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# Weak and strong type estimates for multilinear Calderón–Zygmund operators on differential forms

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## Abstract

In this paper, we define the multilinear Calderón–Zygmund operators on differential forms and prove the end-point weak type boundedness of the operators. Based on nonhomogeneous  $A$ -harmonic tensor, the Poincaré-type inequalities for multilinear Calderón–Zygmund operators on differential forms are obtained.

**Keywords:** Multilinear Calderón–Zygmund operators; End-point weak type boundedness;  $A$ -harmonic equations; Differential forms

## 1 Introduction

The multilinear Calderón–Zygmund theory was originally introduced by Coifman and Meyer [1–3] in their study of certain singular integral operators, such as Calderón commutators, paraproducts, and pseudodifferential operators. Afterwards, the multilinear Calderón–Zygmund theory has been further developed by many scholars in the last few decades. For example, Grafakos and Torres studied systematically on the multilinear Calderón–Zygmund operators in [4]. They proved an end-point weak type estimate and obtained the strong type  $L^{p_1} \times \cdots \times L^{p_m} \rightarrow L^p$  boundedness results for multilinear Calderón–Zygmund operators by the classical interpolation method. In [5] and [6], the maximal operator associated with multilinear Calderón–Zygmund singular integrals was introduced and used to obtain the weighted norm estimates for multilinear singular integrals. More recently, Stockdale and Wick [7] provided an alternative proof of the weak-type  $(1, \dots, 1; 1/m)$  estimates for  $m$ -multilinear Calderón–Zygmund operators on  $\mathbb{R}^n$  first proved by Grafakos and Torres. Subsequent results on end-point estimates of the bilinear Calderón–Zygmund operators can be found in [8]. Motivated by the research and considering that differential forms as the generalizations of functions are widely used in physics systems, differential geometry, and PDEs, we aim to establish the boundedness of multilinear Calderón–Zygmund operators on differential forms. In this paper, the definition of multilinear Calderón–Zygmund operators on differential forms is set forth. Moreover, by combining the Calderón–Zygmund decomposition with some skillful techniques, we establish the end-point weak type boundedness of multilinear Calderón–Zygmund operators on differential forms which includes the result of multilinear Calderón–Zygmund

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operators on functions in [4] as a special case. Unfortunately, it is difficult to apply the complex variable theory to differential forms. So the  $T1$  theorem and strong type boundedness of multilinear Calderón–Zygmund operators on differential forms are still some open questions. But with the help of decomposition theorem of differential forms, we derive the Poincaré-type inequalities for multilinear Calderón–Zygmund operators on  $A$ -harmonic tensors which are the generalized solutions to  $A$ -harmonic equations on differential forms. Based on the Poincaré-type inequalities, we can make a further study of the multilinear Calderón–Zygmund operators in Orlicz spaces and establish the  $L^\varphi$  norm inequalities. More results on the Poincaré-type inequalities and singular integral operators on differential forms can be found in [9–13].

This work is organized as follows. To state our results, we first recall some necessary notations and lemmas in Sect. 2. Then, in Sect. 3, we define the multilinear Calderón–Zygmund operators on differential forms and prove the end-point weak type boundedness of multilinear Calderón–Zygmund operators on differential forms in Theorem 1. Using the weak type inequality, we derive the Poincaré-type inequality for multilinear Calderón–Zygmund operator on a local domain in Theorem 2 in Sect. 4. Finally, the result is extended to obtain the Poincaré-type inequality for the multilinear Calderón–Zygmund operator on a bounded convex domain in Theorem 3.

### 2 Preliminaries

Throughout this paper, let  $\Omega \subset \mathbb{R}^n$  be a bounded domain,  $n \geq 2$ ,  $B$  and  $\sigma B$  be the balls with the same center and  $\text{diam}(\sigma B) = \sigma \text{diam}(B)$ . We use  $|E|$  to denote the Lebesgue measure of a set  $E \subset \mathbb{R}^n$ . Let  $\Lambda^l(\mathbb{R}^n) = \Lambda^l$ ,  $l = 1, 2, \dots, n$ , be the set of all  $l$ -forms  $u(x) = \sum_I u_I(x) dx_I = \sum u_{i_1 \dots i_l}(x) dx_{i_1} \wedge \dots \wedge dx_{i_l}$  with summation over all ordered  $l$ -tuples  $I = (i_1, i_2, \dots, i_l)$ ,  $1 \leq i_1 < \dots < i_l \leq n$ .  $D'(\Omega, \Lambda^l)$  is the space of all differential  $l$ -forms on  $\Omega$ , namely, the coefficient of the  $l$ -forms is differential on  $\Omega$ . The operator  $\star : \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^{n-l}(\mathbb{R}^n)$  is the Hodge–Star operator as usual, and the linear operator  $d : D'(\Omega, \Lambda^l) \rightarrow D'(\Omega, \Lambda^{l+1})$ ,  $0 \leq l \leq n - 1$ , is called the exterior differential operator. The Hodge codifferential operator  $d^\star : D'(\Omega, \Lambda^{l+1}) \rightarrow D'(\Omega, \Lambda^l)$ , the formal adjoint of  $d$ , is defined by  $d^\star = (-1)^{n-l+1} \star d \star$ , see [14] for more introduction. We shall denote by  $L^p(\Omega, \Lambda^l)$  the space of differential  $l$ -forms with the coefficients in  $L^p(\Omega, \mathbb{R}^n)$  and with the norm  $\|u\|_{p,\Omega} = (\int_\Omega (\sum_I |u_I(x)|^2)^{\frac{p}{2}} dx)^{\frac{1}{p}}$ . The homotopy operator  $T : C^\infty(\Omega, \Lambda^l) \rightarrow C^\infty(\Omega, \Lambda^{l-1})$  is a very important operator in the theory of differential forms, given by

$$Tu = \int_\Omega \psi(y) K_y u dy,$$

where  $\psi \in C_0^\infty(\Omega)$  is normalized by  $\int_\Omega \psi(y) dy = 1$ , and  $K_y$  is a linear operator defined by

$$(K_y u)(x; \xi_1, \dots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y; \xi_1, \dots, \xi_{l-1}) dt.$$

See [15] for more of the function  $\psi$  and the operator  $K_y$ . About the homotopy operator  $T$ , we have the following decomposition:

$$u = d(Tu) + T(du)$$

for any differential form  $u \in L^p(\Omega, \Lambda^l)$ ,  $1 \leq p < \infty$ . We also call it the decomposition theorem for differential form which will be used repeatedly in the proof of this paper. A closed form  $u_\Omega$  is defined by  $u_\Omega = d(Tu)$ ,  $l = 1, \dots, n$ , and when  $u$  is a differential 0-form,  $u_\Omega = |\Omega|^{-1} \int_\Omega u(y) dy$ .

The following lemma is the  $L^p$  estimate for the homotopy operator  $T$  from [15].

