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Positive periodic solutions for multiparameter nonlinear differential systems with delays

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Abstract

We establish several criteria for the existence of positive periodic solutions of the multi-parameter differential systems

$$\begin{cases} u'(t) + a_1(t)g_1(u(t))u(t) = \lambda b_1(t)f(u(t-\tau_1(t)), v(t-\zeta_1(t))), \\ v'(t) + a_2(t)g_2(v(t))v(t) = \mu b_2(t)g(u(t-\tau_2(t)), v(t-\zeta_2(t))), \end{cases}$$

where the functions $g_1, g_2 : [0,\infty) \to [0,\infty)$ are assumed to be unbounded. The analysis in the paper relies on the classical fixed point index theory. Our main findings improve and complement some existing results in the literature.

MSC: 34B15

Keywords: Positive periodic solutions; Existence; Multiparameter systems; Fixed point

1 Introduction

Let $\omega > 0$ be a constant. In this article we shall seek some criterion to guarantee that the multiparameter system

$$\begin{cases} u'(t) = a_1(t)g_1(u(t))u(t) - \lambda b_1(t)f(u(t-\tau_1(t)), v(t-\zeta_1(t))), \\ v'(t) = a_2(t)g_2(v(t))v(t) - \mu b_2(t)g(u(t-\tau_2(t)), v(t-\zeta_2(t))) \end{cases} \quad (1.1)$$

admits a positive ω-periodic solution, where the functions $a_i, b_i, \tau_i, \zeta_i \in C(\mathbb{R},\mathbb{R})$ are ω-periodic, and $g_i \in C([0,\infty),[0,\infty))$ are unbounded, $i = 1,2$. In addition, we assume that the nonlinear terms $f, g \in C([0,\infty)\times[0,\infty),[0,\infty))$ and λ, μ are positive parameters.

Here a positive periodic solution of (1.1) means a solution $(u,v) \in E := X^2$ of (1.1) satisfying $u > 0$, $v > 0$ on $[0,\omega]$, where

$$X = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t)\}$$

is a Banach space, and the norm of $x \in X$ is

$$\|x\| = \max_{t\in[0,\omega]} \left|x(t)\right|.$$

Moreover, for $(x,y) \in E$, we denote $\|(x,y)\| = \|x\| + \|y\|$, and write $(x,y) \geq (0,0)$ if $(x,y) \in E$ fulfills $x(t) \geq 0$, $y(t) \geq 0$, $t \in [0,\omega]$.

Obviously, the first equation of (1.1) reduces in some special circumstances to

$$u'(t) = a(t)g\bigl(u(t)\bigr)u(t) - \lambda b(t)f\bigl(u\bigl(t-\tau(t)\bigr)\bigr), \tag{1.2}$$

and when $\lambda = 0$, $g(u) \equiv 1$, Eq. (1.2) becomes $u'(t) = a(t)u(t)$, which is famous in Malthusian population dynamics. In recent decades, (1.2) has also been extensively applied to describe various physiological processes emerging in practical applications, for instance, the production of blood cells, respiration, cardiac arrhythmias, etc. One may refer to [1–6] and references therein. Nevertheless, the research work in the above mentioned papers is mainly dependent on the condition that $g(u)$ is positive and bounded, that is, there are constants $L > l > 0$ such that $0 < l \leq g(u) \leq L$, $u \in [0,\infty)$. Jin and Wang [7] have recently studied the spectral problem

$$u'(t) = a(t)e^{u(t)}u(t) - \lambda b(t)f\bigl(u\bigl(t-\tau(t)\bigr)\bigr),$$

and they obtained some existence results on positive periodic solutions by means of the fixed point theory. It is worth noting the function e^u is unbounded on $[0,\infty)$. Since then, Eq. (1.2) has been extensively investigated under the more general case that $g(u)$ is unbounded on $[0,\infty)$, by applying the lower and upper solutions method, fixed point theory, and so on. See, for example, [7–10].

