The authors would like to express their sincere thanks to the support of National Natural Science Foundation of China.

Funding
The research was supported by the National Natural Science Foundation of China (Grant Nos. 11971142, 61673169).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 29 October 2020 Accepted: 15 March 2021 Published online: 13 July 2021

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