**Lemma 1** *Let  $u \in L^p_{loc}(\Omega, \Lambda^l)$ ,  $l = 1, 2, \dots, n$ ,  $1 < p < \infty$ , be a differential form in  $\Omega$ , and  $T$  be the homotopy operator defined on differential forms. Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|Tu\|_{p,\Omega} \leq C \text{diam}(\Omega) \|u\|_{p,\Omega}.$$

The following nonlinear partial differential equation for differential forms

$$d^*A(x, du) = B(x, du) \tag{1}$$

is called nonhomogeneous  $A$ -harmonic equation, where  $A : \Omega \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^l(\mathbb{R}^n)$  and  $B : \Omega \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^{l-1}(\mathbb{R}^n)$  satisfy the conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \quad A(x, \xi) \cdot \xi \geq |\xi|^p, \quad \text{and} \quad |B(x, \xi)| \leq b|\xi|^{p-1}$$

for  $x \in \Omega$  a.e. and all  $\xi \in \Lambda^l(\mathbb{R}^n)$ . Here,  $p > 1$  is a constant related to equation (1), and  $a, b > 0$ .

In general, we call the differential form satisfying nonhomogeneous  $A$ -harmonic equation the nonhomogeneous  $A$ -harmonic tensor. In the proof of the strong type inequality for the multilinear Calderón–Zygmund operator on differential forms, we let the differential form be the nonhomogeneous  $A$ -harmonic tensor to get the desired result. See [16–20] for a list of recent results on the  $A$ -harmonic equations and related topics. We also need the following weak inverse Hölder inequality for nonhomogeneous  $A$ -harmonic tensor, see [21] for more introduction.

**Lemma 2** *Let  $u \in \Omega$  satisfy equation (1),  $\sigma > 1$ ,  $0 < s, t < \infty$ . Then there exists a constant  $C$ , independent of  $u$ , such that*

$$\|u\|_{s,B} \leq C|B|^{(t-s)/st} \|u\|_{t,\sigma B}$$

for  $\sigma B \subset \Omega$ .

### 3 End-point estimate

In the classical theory of singular integrals, it is important to prove the end-point weak type inequality which is the core link to get the strong boundedness of singular integral operators. So the focus of this section is to establish the end-point weak type inequality of multilinear Calderón–Zygmund operators on differential forms. Before giving the definition of multilinear Calderón–Zygmund operators on differential forms, we first define the kernel function in multilinear Calderón–Zygmund operators, see [4] for more introduction about the kernel function.

**Definition 1** Let  $K(x, y_1, \dots, y_m)$  be a locally integrable function which is defined on

$$(\mathbb{R}^n)^{m+1} \setminus \{ \tau \in (\mathbb{R}^n)^{m+1}, x = y_1 = \dots = y_m \}$$

and satisfy the following conditions:

(1) For some  $A > 0$  and all points in the domain of definition, the function  $K(x, y_1, \dots, y_m)$  satisfies

$$|K(x, y_1, \dots, y_m)| \leq \frac{A}{(|x - y_1| + \dots + |x - y_m|)^{mn}};$$

(2) For  $\varepsilon > 0$ , we have

$$\begin{aligned} &|K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)| \\ &\leq \frac{A|x - x'|^\varepsilon}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}}, \end{aligned} \tag{2}$$

where  $|x - x'| \leq \frac{1}{2} \max_{1 \leq j \leq m} |x - y_j|$ . For other  $y_j$ , we also have

$$\begin{aligned} &|K(x, y_1, \dots, y_j, \dots, y_m) - K(x, y_1, \dots, y'_j, \dots, y_m)| \\ &\leq \frac{A|y_j - y'_j|^\varepsilon}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}}, \end{aligned}$$

where  $|y_j - y'_j| \leq \frac{1}{2} \max_{1 \leq j \leq m} |x - y_j|$ . For convenience, we call conditions (1), (2) for kernel function the  $m - CZK(A, \varepsilon)$  conditions.

Next, we give the definition of multilinear Calderón–Zygmund operators on differential forms.

**Definition 2** The operator  $\mathcal{L} : \Lambda^l(\mathbb{R}^n) \times \Lambda^l(\mathbb{R}^n) \times \dots \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^l(\mathbb{R}^n)$  is called multilinear operator on differential forms if

$$\begin{aligned} &\mathcal{L}(u^{(1)}, \dots, u^{(m)})(x; \xi) \\ &= \left( \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) u^{(1)}(x; \xi) \cdots u^{(m)}(x; \xi) dy_1 \cdots dy_m \right), \end{aligned}$$

where  $x \notin \bigcap_{i=1}^m \text{supp } u_i^{(i)}, \xi = (\xi_1, \xi_2, \dots, \xi_l)$ .

Now, we establish the end-point weak type inequality for multilinear Calderón–Zygmund operators on differential forms.

**Theorem 1** Let  $\mathcal{L}$  be a multilinear Calderón–Zygmund operator, the kernel function  $K$  satisfies the  $m - CZK(A, \varepsilon)$  conditions, and  $u^{(1)}, \dots, u^{(m)} \in D'(\Omega, \Lambda^l)$ . Assume that, for  $1 \leq q_1, q_2, \dots, q_m \leq \infty$  and  $0 < q < \infty$  with

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m},$$

if the operator  $\mathcal{L}$  is weak- $(q_1, \dots, q_m, q)$ , that is,

$$|\{x : |\mathcal{L}(u^{(1)}, \dots, u^{(m)})| > \lambda\}| \leq \left(\frac{B}{\lambda} \|u^{(1)}\|_{q_1} \cdots \|u^{(m)}\|_{q_m}\right)^q$$

for all  $\lambda > 0$ , then  $\mathcal{L}$  is also a weak- $(1, 1, \dots, 1, \frac{1}{m})$  operator. Especially, we have

$$[\mathcal{L}]_{L^1 \times \dots \times L^1 \rightarrow L^{1/m}} \leq C(A + [\mathcal{L}]_{L^{q_1} \times \dots \times L^{q_m} \rightarrow L^q}),$$

where  $[\cdot]$  means the weak norm of an operator and  $C > 0$  is a constant that depends only on the parameters  $n, m$ .

*Proof* Without loss of generality, suppose

$$\|u^{(1)}\|_1 = \|u^{(2)}\|_1 = \dots = \|u^{(m)}\|_1 = 1.$$

In order to prove the conclusion, we need to show that there exists a constant  $C$  independent of  $u^{(1)}, \dots, u^{(m)}$  such that

$$\begin{aligned} (\mathcal{L}(u^{(1)}, \dots, u^{(m)}))_*(\lambda) &= |\{x : |\mathcal{L}(u^{(1)}, \dots, u^{(m)})| > \lambda\}| \\ &\leq \left\{ \frac{C}{\lambda} (A + B) \right\}^{\frac{1}{m}}. \end{aligned} \tag{3}$$