Besides, researchers have focused on the differential systems associated to (1.2), namely,

$$u_i'(t) = a_i(t)g_i\bigl(u_i(t)\bigr)u_i(t) - \lambda b_i(t)f_i\bigl(u_1(t),u_2(t),\ldots,u_n(t)\bigr), \quad i = 1,2,\ldots,n. \tag{1.3}$$

One can see [11–14] for some related results. However, in [11–13], the authors have only dealt with the special case $g_i(u_i) \equiv 1$, $i = 1,2,\ldots,n$. Indeed in that case, the Green's function corresponding to $u_i'(t) = a_i(t)u_i(t)$ is simple, and some suitable cones could be easily constructed. Furthermore, system (1.3) investigated in above papers includes only one positive parameter λ. Hence, it will be interesting to study the multiparameter systems (1.1) with g_i $(i = 1,2)$ being unbounded. On the other hand, what is worth mentioning is that Zhang et al. [14] considered system (1.1) for the special case $g_i \equiv 1$, $i = 1,2$, where nonlinearities $f(u,v)$ and $g(u,v)$ were assumed to be nondecreasing, and only the case $f(0,0) > 0$, $g(0,0) > 0$ was treated. Therefore, we want to know whether or not (1.1) has a positive periodic solution under more relaxed assumption $f(0,0) = 0$, $g(0,0) = 0$. In view of above reasons, we shall concentrate on the existence of positive periodic solutions for system (1.1) in the current paper, to further improve and generalize tho results in the literature. For this purpose, we assume

(C1) $a_i, b_i, \tau_i, \zeta_i \in C(\mathbb{R},[0,\infty))$ are ω-periodic with $\int_0^\omega a_i(t)\,dt > 0$, $\int_0^\omega b_i(t)\,dt > 0$, $i = 1,2$.
(C2) There is $l_i > 0$ such that $0 < l_i \leq g_i(s) < \infty$, $s \in [0,\infty)$.
(C3) $f, g \in C([0,\infty) \times [0,\infty), [0,\infty))$ with $f(u,v) > 0, g(u,v) > 0$ for $(u,v) \neq (0,0)$.

Remark 1.1 For other research work on periodic solutions of functional differential equations and systems, we refer the readers to [15–17] and references therein.

The remainder of the paper is arranged as follows. In Sect. 2, we introduce some preliminaries needed in our proof. Section 3 is devoted to stating and proving our main findings. Meanwhile, some related results and remarks will be given.

2 Preliminaries

Recall that $E = X^2$ is the Banach space defined as in Sect. 1. We first give the following lemma.

Lemma 2.1 *Assume* (C1)–(C3). *If* $(u,v) \in E$ *is a solution of* (1.1), *then*

$$u(t) = \lambda \int_t^{t+\omega} G_1(t,s)b_1(s)f\big(u\big(s-\tau_1(s)\big), v\big(s-\zeta_1(s)\big)\big)\,ds,$$
$$v(t) = \mu \int_t^{t+\omega} G_2(t,s)b_2(s)g\big(u\big(s-\tau_2(s)\big), v\big(s-\zeta_2(s)\big)\big)\,ds,$$

where

$$G_1(t,s) = \frac{e^{-\int_t^s a_1(\theta)g_1(u(\theta))\,d\theta}}{1-e^{-\int_0^\omega a_1(\theta)g_1(u(\theta))\,d\theta}}, \qquad G_2(t,s) = \frac{e^{-\int_t^s a_2(\theta)g_2(v(\theta))\,d\theta}}{1-e^{-\int_0^\omega a_2(\theta)g_2(v(\theta))\,d\theta}}, \quad t \le s \le t+\omega.$$

Proof Multiplying the both sides of the first equation of (1.1) with $e^{-\int_0^t a_1(s)g_1(u(s))\,ds}$, we can obtain

$$\big(u(t)e^{-\int_0^t a_1(s)g_1(u(s))\,ds}\big)' = -\lambda b_1(t)f\big(u\big(t-\tau_1(t)\big), v\big(t-\zeta_1(t)\big)\big)\cdot e^{-\int_0^t a_1(s)g_1(u(s))\,ds}.$$