According to the algorithms of differential forms, for  $i_1 < i_2 < \dots < i_k$  and  $j_1 < j_2 < \dots < j_k$ , we obtain

$$dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}(e_{j_1}, e_{j_2}, \dots, e_{j_k}) = \begin{cases} 1, & \text{if } i_1 = j_1, i_2 = j_2, \dots, i_k = j_k, \\ 0, & \text{others,} \end{cases}$$

where  $e_1, e_2, \dots, e_n$  are orthogonal basis for the tangent space of  $\mathbb{R}^n$ . So, for a differential form

$$u(x) = \sum_I u_I(x) dx_I$$

with  $I = (i_1, i_2, \dots, i_l), 1 \leq i_1 < i_2 < \dots < i_l \leq n$ , we have

$$u_I = u(x)(e_I),$$

where  $e_I = (e_{i_1}, e_{i_2}, \dots, e_{i_l})$ . It is also a differential form for the image of the multilinear Calderón–Zygmund operator on a differential form. So, there exist  $a_I, I = (i_1, i_2, \dots, i_l), 1 \leq i_1 < i_2 < \dots < i_l \leq n$  satisfying

$$\mathcal{L}(u^{(1)}, \dots, u^{(m)}) = \sum_I a_I dx_I.$$

If we write  $dy = dy_1 \cdots dy_m$ , then it follows that

$$|\mathcal{L}(u^{(1)}, \dots, u^{(m)})(x)| = \left| \sum_I a_I dx_I \right|$$

$$\begin{aligned}
 &= \left( \sum_I |a_I|^2 \right)^{1/2} = \left( \sum_I |\mathcal{L}(u^{(1)}, \dots, u^{(m)})(x; e_I)|^2 \right)^{1/2} \\
 &= \left( \sum_I \left| \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) u^{(1)}(x; e_I) \cdots u^{(m)}(x; e_I) \, dy \right|^2 \right)^{1/2} \\
 &= \left( \sum_I \left| \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) u_I^{(1)} \cdots u_I^{(m)} \, dy \right|^2 \right)^{1/2}.
 \end{aligned}$$

Applying the Calderón–Zygmund decomposition to each function  $u_I^{(i)}$  at height  $\alpha = (\lambda\rho)^{1/m}$ , we obtain a sequence of pairwise disjoint cubes  $\{Q_{i,k_i}\}_{k_i=1}^\infty$  and a decomposition

$$u_I^{(i)} = g_I^{(i)} + b_I^{(i)} = g_I^{(i)} + \sum_{k_i} b_I^{(i,k_i)}$$

such that, for all  $i = 1, \dots, m$ ,

- (a)  $|g_I^{(i)}| \leq C_1\alpha$ ,
- (b)  $\text{supp}(b_I^{(i,k_i)}) \subset Q_{i,k_i}$ ,  $\int_{Q_{i,k_i}} b_I^{(i,k_i)}(x) \, dx = 0$ , and

$$\int_{Q_{i,k_i}} |b_I^{(i,k_i)}(x)| \, dx \leq C_1\alpha|Q_{i,k_i}|,$$

- (c)  $\sum_{k_i} |Q_{i,k_i}| \leq \frac{C_1}{\alpha} \int_{\mathbb{R}^n} |u_I^{(i)}(x)| \, dx$ .

Applying the Calderón–Zygmund decomposition above to the operator  $\mathcal{L}$ , we have

$$\begin{aligned}
 &(\mathcal{L}(u^{(1)}, \dots, u^{(m)}))_*(\lambda) \\
 &= |\{x : |\mathcal{L}(u^{(1)}, \dots, u^{(m)})| > \lambda\}| \\
 &= \left| \left\{ x : \left( \sum_I (\mathcal{L}(u_I^{(1)}, \dots, u_I^{(m)}))^2 \right)^{\frac{1}{2}} > \lambda \right\} \right| \\
 &\leq \sum_I \left| \left\{ x : \mathcal{L}(u_I^{(1)}, \dots, u_I^{(m)}) > \frac{\lambda}{C'_n} \right\} \right| \\
 &\leq \sum_I \left( \left| \left\{ x : \mathcal{L}(g_I^{(1)}, \dots, g_I^{(m)}) > \frac{\lambda}{2^m C'_n} \right\} \right| \right. \\
 &\quad \left. + \sum_I \left( \sum_{\exists h_I^{(i)} = b_I^{(i)}} \left| \left\{ x : \mathcal{L}(h_I^{(1)}, \dots, h_I^{(m)}) > \frac{\lambda}{2^m C'_n} \right\} \right| \right) \right), \tag{4}
 \end{aligned}$$

where  $h_I^{(i)} \in \{b_I^{(i)}, g_I^{(i)}\}$ . Then, by properties (a) and (c), one has

$$\begin{aligned}
 \|g_I^{(i)}\|_{q_i}^{q_i} &= \int_{(\cup Q_{i,k_i})^c} |g_I^{(i)}(x)|^{q_i} \, dx + \int_{\cup Q_{i,k_i}} |g_I^{(i)}(x)|^{q_i} \, dx \\
 &= \int_{(\cup Q_{i,k_i})^c} |g_I^{(i)}(x)|^{q_i-1} |u_I^{(i)}(x)| \, dx + \int_{\cup Q_{i,k_i}} |g_I^{(i)}(x)|^{q_i} \, dx \\
 &\leq C_1\alpha^{q_i-1} \int |u_I^{(i)}(x)| \, dx + C_1\alpha^{q_i} \left| \left\{ \cup Q_{i,k_i} \right\} \right|
 \end{aligned}$$

$$\begin{aligned} &\leq C_1 \alpha^{q_i-1} \|u_I^{(i)}\|_1 \\ &\leq C_1 \alpha^{q_i-1} \|u^{(i)}\|_1. \end{aligned} \tag{5}$$

Using (5) and the weak  $-(q_1, q_2, \dots, q_m, q)$  boundedness to the first item of the right-hand side of inequality (4), we get

$$\begin{aligned} &\left| \left\{ x : \mathcal{L}(g_I^{(1)}, \dots, g_I^{(m)}) > \frac{\lambda}{2^m C_n^l} \right\} \right| \\ &\leq \left( \frac{B 2^m C_n^l}{\lambda} \|g_I^{(1)}\|_{q_1} \cdots \|g_I^{(m)}\|_{q_m} \right)^q \\ &\leq \frac{C_1 B^q}{\lambda^q} \prod_{i=1}^m \alpha^{\frac{q(q_i-1)}{q_i}} \|u^{(i)}\|_1^{q/q_i} \\ &= \frac{C_1 B^q}{\lambda^q} (\lambda \rho)^{\frac{q}{m}(m-\frac{1}{q})} \\ &= C_1 B^q \lambda^{-\frac{1}{m}} \rho^{q-\frac{1}{m}}. \end{aligned} \tag{6}$$

Now we begin to estimate the measure of the following set which appeared in (4):

$$|E_s| = \left| \left\{ x : \mathcal{L}(h_I^{(1)}, \dots, h_I^{(m)}) > \frac{\lambda}{2^m C_n^l} \right\} \right|,$$

where  $h_I^{(r)} = b_I^{(r)}$ ,  $1 \leq r \leq t$ , and  $h_I^{(i_{t+j})} = g_I^{(i_{t+j})}$ ,  $1 \leq j \leq m-t$ ,  $1 \leq t \leq m$ . For the sake of simplicity, there is no harm in the setting  $b_I^{(ir)}$  appearing with superscript  $1, \dots, t$ . We set that  $\Omega = \bigcup_{r=1}^t \Omega_r = \bigcup_{r=1}^t \bigcup_{k_r} \tilde{Q}_{r,k_r}$  with  $\tilde{Q}_{r,k_r}$  is a concentric and double diameter cube to  $Q_{r,k_r}$ . Then we obtain

$$\begin{aligned} |E_s| &\leq |\Omega| + \left| \left\{ x \in \mathbb{R}^n \setminus \Omega : \mathcal{L}(b_I^{(1)}, \dots, b_I^{(t)}, g_I^{(t+1)}, \dots, g_I^{(m)}) > \frac{\lambda}{2^m C_n^l} \right\} \right| \\ &= |\Omega| + |\tilde{E}_s|. \end{aligned} \tag{7}$$