Integrating above equation from t to $t+\omega$ and by elementary calculation, we can easily get

$$u(t) = \lambda \int_t^{t+\omega} G_1(t,s)b_1(s)f\big(u\big(s-\tau_1(s)\big), v\big(s-\zeta_1(s)\big)\big)\,ds.$$

Similar evaluation shows

$$v(t) = \mu \int_t^{t+\omega} G_2(t,s)b_2(s)g\big(u\big(s-\tau_2(s)\big), v\big(s-\zeta_2(s)\big)\big)\,ds. \qquad \square$$

Let $q > 0$ be a fixed constant. Then we can establish a series of lemmas required in the subsequent discussion.

Lemma 2.2 *Assume* (C1)–(C3). *Let* $\sigma_i = e^{-\int_0^\omega a_i(\theta)\,d\theta}$, $i = 1,2$. *Then for any* $(u,v) \in E$ *satisfying* $(u,v) \ge (0,0)$ *and* $\|(u,v)\| \le q$,

$$0 < \frac{\sigma_i^{g_i^*(q)}}{1-\sigma_i^{g_i^*(q)}} \le G_i(t,s) \le \frac{1}{1-\sigma_i^{g_{i*}(q)}}, \quad i = 1,2, \tag{2.1}$$

where

$$g_i^*(q) = \max_{0\le s\le q} g_i(s), \qquad g_{i*}(q) = \min_{0\le s\le q} g_i(s), \quad i = 1,2.$$

Proof Clearly, for $(u,v) \in E$ with $(u,v) \geq (0,0)$ and $\|(u,v)\| \leq q$, we have $0 \leq u \leq \|u\| \leq q$. Thus,

$$g_{1*}(q) \leq g_1(u) \leq g_1^*(q),$$

and then simple estimation shows (2.1) holds for $i = 1$. The case $i = 2$ is similar. □

Defining for $i = 1,2$,

$$m_i(q) = \frac{\sigma_i^{g_i^*(q)}}{1-\sigma_i^{g_i^*(q)}}, \qquad M_i(q) = \frac{1}{1-\sigma_i^{g_{i*}(q)}}, \qquad \eta_i(q) = \frac{m_i(q)}{M_i(q)}.$$

Then it is not hard to verify $\eta_i(q) \in (0,1)$, and accordingly,

$$\eta(q) := \min\{\eta_1(q), \eta_2(q)\} \in (0,1).$$

Set

$$P = \{(u,v) \in E : u(t) \geq 0, v(t) \geq 0, t \in [0,\omega]\},$$
$$K_q = \{(u,v) \in P : u(t) + v(t) \geq \eta(q)\|(u,v)\|, t \in [0,\omega]\},$$

and for $r > 0$,

$$\Omega_r = \{(u,v) \in K_q : \|(u,v)\| < r\}, \partial\Omega_r = \{(u,v) \in K_q : \|(u,v)\| = r\}.$$

Then P and K_q are cones in E.

Lemma 2.3 *Assume* (C1)–(C3). *Let* $0 < r \leq q$. *Then for any* $(u,v) \in \bar{\Omega}_r$,

$$\frac{\sigma_i^{g_i^*(q)}}{1-\sigma_i^{g_i^*(q)}} \leq \frac{\sigma_i^{g_i^*(r)}}{1-\sigma_i^{g_i^*(r)}} \leq G_i(t,s) \leq \frac{1}{1-\sigma_i^{g_{i*}(r)}} \leq \frac{1}{1-\sigma_i^{g_{i*}(q)}}, \quad i = 1,2. \tag{2.2}$$

Proof Similar to the proof of Lemma 2.2, we obtain for $t \leq s \leq t+\omega$,

$$\frac{\sigma_i^{g_i^*(r)}}{1-\sigma_i^{g_i^*(r)}} \leq G_i(t,s) \leq \frac{1}{1-\sigma_i^{g_{i*}(r)}}, \quad i = 1,2.$$

Moreover, since $\varphi(t) := \frac{\sigma_i^t}{1-\sigma_i^t}$ and $\psi(t) := \frac{1}{1-\sigma_i^t}$ are strictly decreasing on $[0,\infty)$, one can easily see that (2.2) holds true. □

Define, for given $(u,v) \in E$,

$$T_{\lambda,\mu}(u,v)(t) = (A_\lambda(u,v)(t), B_\mu(u,v)(t)),$$

where

$$A_\lambda(u,v)(t)=\lambda\int_t^{t+\omega}G_1(t,s)b_1(s)f\bigl(u\bigl(s-\tau_1(s)\bigr),v\bigl(s-\zeta_1(s)\bigr)\bigr)\,ds$$

and

$$B_\mu(u,v)(t)=\mu\int_t^{t+\omega}G_2(t,s)b_2(s)g\bigl(u\bigl(s-\tau_2(s)\bigr),v\bigl(s-\zeta_2(s)\bigr)\bigr)\,ds.$$

Then we have

Lemma 2.4 *Assume* (C1)–(C3) *and* $0<r\le q$. *Then* $T_{\lambda,\mu}(\bar{\Omega}_r)\subseteq K_q$ *and* $T_{\lambda,\mu}:\bar{\Omega}_r\to K_q$ *is completely continuous.*

Proof For $(u,v)\in\bar{\Omega}_r$, we can deduce from Lemma 2.3 that

$$\begin{aligned}A_\lambda(u,v)(t)&=\lambda\int_t^{t+\omega}G_1(t,s)b_1(s)f\bigl(u\bigl(s-\tau_1(s)\bigr),v\bigl(s-\zeta_1(s)\bigr)\bigr)\,ds\\&\le\lambda\frac{1}{1-\sigma_1^{g_{1*}(r)}}\int_0^{\omega}b_1(s)f\bigl(u\bigl(s-\tau_1(s)\bigr),v\bigl(s-\zeta_1(s)\bigr)\bigr)\,ds,\end{aligned}$$

which yields

$$\bigl\|A_\lambda(u,v)\bigr\|\le\lambda\frac{1}{1-\sigma_1^{g_{1*}(r)}}\int_0^{\omega}b_1(s)f\bigl(u\bigl(s-\tau_1(s)\bigr),v\bigl(s-\zeta_1(s)\bigr)\bigr)\,ds.$$

Meanwhile, (2.2) implies

$$\begin{aligned}A_\lambda(u,v)(t)&\ge\lambda\frac{\sigma_1^{g_1^*(r)}}{1-\sigma_1^{g_1^*(r)}}\int_0^{\omega}b_1(s)f\bigl(u\bigl(s-\tau_1(s)\bigr),v\bigl(s-\zeta_1(s)\bigr)\bigr)\,ds\\&=\lambda\frac{\sigma_1^{g_1^*(r)}(1-\sigma_1^{g_{1*}(r)})}{1-\sigma_1^{g_1^*(r)}}\cdot\frac{1}{1-\sigma_1^{g_{1*}(r)}}\int_0^{\omega}b_1(s)f\bigl(u\bigl(s-\tau_1(s)\bigr),v\bigl(s-\zeta_1(s)\bigr)\bigr)\,ds\\&\ge\frac{\sigma_1^{g_1^*(r)}(1-\sigma_1^{g_{1*}(r)})}{1-\sigma_1^{g_1^*(r)}}\bigl\|A(u,v)\bigr\|\\&\ge\eta_1(q)\bigl\|A_\lambda(u,v)\bigr\|\\&\ge\eta(q)\bigl\|A_\lambda(u,v)\bigr\|.\end{aligned}\tag{2.3}$$

In an analogous manner, we get

$$B_\mu(u,v)(t)\ge\eta(q)\bigl\|B_\mu(u,v)\bigr\|,\quad(u,v)\in\bar{\Omega}_r.$$

Hence $T_{\lambda,\mu}(\bar{\Omega}_r)\subseteq K_q$. The completely continuity of $T_{\lambda,\mu}$ is obvious. □

It is obvious that if (u,v) is a fixed point of the completely continuous operator $T_{\lambda,\mu}$ in K_q, then (u,v) is a positive periodic solution of (1.1). We conclude this section by giving the main tool employed in proving our main results.