Using properties (a), (b) and the Chebyshev inequality, we have

$$\begin{aligned} &|\tilde{E}_s| \\ &\leq \frac{2^m C_n^l}{\lambda} \int_{\mathbb{R}^n \setminus \Omega} |\mathcal{L}(b_I^{(1)}, \dots, b_I^{(t)}, g_I^{(t+1)}, \dots, g_I^{(m)})(x)| \, dx \\ &\leq \frac{2^m C_n^l}{\lambda} \int_{\mathbb{R}^n \setminus \Omega} \left| \sum_{k_1, \dots, k_t} \int_{(\mathbb{R}^n)^m} (K(x, y_1, \dots, y_m) - K(x, c_{1,k_1}, \dots, y_m)) \right. \\ &\quad \left. \times b_I^{(1,k_1)} \cdots b_I^{(t,k_t)} g_I^{(t+1)} \cdots g_I^{(m)} \right| \, dy \, dx \\ &\leq \frac{2^m C_n^l}{\lambda} (\lambda \rho)^{\frac{m-t}{m}} \int_{\mathbb{R}^n \setminus \Omega} \sum_{k_1, \dots, k_t} \int_{(\mathbb{R}^n)^m} |K(x, y_1, \dots, y_m) - K(x, c_{1,k_1}, \dots, y_m)| \\ &\quad \times |b_I^{(1,k_1)} \cdots b_I^{(t,k_t)}| \, dy \, dx \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{2^m C_n^l}{\lambda} (\lambda \rho)^{\frac{m-t}{m}} \int_{(\mathbb{R}^n)^m} \sum_{k_1, \dots, k_t} \left( \int_{\mathbb{R}^n \setminus \Omega} |K(x, y_1, \dots, y_m) - K(x, c_{1,k_1}, \dots, y_m)| \, dx \right) \\
 &\quad \times |b_I^{(1,k_1)} \dots b_I^{(t,k_t)}| \, dy \\
 &\leq \frac{2^m C_n^l}{\lambda} (\lambda \rho)^{\frac{m-t}{m}} \int_{(\mathbb{R}^n)^m} \sum_{k_1, \dots, k_t} \left( \int_{\mathbb{R}^n \setminus \Omega} \left| \frac{A |y_1 - c_{1,k_1}|^\varepsilon}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}} \right| \, dx \right) \\
 &\quad \times |b_I^{(1,k_1)} \dots b_I^{(t,k_t)}| \, dy, \tag{8}
 \end{aligned}$$

where  $c_{1,k_1}$  is the center of cube  $\tilde{Q}_{1,k_1}$ . We represent the difference set  $(2^{\tau_r+1}Q_{r,k_r}) \setminus (2^{\tau_r}Q_{r,k_r})$  by  $H_{r,k_r}^{\tau_r}$  for  $\tau_r = 1, 2, \dots$ , and  $r = 1, \dots, t$ . Then the difference set of  $\mathbb{R}^n$  and  $\Omega$  satisfies

$$\mathbb{R}^n \setminus \Omega \subset \bigcup_{\tau_1, \dots, \tau_t=1}^{\infty} \bigcap_{r=1}^t (H_{r,k_r}^{\tau_r}).$$

Note that, for the arbitrary  $x \in \bigcap_{r=1}^t (H_{r,k_r}^{\tau_r})$  and  $y_r \in Q_{r,k_r}$ , we have

$$|y_1 - c_{1,k_1}| \leq 2^{-1} \tilde{l}(Q_{1,k_1})$$

and

$$2^{\tau_r} \tilde{l}(Q_{r,k_r}) \leq |x - y_r| \leq 2^{\tau_r+1} \tilde{l}(Q_{r,k_r}), \tag{9}$$

where  $\tilde{l}(Q_{r,k_r})$  denotes the diameter of the cube  $\tilde{Q}_{r,k_r}$ . Hence, it follows that

$$\begin{aligned}
 \frac{|y_1 - c_{1,k_1}|^\varepsilon}{(|x - y_1| + \dots + |x - y_m|)^\varepsilon} &\leq \frac{|y_1 - c_{1,k_1}|^\varepsilon}{(|x - y_1|)^\varepsilon} \\
 &\leq \frac{|2^{-1} \tilde{l}(Q_{1,k_1})|^\varepsilon}{|2^{\tau_1} \tilde{l}(Q_{1,k_1})|^\varepsilon} \\
 &= \frac{1}{2^{\tau_1 \varepsilon + \varepsilon}}. \tag{10}
 \end{aligned}$$

Bringing formula (10) into (8) and combining with inequality (9), we get

$$\begin{aligned}
 &\int_{(\mathbb{R}^n)^m} \left( \int_{\mathbb{R}^n \setminus \Omega} \left| \frac{A |y_1 - c_{1,k_1}|^\varepsilon}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}} \right| \, dx \right) |b_I^{(1,k_1)} \dots b_I^{(t,k_t)}| \, dy \\
 &\leq \int_{(\mathbb{R}^n)^m} \int_{\mathbb{R}^n \setminus \Omega} \frac{A \frac{1}{2^{\tau_1 \varepsilon + \varepsilon}} \, dx}{(\sum_{j=1}^m |x - y_j|)^{mn}} |b_I^{(1,k_1)} \dots b_I^{(t,k_t)}| \, dy \\
 &\leq \int_{(\mathbb{R}^n)^t} \int_{\mathbb{R}^n \setminus \Omega} \frac{A \frac{1}{2^{\tau_1 \varepsilon + \varepsilon}} \, dx}{(\sum_{j=1}^t |x - y_j|)^{tn}} |b_I^{(1,k_1)} \dots b_I^{(t,k_t)}| \, dy_1 \dots dy_t \\
 &\leq \int_{Q_{1,k_1}} \dots \int_{Q_{t,k_t}} \sum_{\tau_1, \dots, \tau_t=1}^{\infty} \int_{\bigcap_{r=1}^t H_{r,k_r}^{\tau_r}} \frac{A \frac{1}{2^{\tau_1 \varepsilon + \varepsilon}} \, dx}{(\sum_{j=1}^t |x - y_j|)^{tn}} |b_I^{(1,k_1)} \dots b_I^{(t,k_t)}| \, dy_1 \dots dy_t \\
 &\leq \int_{Q_{1,k_1}} \dots \int_{Q_{t,k_t}} \sum_{\tau_1, \dots, \tau_t=1}^{\infty} \int_{\bigcap_{r=1}^t H_{r,k_r}^{\tau_r}} \frac{A \frac{1}{2^{\tau_1 \varepsilon + \varepsilon}} \, dx}{(\sum_{j=1}^t |2^{\tau_r} \tilde{l}(Q_{r,k_r})|)^{tn}} |b_I^{(1,k_1)} \dots b_I^{(t,k_t)}| \, dy_1 \dots dy_t
 \end{aligned}$$



$$\begin{aligned}
 &\leq C_2 \alpha^t \sum_{\tau_1, \dots, \tau_t=1}^{\infty} \int_{\cap_{r=1}^t H_{r, k_r}^{\tau_r}} \frac{A \frac{1}{2^{\tau_1 \varepsilon + \varepsilon}} dx}{(\sum_{j=1}^t |2^{\tau_r} \tilde{l}(Q_{r, k_r})|)^{tn}} \prod_{r=1}^t |Q_{r, k_r}| \\
 &\leq C_2 \alpha^t \sum_{\tau_1, \dots, \tau_t=1}^{\infty} \frac{A}{2^{\tau_1 \varepsilon + \varepsilon}} \int_{Q_{1, k_1}} \cdots \int_{Q_{t, k_t}} \int_{\cap_{r=1}^t H_{r, k_r}^{\tau_r}} \frac{dx}{(\sum_{j=1}^t |2^{\tau_r} \tilde{l}(Q_{r, k_r})|)^{tn}} dy_1 \cdots dy_t \\
 &\leq C_3 \alpha^t \sum_{\tau_1, \dots, \tau_t=1}^{\infty} \frac{A}{2^{\tau_1 \varepsilon + \varepsilon}} \int_{Q_{1, k_1}} \cdots \int_{Q_{t, k_t}} \int_{\cap_{r=1}^t H_{r, k_r}^{\tau_r}} \frac{dx}{(\sum_{j=1}^t |x - y_j|)^{tn}} dy_1 \cdots dy_t. \tag{11}
 \end{aligned}$$