Lemma 2.5 ([18, 19]) *Assume E is a Banach space and $K \subseteq E$ is a cone. For $r > 0$, let $K_r = \{u \in K : \|u\| < r\}$ and $\partial K_r = \{u \in K : \|u\| = r\}$. Suppose $T : \bar{K}_r \to K$ is a completely continuous operator satisfying $Tu \neq u$, $u \in \partial K_r$. Then*

(i) *If $\|Tu\| < \|u\|$, $u \in \partial K_r$, then $i(T, \bar{K}_r, K) = 1$;*

(ii) *If $\|Tu\| > \|u\|$, $u \in \partial K_r$, then $i(T, \bar{K}_r, K) = 0$.*

3 Main results

Let

$$f_0 = \lim_{(u,v)\to 0} \frac{f(u,v)}{u+v}, \qquad g_0 = \lim_{(u,v)\to 0} \frac{g(u,v)}{u+v}.$$

Theorem 3.1 *Assume* (C1)–(C3) *hold and $f_0 = 0 = g_0$. Then for every $q > 0$, there is a constant $\gamma_q > 0$ such that for all $\lambda, \mu > \gamma_q$, system* (1.1) *admits a positive periodic solution (u, v) satisfying $\|(u, v)\| \leq q$.*

Proof Choose $r_1 = q$ and define

$$\psi_f(q) = \min\bigl\{f(u,v) : \eta(q)q \leq u + v \leq q\bigr\},$$
$$\psi_g(q) = \min\bigl\{g(u,v) : \eta(q)q \leq u + v \leq q\bigr\}.$$

Take

$$\gamma_q = q \cdot \max\left\{\frac{1}{2\psi_f(q)m_1(q)\int_0^\omega b_1(s)\,ds}, \frac{1}{2\psi_g(q)m_2(q)\int_0^\omega b_2(s)\,ds}\right\}.$$

By Lemma 2.4, we know $T_{\lambda,\mu}(\bar{\Omega}_q) \subseteq K_q$ and $T_{\lambda,\mu} : \bar{\Omega}_q \to K_q$ is completely continuous. Fix $\lambda, \mu > \gamma_q$. Then for $(u, v) \in \partial\Omega_q$, we have $\eta(q)q \leq u + v \leq q$, and so

$$\begin{aligned} A_\lambda(u,v)(t) &\geq \lambda m_1(q) \int_0^\omega b_1(s) f\bigl(u\bigl(s - \tau_1(s)\bigr), v\bigl(s - \zeta_1(s)\bigr)\bigr)\,ds \\ &= \lambda m_1(q)\psi_f(q) \cdot \int_0^\omega b_1(s)\,ds \\ &> \frac{q}{2} = \frac{\|(u,v)\|}{2}, \end{aligned}$$

which implies

$$\bigl\|A_\lambda(u,v)\bigr\| > \frac{\|(u,v)\|}{2}, \quad (u,v) \in \partial\Omega_q.$$

Similarly,

$$\bigl\|B_\mu(u,v)\bigr\| > \frac{\|(u,v)\|}{2}, \quad (u,v) \in \partial\Omega_q.$$

Hence $\|T_{\lambda,\mu}(u,v)\| > \|(u,v)\|$ on $\partial\Omega_q$, and then Lemma 2.5 gives $i(T_{\lambda,\mu}, \bar{\Omega}_q, K_q) = 0$.