Owing to the fact that the sets  $H_{r, k_r}^{\tau_r}$  and  $\tau_r = 1, \dots, \infty$  are disjoint, we know

$$\begin{aligned}
 &\sum_{\tau_2, \dots, \tau_t=1}^{\infty} \int_{\cap_{r=1}^t H_{r, k_r}^{\tau_r}} \frac{dx}{(\sum_{j=1}^t |x - y_j|)^{tn}} \\
 &\leq \sum_{\tau_2, \dots, \tau_{t-1}=1}^{\infty} \int_{\cap_{r=1}^{t-1} H_{r, k_r}^{\tau_r}} \frac{dx}{(\sum_{j=1}^t |x - y_j|)^{tn}} \\
 &\leq \int_{H_{1, k_1}^{\tau_1}} \frac{dx}{(\sum_{j=1}^t |x - y_j|)^{tn}}. \tag{12}
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 &\sum_{k_2, \dots, k_t} \int_{Q_{1, k_1}} \cdots \int_{Q_{t, k_t}} \frac{dy_1 \cdots dy_t}{(\sum_{j=1}^t |x - y_j|)^{tn}} \\
 &\leq \sum_{k_2, \dots, k_{t-1}} \int_{Q_{1, k_1}} \cdots \int_{Q_{t-1, k_{t-1}}} \int_{\mathbb{R}^n} \frac{dy_1 \cdots dy_t}{(\sum_{j=1}^t |x - y_j|)^{tn}} \\
 &\leq \sum_{k_2, \dots, k_{t-1}} \int_{Q_{1, k_1}} \cdots \int_{Q_{t-1, k_{t-1}}} \frac{dy_1 \cdots dy_{t-1}}{(\sum_{j=1}^{t-1} |x - y_j|)^{(t-1)n}} \\
 &\leq \int_{Q_{1, k_1}} \frac{dy_1}{(|x - y_1|)^n}. \tag{13}
 \end{aligned}$$

Combining (8), (11), (12), and (13), we obtain

$$\begin{aligned}
 |\tilde{E}_s| &\leq C_3 \frac{2^m C_n^l}{\lambda} \alpha^t (\lambda \rho)^{\frac{m-t}{m}} \sum_{\tau_1=1}^{\infty} \sum_{k_1} \int_{Q_{1, k_1}} \int_{H_{1, k_1}^{\tau_1}} \frac{\frac{A}{2^{\tau_1 \varepsilon + \varepsilon}} dy_1 dx}{(|x - y_1|)^n} \\
 &\leq C_3 \frac{2^m C_n^l}{\lambda} \alpha^t (\lambda \rho)^{\frac{m-t}{m}} \sum_{\tau_1=1}^{\infty} \frac{A}{2^{\tau_1 \varepsilon + \varepsilon}} \sum_{k_1} \int_{Q_{1, k_1}} \int_{H_{1, k_1}^{\tau_1}} \frac{dx}{|x - y_1|^n} dy_1 \\
 &\leq C_3 \frac{2^m C_n^l}{\lambda} \alpha^t (\lambda \rho)^{\frac{m-t}{m}} \sum_{\tau_1=1}^{\infty} \frac{A}{2^{\tau_1 \varepsilon + \varepsilon}} \sum_{k_1} \int_{Q_{1, k_1}} \int_{H_{1, k_1}^{\tau_1}} \frac{dx}{|2^{\tau_r} \tilde{l}(Q_{r, k_r})|^n} dy_1 \\
 &\leq C_3 \frac{2^m C_n^l}{\lambda} \alpha^t (\lambda \rho)^{\frac{m-t}{m}} \sum_{\tau_1=1}^{\infty} \frac{A}{2^{\tau_1 \varepsilon + \varepsilon}} \sum_{k_1} \int_{Q_{1, k_1}} \int_{2^{\tau_1+1} Q_{r, k_r}} \frac{dx}{|2^{\tau_r} \tilde{l}(Q_{r, k_r})|^n} dy_1 \\
 &\leq C_3 \frac{2^m C_n^l}{\lambda} \alpha^t (\lambda \rho)^{\frac{m-t}{m}} \sum_{\tau_1=1}^{\infty} \frac{A}{2^{\tau_1 \varepsilon + \varepsilon}} \sum_{k_1} |Q_{1, k_1}|
 \end{aligned}$$

$$\begin{aligned}
 &\leq C_3 \frac{2^m C_n^l}{\lambda} \alpha^t (\lambda \rho)^{\frac{m-t}{m}} \sum_{\tau_1=1}^{\infty} \frac{A}{2^{\tau_1 \varepsilon + \varepsilon}} \frac{c}{\alpha} \int_{\mathbb{R}^n} |u_I^{(1)}(y_1)| \, dy_1 \\
 &\leq C_3 A (\lambda \rho)^{\frac{m-1}{m}} \frac{1}{\lambda} \int_{\mathbb{R}^n} |u_I^{(1)}(y_1)| \, dy_1 \\
 &= C_3 A \rho \frac{1}{\alpha} \int_{\mathbb{R}^n} |u_I^{(1)}(y_1)| \, dy_1 \\
 &\leq C_3 A \rho \frac{1}{\alpha} \int_{\mathbb{R}^n} |u^{(1)}(y_1)| \, dy_1.
 \end{aligned} \tag{14}$$

Choose  $\rho = \frac{1}{A+B}$ , then inequality (6) shows that

$$\begin{aligned}
 &\left| \left\{ x : \mathcal{L}(g_I^{(1)}, \dots, g_I^{(m)}) > \frac{\lambda}{2^m C_n^l} \right\} \right| \\
 &\leq C_1 B^q \lambda^{-\frac{1}{m}} \rho^{q-\frac{1}{m}} \|u^{(i)}\|_1^{q/q_i} \\
 &= C_1 B^q \lambda^{-\frac{1}{m}} \left( \frac{1}{A+B} \right)^{q-\frac{1}{m}} \|u^{(i)}\|_1^{q/q_i} \\
 &= C_1 \left( \frac{B}{A+B} \right)^q \lambda^{-\frac{1}{m}} \left( \frac{1}{A+B} \right)^{-\frac{1}{m}} \|u^{(i)}\|_1^{q/q_i} \\
 &= C_1 \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}} \|u^{(i)}\|_1^{q/q_i}.
 \end{aligned} \tag{15}$$