On the other hand, since $f_0 = g_0 = 0$, there exists a constant r_2 with $0 < r_2 < q$, such that for (u, v) satisfying $0 < u + v \leq r_2$,

$$f(u,v) \leq \varepsilon(u+v), \qquad g(u,v) \leq \varepsilon(u+v),$$

where $\varepsilon > 0$ is a constant satisfying

$$\frac{2\lambda\varepsilon\int_0^\omega b_1(s)\,ds}{1-\sigma_1^{g_1^*(q)}} < 1, \qquad \frac{2\mu\varepsilon\int_0^\omega b_2(s)\,ds}{1-\sigma_2^{g_2^*(q)}} < 1. \tag{3.1}$$

For $(u,v) \in \partial\Omega_{r_2}$, we can deduce by (2.2) and (3.1) that

$$\begin{aligned} A_\lambda(u,v)(t) &\le \lambda\frac{1}{1-\sigma_1^{g_{1*}(r_2)}}\int_t^{t+\omega} b_1(s)f\bigl(u\bigl(s-\tau_1(s)\bigr),v\bigl(s-\zeta_1(s)\bigr)\bigr)\,ds \\ &\le \frac{\lambda\varepsilon}{1-\sigma_1^{g_{1*}(q)}}\cdot\int_0^\omega b_1(s)\,ds\cdot\bigl\|(u,v)\bigr\| \\ &< \frac{\|(u,v)\|}{2}, \end{aligned}$$

and hence

$$\bigl\|A_\lambda(u,v)\bigr\| < \frac{\|(u,v)\|}{2}, \quad (u,v)\in\partial\Omega_{r_2}.$$

In an analogous way, we get

$$\bigl\|B_\mu(u,v)\bigr\| < \frac{\|(u,v)\|}{2}, \quad (u,v)\in\partial\Omega_{r_2}.$$

Thus $\|T_{\lambda,\mu}(u,v)\| < \|(u,v)\|$ on $\partial\Omega_{r_2}$. Lemma 2.5 ensures $i(T_{\lambda,\mu},\bar{\Omega}_{r_2},K_q) = 1$.

Consequently, $i(T_{\lambda,\mu},\bar{\Omega}_q\setminus\Omega_{r_2},K_q) = -1$. Therefore, $T_{\lambda,\mu}$ possesses a fixed point (u,v) in $\bar{\Omega}_q\setminus\Omega_{r_2}$, and system (1.1) has a positive periodic solution (u,v) with $\|(u,v)\|\le q$. □

Theorem 3.2 *Assume* (C1)–(C3) *hold and* $f_0 = \infty$. *Then for every* $q > 0$, *there is a constant* $\gamma_q > 0$ *such that for all* $\lambda,\mu < \gamma_q$, *system* (1.1) *admits a positive periodic solution* (u,v) *satisfying* $\|(u,v)\|\le q$.

Proof Fix $r_1 = q$ and set

$$\begin{aligned} \Psi_f(q) &= \max\bigl\{f(u,v):\eta(q)q\le u+v\le q\bigr\}, \\ \Psi_g(q) &= \max\bigl\{g(u,v):\eta(q)q\le u+v\le q\bigr\}. \end{aligned}$$

Define

$$\gamma_q = q\cdot\min\biggl\{\frac{1}{2\Psi_f(q)M_1(q)\int_0^\omega b_1(s)\,ds},\frac{1}{2\Psi_g(q)M_2(q)\int_0^\omega b_2(s)\,ds}\biggr\}.$$

By Lemma 2.4, $T_{\lambda,\mu}(\bar{\Omega}_q)\subseteq K_q$ and $T_{\lambda,\mu}:\bar{\Omega}_q\to K_q$ is completely continuous. Thus, for fixed $\lambda,\mu<\gamma_q$ and $(u,v)\in\partial\Omega_q$,

$$\begin{aligned} A_\lambda(u,v)(t) &\le \lambda M_1(q)\int_0^\omega b_1(s)f\bigl(u\bigl(s-\tau_1(s)\bigr),v\bigl(s-\zeta_1(s)\bigr)\bigr)\,ds \\ &= \lambda M_1(q)\psi_f(q)\cdot\int_0^\omega b_1(s)\,ds \\ &< \frac{q}{2} = \frac{\|(u,v)\|}{2}, \end{aligned}$$

and then

$$\left\|A_\lambda(u,v)\right\| < \frac{\|(u,v)\|}{2}, \quad (u,v) \in \partial\Omega_q.$$