On the other hand, property (c) implies that

$$\begin{aligned}
 |\Omega| &= \left| \bigcup_r \Omega_r \right| \\
 &= \left| \bigcup_r \bigcup_{k_i} \tilde{B}_{i,k_i} \right| \\
 &\leq \frac{C_4}{\alpha} \int_{\mathbb{R}^n} |u_I^{(s)}(x)| \, dx \\
 &= C_4 \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}} \int_{\mathbb{R}^n} |u_I^{(s)}(x)| \, dx,
 \end{aligned} \tag{16}$$

where

$$\int_{\mathbb{R}^n} |u_I^{(s)}(x)| \, dx = \max \left\{ \int_{\mathbb{R}^n} |u_I^{(1)}(x)| \, dx, \dots, \int_{\mathbb{R}^n} |u_I^{(m)}(x)| \, dx \right\}.$$

And it should be noted that

$$\begin{aligned}
 |\tilde{E}_s| &\leq C_3 A \rho \frac{1}{\alpha} \int_{\mathbb{R}^n} |u^{(1)}(y_1)| \, dy_1 \\
 &= C_3 \left( \frac{A}{A+B} \right) \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}} \int_{\mathbb{R}^n} |u^{(1)}(y_1)| \, dy_1 \\
 &\leq C_5 \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}} \int_{\mathbb{R}^n} |u^{(1)}(y_1)| \, dy_1.
 \end{aligned} \tag{17}$$

Finally, combining (15), (16), and (17), we have

$$\begin{aligned}
 & (\mathcal{L}(u^{(1)}, \dots, u^{(m)}))_*(\lambda) \\
 &= |\{x : |\mathcal{L}(u^{(1)}, \dots, u^{(m)})| > \lambda\}| \\
 &\leq \sum_I \left( \left| \left\{ x : \mathcal{L}(g_I^{(1)}, \dots, g_I^{(m)}) > \frac{\lambda}{2^m C_n^I} \right\} \right| \right) \\
 &\quad + \sum_I \left( \sum_{\exists h_I^{(i)} = b_I^{(i)}} \left| \left\{ x : \mathcal{L}(h_I^{(1)}, \dots, h_I^{(m)}) > \frac{\lambda}{2^m C_n^I} \right\} \right| \right) \\
 &\leq \sum_I \left( C_1 \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}} \|u^{(i)}\|_1^{q/q_i} \right) + \sum_I \left( \sum_{\exists h_I^{(i)} = b_I^{(i)}} (|\Omega| + |\tilde{E}_s|) \right) \\
 &\leq \sum_I \left( C_1 \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}} \|u^{(i)}\|_1^{q/q_i} \right) \\
 &\quad + \sum_I \left( \sum_{\exists h_I^{(i)} = b_I^{(i)}} \left( C_4 \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}} \int |u_I^{(s)}(x)| \, dx \right) \right) \\
 &\quad + \sum_I \left( \sum_{\exists h_I^{(i)} = b_I^{(i)}} \left( C_5 \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}} \int |u^{(1)}(y_1)| \, dy_1 \right) \right) \\
 &\leq C_6 \left( \frac{A+B}{\lambda} \right)^{\frac{1}{m}}. \tag{18}
 \end{aligned}$$

This ends the proof of Theorem 1. □

#### 4 Poincaré-type inequalities

In this section, we establish the Poincaré-type inequalities for multilinear Calderón–Zygmund operators on differential forms in the local and global domain. So we first define the operator on differential  $l$ -form in a local domain as follows:

$$\begin{aligned}
 & \tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x; \xi) \\
 &= \left( \int_{(B)^m} K(x, y_1, \dots, y_m) u^{(1)}(x; \xi) \cdots u^{(m)}(x; \xi) \, dy_1 \cdots dy_m \right) \tag{19}
 \end{aligned}$$

for  $x \notin \bigcap_{i=1}^m \text{supp } u_I^{(i)}$ ,  $B \subset \Omega$ , and the kernel function  $K(y_1, \dots, y_m)$  satisfies

$$\begin{aligned}
 & |K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)| \\
 &\leq \frac{A|x - x'|}{(|x - y_1| + \cdots + |x - y_m|)^{m-1}} \tag{20}
 \end{aligned}$$

for  $|x - x'| \leq \frac{1}{2} \max_{1 \leq j \leq m} |x - y_j|$ .

Next, we give the Poincaré-type inequality for the multilinear Calderón–Zygmund operator  $\tilde{\mathcal{L}}$  on a differential form in the local domain.

**Theorem 2** Let  $\tilde{\mathcal{L}}$  be a multilinear Calderón–Zygmund operator defined by (19),  $B \subset \Omega$ , the kernel function  $K$  satisfies condition (2) in Definition 1, and  $u^{(1)}, \dots, u^{(m)} \in L^p_{loc}(\Omega, \Lambda^l)$  satisfy  $A$ -harmonic equation (1), then for  $\theta = 1 + \frac{2}{n} - \frac{m-1}{p}$ , we have

$$\begin{aligned} & \|\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x) - (\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x))_B\|_{p,B} \\ & \leq C|B|^\theta \|u^{(1)}\|_{p,\sigma B} \cdots \|u^{(m-1)}\|_{p,\sigma B} \|u^{(m)}\|_{p,B}. \end{aligned}$$

*Proof* Applying the decomposition theorem for a differential form and Lemma 1, we get

$$\begin{aligned} & \|\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x) - (\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x))_B\|_{p,B} \\ & = \|T(d\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x)) + d(T\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x)) - (\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x))_B\|_{p,B} \\ & = \|T(d(\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x)))\|_{p,B} \\ & \leq C|B| \operatorname{diam}(B) \|d(\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x))\|_{p,B}. \end{aligned} \tag{21}$$

For convenience, we write

$$\tilde{\mathcal{L}}_I(u^{(1)}, \dots, u^{(m)})(x) = \int_{B^m} K(x, y_1, \dots, y_m) u^{(1)}_I u^{(2)}_I \cdots u^{(m)}_I dy_1 \cdots dy_m.$$

Then,

$$\begin{aligned} & d(\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x)) \\ & = d\left(\sum_I ((\tilde{\mathcal{L}}_I(u^{(1)}, \dots, u^{(m)})(x)) dx_I)\right) \\ & = \sum_{k=1}^n \sum_I \frac{\partial \tilde{\mathcal{L}}_I(u^{(1)}, \dots, u^{(m)})(x)}{\partial x_k} dx_k \wedge dx_I \\ & = \sum_{k=1}^n \sum_I \lim_{\eta \rightarrow 0} \frac{\tilde{\mathcal{L}}_I(u^{(1)}, \dots, u^{(m)})(x + \eta e_k) - \tilde{\mathcal{L}}_I(u^{(1)}, \dots, u^{(m)})(x)}{\eta} dx_k \wedge dx_I \\ & = \sum_{k=1}^n \sum_I \lim_{\eta \rightarrow 0} \int_{B^m} \left( \frac{K(x + \eta e_k, y_1, \dots, y_m) - K(x, y_1, \dots, y_m)}{\eta} \right. \\ & \quad \left. \times u^{(1)}_I \cdots u^{(m)}_I dy_1 \cdots dy_m \right) dx_k \wedge dx_I. \end{aligned} \tag{22}$$