By a similar argument, we can also obtain

$$\left\|B_\mu(u,v)\right\| < \frac{\|(u,v)\|}{2}, \quad (u,v) \in \partial\Omega_q.$$

Therefore, $\|T_{\lambda,\mu}(u,v)\| < \|(u,v)\|$ for $(u,v) \in \partial\Omega_q$. Using Lemma 2.5 again, we can easily get $i(T_{\lambda,\mu}, \bar{\Omega}_q, K_q) = 1$.

By the assumption $f_0 = \infty$, there exists a constant $r_2 \in (0,q)$, such that for (u,v) satisfying $0 < u + v \le r_2$,

$$f(u,v) \ge \Upsilon(u+v),$$

where $\Upsilon > 0$ satisfies

$$\lambda\Upsilon\eta(q)\frac{\sigma_1^{g_1^*(q)}}{1-\sigma_1^{g_1^*(q)}}\int_0^\omega b_1(s)\,ds > 1. \tag{3.2}$$

Thus for $(u,v) \in \partial\Omega_{r_2}$, we get by (2.2) and (3.2) that

$$\begin{aligned} A_\lambda(u,v)(t) &\ge \lambda\frac{\sigma_1^{g_1^*(r_2)}}{1-\sigma_1^{g_1^*(r_2)}}\int_0^\omega b_1(s)f\big(u\big(s-\tau_1(s)\big), v\big(s-\zeta_1(s)\big)\big)\,ds \\ &\ge \lambda\Upsilon\eta(q)\frac{\sigma_1^{g_1^*(q)}}{1-\sigma_1^{g_1^*(q)}}\int_0^\omega b_1(s)\,ds\cdot\left\|(u,v)\right\| \\ &> \left\|(u,v)\right\|, \end{aligned}$$

which means $\|A_\lambda(u,v)\| > \|(u,v)\|$ on $\partial\Omega_{r_2}$. Hence

$$\left\|T_{\lambda,\mu}(u,v)\right\| \ge \left\|A_\lambda(u,v)\right\| > \left\|(u,v)\right\|, \quad (u,v) \in \partial\Omega_{r_2},$$

and Lemma 2.5 again implies $i(T_{\lambda,\mu}, \bar{\Omega}_{r_2}, K_q) = 0$.

Consequently, $i(T_{\lambda,\mu}, \bar{\Omega}_q \setminus \Omega_{r_2}, K_q) = 1$. Thus, $T_{\lambda,\mu}$ has a fixed point (u,v) in $\bar{\Omega}_q \setminus \Omega_{r_2}$, and (1.1) has a positive periodic solution (u,v) with $\|(u,v)\| \le q$. □

Similarly to Theorems 3.1 and 3.2, we can prove the following

Theorem 3.3 *Assume* (C1)–(C3) *and* $g_0 = \infty$. *Then for every* $q > 0$, *there is a constant* $\gamma_q > 0$ *such that for all* $\lambda, \mu < \gamma_q$, *system* (1.1) *admits a positive periodic solution* (u,v) *satisfying* $\|(u,v)\| \le q$.

Remark 3.1 Clearly, the results of Theorems 3.1–3.3 generalize and complement the corresponding ones in [7, 9, 12–14].

To illustrate our main findings, we may choose $\omega = 2\pi$ and $\tau_i \equiv 0$, $\zeta_i \equiv 0$ $(i = 1, 2)$ in the subsequent discussion. Let

$$a_1(t) = \sin t + 1, \qquad a_2(t) = \sin t + 2, \quad t \in [0, 2\pi],$$
$$b_1(t) = \cos t + 2, \qquad b_2(t) = \cos t + 1, \quad t \in [0, 2\pi].$$

Then it is not hard to check that (C1) is satisfied. Moreover, define

$$g_1(s) = e^s, \qquad g_2(s) = 2e^s, \quad s \in [0, \infty),$$

then there are constants $l_1 = 1$ and $l_2 = 2$ such that

$$0 < 1 = l_1 \le g_1(s) < \infty, \qquad 0 < 2 = l_2 \le g_2(s) < \infty, \quad s \in [0, \infty).$$

Hence (C2) is also satisfied.

Example 3.1 For $(u, v) \in [0, \infty) \times [0, \infty)$, let

$$f(u, v) = 3(u + v)^2 \big(u^2 + v^2 + 1\big)^2, \qquad g(u, v) = 2(u + v)^4 \big(u^2 + v^2 + 5\big)^2.$$

Then $f, g \in C([0, \infty) \times [0, \infty), [0, \infty))$ with $f(u, v) > 0, g(u, v) > 0$ for $(u, v) \neq (0, 0)$. Thus (C3) holds true. Furthermore, simple calculation gives $f_0 = 0 = g_0$. Consequently, the results of Theorem 3.1 are valid.

Example 3.2 We shall follow the same notations and definitions as before. Let us redefine

$$f(u, v) = \sqrt{u + v} \cdot \big(u^2 + v^2 + 1\big)^2, \quad (u, v) \in [0, \infty) \times [0, \infty).$$

Clearly, f verifies (C3). Moreover, it is not difficult to see $f_0 = \infty$, and accordingly the results of Theorem 3.2 are also valid.

At the end of the section, we list some related results and remarks.

Let us consider the multiparameter differential systems

$$\begin{cases} u'(t) = -a_1(t) g_1(u(t)) u(t) + \lambda b_1(t) f(u(t - \tau_1(t)), v(t - \zeta_1(t))), \\ v'(t) = -a_2(t) g_2(v(t)) v(t) + \mu b_2(t) g(u(t - \tau_2(t)), v(t - \zeta_2(t))), \end{cases} \tag{3.3}$$

where $\lambda, \mu > 0$ are parameters. Under the same assumptions as before, one can check that system (3.3) is equivalent to

$$u(t) = \lambda \int_t^{t+\omega} G_1(t, s) b_1(s) f\big(u\big(s - \tau_1(s)\big), v\big(s - \zeta_1(s)\big)\big)\, ds,$$
$$v(t) = \mu \int_t^{t+\omega} G_2(t, s) b_2(s) g\big(u\big(s - \tau_2(s)\big), v\big(s - \zeta_2(s)\big)\big)\, ds,$$

where

$$G_1(t, s) = \frac{e^{\int_t^s a_1(\theta) g_1(u(\theta))\, d\theta}}{e^{\int_0^\omega a_1(\theta) g_1(u(\theta))\, d\theta} - 1}, \qquad G_2(t, s) = \frac{e^{\int_t^s a_2(\theta) g_2(v(\theta))\, d\theta}}{e^{\int_0^\omega a_2(\theta) g_2(v(\theta))\, d\theta} - 1}, \quad t \le s \le t + \omega.$$

Furthermore, by a similar argument as above, it is not difficult to see that the results of Theorems 3.1–3.3 remain true for system (3.3).

Remark 3.2 It is worth remarking that, under some reasonable assumptions, the results of the paper are still valid for the more general coupled systems

$$u_i'(t) + a_i(t)g_i\big(u_i(t)\big)u_i(t) = \lambda_i b_i(t)f_i\big(u_1\big(t-\tau_{i1}(t)\big),\dots,u_n\big(t-\tau_{in}(t)\big)\big), \quad i=1,2,\dots,n$$

and

$$u_i'(t) = a_i(t)g_i\big(u_i(t)\big)u_i(t) - \lambda_i b_i(t)f_i\big(u_1\big(t-\tau_{i1}(t)\big),\dots,u_n\big(t-\tau_{in}(t)\big)\big), \quad i=1,2,\dots,n.$$

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Authors' contributions
RC analyzed and proved the main results, and was a major contributor in writing the manuscript. XL checked the English grammar and typing errors in the full text. All authors read and approved the final manuscript.

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