In terms of the conditions of kernel function  $K$ , we know

$$\begin{aligned} & K(x + \eta e_k, y_1, \dots, y_m) - K(x, y_1, \dots, y_m) \\ & \leq \frac{A|x + \eta e_k - x|}{(|x - y_1| + \cdots + |x - y_m|)^{m-1}} \\ & = \frac{A\eta}{(|x - y_1| + \cdots + |x - y_m|)^{m-1}}. \end{aligned} \tag{23}$$

It follows from (23) and (22) that

$$\begin{aligned}
 & d(\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x)) \\
 & \leq \sum_{k=1}^n \sum_I \lim_{\eta \rightarrow 0} \frac{\int_{B^m} (\frac{A\eta}{(|x-y_1|+\dots+|x-y_m|)^{m-1}}) u_I^{(1)} \dots u_I^{(m)} dy_1 \dots dy_m}{\eta} dx_k \wedge dx_I \\
 & = \sum_{k=1}^n \sum_I \lim_{\eta \rightarrow 0} \int_{B^m} \frac{A u_I^{(1)} \dots u_I^{(m)} dy_1 \dots dy_m}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} dx_k \wedge dx_I \\
 & = \sum_{k=1}^n \sum_I \int_{B^m} \frac{A u_I^{(1)} \dots u_I^{(m)} dy_1 \dots dy_m}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} dx_k \wedge dx_I.
 \end{aligned}$$

Further, we get

$$\begin{aligned}
 & \|d(\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x))\|_{p,B} \\
 & \leq \left\| \sum_{k=1}^n \sum_I \int_{B^m} \frac{A u_I^{(1)} \dots u_I^{(m)} dy_1 \dots dy_m}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} dx_k \wedge dx_I \right\|_{p,B} \\
 & \leq \left\| \left( \sum_{k=1}^n \sum_I \left| \int_{B^m} \frac{A u_I^{(1)} \dots u_I^{(m)} dy_1 \dots dy_m}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} \right|^2 \right)^{\frac{1}{2}} \right\|_{p,B} \\
 & \leq \left\| C_1 \sum_{k=1}^n \sum_I \left| \int_{B^m} \frac{A u_I^{(1)} \dots u_I^{(m)} dy_1 \dots dy_m}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} \right| \right\|_{p,B} \\
 & \leq \left\| C_1 A \sum_{k=1}^n \sum_I \left| \int_{B^{m-1}} \left( \int_B \frac{u_I^{(1)} dy_1}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} \right) \right. \right. \\
 & \quad \left. \left. \times u_I^{(2)} \dots u_I^{(m)} dy_2 \dots dy_m \right| \right\|_{p,B} \\
 & \leq C_1 A \sum_{k=1}^n \sum_I \left\| \int_{B^{m-1}} \left( \int_B \frac{u_I^{(1)} dy_1}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} \right) \right. \\
 & \quad \left. \times u_I^{(2)} \dots u_I^{(m)} dy_2 \dots dy_m \right\|_{p,B}.
 \end{aligned}$$

Apply Hölder’s inequality to the innermost layer integral of the right-hand side of the inequality above with Hölder index satisfying  $1 = \frac{n(m-1)-\varepsilon}{n(m-1)} + \frac{\varepsilon}{n(m-1)}$ ,  $0 < \varepsilon < 1$ . For the convenience of writing, we write  $\frac{1}{\zeta} = \frac{n(m-1)-\varepsilon}{n(m-1)}$ ,  $\frac{1}{\eta} = \frac{\varepsilon}{n(m-1)}$ , then it follows that

$$\begin{aligned}
 & \int_B \frac{u_I^{(1)} dy_1}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} \\
 & \leq \int_B \left| \frac{u_I^{(1)}}{(|x-y_1|+\dots+|x-y_m|)^{m-1}} \right| dy_1 \\
 & \leq \left( \int_B \left| \frac{1}{(|x-y_1|+\dots+|x-y_m|)^{\zeta m n - \zeta}} \right| dy_1 \right)^{\frac{1}{\zeta}} \|u_I^{(1)}\|_{\eta, B}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \left( \int_{\mathbb{R}^n} \frac{1}{(|x - y_1| + \dots + |x - y_m|)^{\zeta m n - \zeta}} \, dy_1 \right)^{\frac{1}{\zeta}} \|u_I^{(1)}\|_{\eta, B} \\
 &= \left( \omega_n \int_0^\infty \frac{r^{n-1}}{(r + |x - y_2| + \dots + |x - y_m|)^{\zeta m n - \zeta}} \, dr \right)^{\frac{1}{\zeta}} \|u_I^{(1)}\|_{\eta, B} \\
 &\leq \left( \omega_n \int_0^\infty \frac{dr}{(r + |x - y_2| + \dots + |x - y_m|)^{\zeta m n - \zeta - n + 1}} \right)^{\frac{1}{\zeta}} \|u_I^{(1)}\|_{\eta, B} \\
 &\leq \frac{C_2}{(|x - y_2| + \dots + |x - y_m|)^{m n - n + \frac{\varepsilon}{m-1} - 1}} \|u_I^{(1)}\|_{\eta, B}.
 \end{aligned}$$

Similarly, applying Hölder’s inequalities  $(m - 2)$  times with the indexes satisfying  $1 = \frac{1}{\zeta} + \frac{1}{\eta}$ , we have

$$\begin{aligned}
 &\int_{B^m} \frac{u_I^{(1)} \dots u_I^{(m)} \, dy_1 \dots dy_m}{(|x - y_1| + \dots + |x - y_m|)^{m n - 1}} \\
 &\leq \int_B \frac{u_I^{(m)} \, dy_m}{|x - y_m|^{n + \varepsilon - 1}} \|u_I^{(1)}\|_{\eta, B} \dots \|u_I^{(m-1)}\|_{\eta, B}.
 \end{aligned}$$

Let  $\chi_B$  be a characteristic function as

$$\chi_B = \begin{cases} 1, & x \in B, \\ 0, & x \notin B, \end{cases}$$

then we get

$$\begin{aligned}
 &\left\| \int_B \frac{u_I^{(m)} \, dy_m}{|x - y_m|^{n + \varepsilon - 1}} \right\|_{p, B} \\
 &= \left\| \int_{\mathbb{R}^n} u_I^{(m)} \chi_B(y_m) \frac{\chi_B(x)}{(|x - y_m|^{n + \varepsilon - 1})^{\frac{1}{p}}} \frac{\chi_B(y_m)}{(|x - y_m|^{n + \varepsilon - 1})^{\frac{1}{q}}} \, dy_m \right\|_{p, B}. \tag{24}
 \end{aligned}$$

Applying Hölder’s inequalities again with the indexes satisfying  $1 = \frac{1}{p} + \frac{1}{q}$ , we obtain

$$\begin{aligned}
 &\int_{\mathbb{R}^n} u_I^{(m)} \chi_B(y_m) \frac{\chi_B(x)}{(|x - y_m|^{n + \varepsilon - 1})^{\frac{1}{p}}} \frac{\chi_B(y_m)}{(|x - y_m|^{n + \varepsilon - 1})^{\frac{1}{q}}} \, dy_m \\
 &\leq \left( \int_{\mathbb{R}^n} \left| u_I^{(m)} \chi_B(y_m) \frac{\chi_B(x)}{(|x - y_m|^{n + \varepsilon - 1})^{\frac{1}{p}}} \right|^p \, dy_m \right)^{\frac{1}{p}} \\
 &\quad \times \left( \int_{\mathbb{R}^n} \left| \frac{\chi_B(y_m)}{(|x - y_m|^{n + \varepsilon - 1})^{\frac{1}{q}}} \right|^q \, dy_m \right)^{\frac{1}{q}}. \tag{25}
 \end{aligned}$$

The integral in the second bracket above can be simplified by the basic inequality as follows:

$$\begin{aligned}
 &\int_{\mathbb{R}^n} \left| \frac{\chi_B(y_m)}{(|x - y_m|^{n + \varepsilon - 1})^{\frac{1}{q}}} \right|^q \, dy_m \\
 &= \int_{\mathbb{R}^n} \frac{\chi_B^q(z + x)}{|z|^{n + \varepsilon - 1}} \, dz
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{B'} \frac{1}{|z|^{n+\varepsilon-1}} dz \\
 &\leq \omega_n (r_1 + \text{diam}(B))^{1-\varepsilon} - r_1^{1-\varepsilon} \\
 &\leq \omega_n ((r_1 + \text{diam}(B) - r_1)^{1-\varepsilon}) \\
 &= \omega_n (\text{diam}(B))^{1-\varepsilon}.
 \end{aligned} \tag{26}$$

Here,  $B' = B(y'_0)$  and  $B = B(y_0)$  with  $y'_0 = y_0 - x$ ,  $r_1$  is the distance from 0 to  $B'$ . And then, by Fubini’s theorem, we have

$$\begin{aligned}
 &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u_I^{(m)} \chi_B(y_m)|^p \frac{(\chi_B(x))^p}{|x - y_m|^{n+\varepsilon-1}} dy_m dx \\
 &= \int_{\mathbb{R}^n} |u_I^{(m)} \chi_B(y_m)|^p \int_{\mathbb{R}^n} \frac{(\chi_B(x))^p}{|x - y_m|^{n+\varepsilon-1}} dx dy_m \\
 &\leq \omega_n (\text{diam}(B))^{1-\varepsilon} \|u_I^{(m)}\|_{p,B}^p.
 \end{aligned} \tag{27}$$

Combining (24), (25), (26), and (27), we have

$$\left\| \int_B \frac{u_I^{(m)} dy_m}{|x - y_m|^{n+\varepsilon-1}} \right\|_{p,B} \leq C (\text{diam}(B))^{1-\varepsilon} \|u_I^{(m)}\|_{p,B}. \tag{28}$$

Applying Lemma 2 to  $\|u_I^{(i)}\|_{\eta,B}$ ,  $1 \leq i \leq m - 1$ , we get

$$\begin{aligned}
 \|u_I^{(i)}\|_{\eta,B} &\leq |B|^{\frac{p-\eta}{pn}} \|u_I^{(i)}\|_{p,\sigma B} \\
 &= |B|^{\frac{pe-n(m-1)}{pn(m-1)}} \|u_I^{(i)}\|_{p,\sigma B}
 \end{aligned} \tag{29}$$

for  $\sigma > 1$  with  $\sigma B \subset \Omega$ . Combining (21), (28), (29), we obtain

$$\begin{aligned}
 &\|\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x) - (\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x))_B\|_{p,B} \\
 &\leq C |B|^{1+\frac{1}{n}+\frac{1}{n}-\frac{\varepsilon}{n}+\frac{pe-n(m-1)}{pn(m-1)}(m-1)} \|u_I^{(1)}\|_{p,\sigma B} \cdots \|u_I^{(m-1)}\|_{p,\sigma B} \|u_I^{(m)}\|_{p,B} \\
 &= C |B|^{1+\frac{2}{n}-\frac{m-1}{p}} \|u_I^{(1)}\|_{p,\sigma B} \cdots \|u_I^{(m-1)}\|_{p,\sigma B} \|u_I^{(m)}\|_{p,B} \\
 &\leq C |B|^{1+\frac{2}{n}-\frac{m-1}{p}} \|u^{(1)}\|_{p,\sigma B} \cdots \|u^{(m-1)}\|_{p,\sigma B} \|u^{(m)}\|_{p,B}.
 \end{aligned} \tag{30}$$

□

Finally, we give the Poincaré-type inequality for the multilinear Calderón–Zygmund operator  $\tilde{\mathcal{L}}$  in a bounded convex domain. We need the following covering lemma.

**Lemma 3** ([21]) *There exists a cover  $\mathcal{V} = \{B_i\}$  for any bounded subset  $\Omega$  in an  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  satisfying*

$$\bigcup_i B_i = \Omega, \quad \sum_{B_i \in \mathcal{V}} \chi_{\sqrt{\frac{5}{4}} B_i} \leq N \chi_\Omega$$

for a constant  $N > 1$ . And if  $B_i \cap B_j \neq \emptyset$ , then there exists a cube  $Q$  included in  $B_i \cap B_j$  with  $B_i \cup B_j \subset NR$ , and the cube  $Q$  does not have to be an element in the set family  $\mathcal{V}$ .

**Theorem 3** Let  $\tilde{\mathcal{L}}$  be a multilinear Calderón–Zygmund operator defined by (19),  $\Omega_1 \subset \subset \Omega$ , kernel function  $K$  satisfies condition (2) in Definition 1,  $u^{(1)}, \dots, u^{(m)} \in L^p_{loc}(\Omega, \Lambda^l)$  satisfy  $A$ -harmonic equation (1) and  $2p + (p + 1 - m)n > 0$ , then

$$\begin{aligned} & \left\| \tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x) - (\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x))_{\Omega} \right\|_{p, \Omega_1} \\ & \leq C \|u^{(1)}\|_{p, \Omega} \cdots \|u^{(m-1)}\|_{p, \Omega} \|u^{(m)}\|_{p, \Omega}. \end{aligned}$$

*Proof* By Lemma 3 and Theorem 2, we have

$$\begin{aligned} & \left\| T(d\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x)) \right\|_{p, \Omega_1} \\ & \leq \sum_{B \in \mathcal{V}} \left\| T(d\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x)) \right\|_{p, B} \\ & \leq \sum_{B \in \mathcal{V}} (C_1 |B|^{1 + \frac{2}{n} - \frac{m-1}{p}} \|u^{(1)}\|_{p, \sigma B} \cdots \|u^{(m-1)}\|_{p, \sigma B} \|u^{(m)}\|_{p, B}) \\ & \leq C_2 N \|u^{(1)}\|_{p, \Omega} \cdots \|u^{(m-1)}\|_{p, \Omega} \|u^{(m)}\|_{p, \Omega}, \end{aligned}$$

where  $N$  is the constant in Lemma 3. Next, from the decomposition theorem of homotopy operator, we get

$$\begin{aligned} & \left\| \tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x) - (\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x))_{\Omega} \right\|_{p, \Omega_1} \\ & \leq \left\| T(d\tilde{\mathcal{L}}(u^{(1)}, \dots, u^{(m)})(x)) \right\|_{p, \Omega_1} \\ & \leq C_2 N \|u^{(1)}\|_{p, \Omega} \cdots \|u^{(m-1)}\|_{p, \Omega} \|u^{(m)}\|_{p, \Omega}, \end{aligned}$$

which completes the proof of Theorem 3. □

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**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors put their efforts together into the research and writing of this manuscript. XL carried out the proofs of all research results in this manuscript and wrote its draft. YX and JN proposed the study, participated in its design, and revised its final version. All authors read and approved the final manuscript.

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