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Inertial hybrid algorithm for variational inequality problems in Hilbert spaces

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Abstract

For a variational inequality problem, the inertial projection and contraction method have been studied. It has a weak convergence result. In this paper, we propose a strong convergence iterative method for finding a solution of a variational inequality problem with a monotone mapping by projection and contraction method and inertial hybrid algorithm. Our result can be used to solve other related problems in Hilbert spaces.

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Keywords: Variational inequality problem; Inertial projection and contraction method; Strong convergence; Monotone mapping; Inertial hybrid algorithm; Hilbert spaces

1 Introduction

The variational inequality (VI) problem plays an important role in nonlinear analysis and optimization. It is a generalization of the nonlinear complementarity problem. Recently, it has had considerable applications in many fields. The VI problem was introduced by Fichera [1, 2] for solving Signorini problem. Later, it was studied by Stampacchia [3] for solving mechanic problems.

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. The variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where F is a mapping of H into H. The solution set of VI (1.1) is denoted by $VI(C,F)$.

Using properties of the metric projection, we can easily see that $x^* \in VI(C,F)$ if and only if

$$x^* = P_C(I - \lambda F)x^*.$$

Many scholars are devoted to the research of variational inequality problems. Some authors have proposed several iterative methods for solving VI (1.1). A simple iterative method [4] is

$$x_{n+1} = P_C(I - \lambda F)x_n, \tag{1.2}$$

or more generally,

$$x_{n+1} = P_C(I - \lambda_n F)x_n. \tag{1.3}$$

The convergence of (1.2) and (1.3) depends on the properties of F. If F is strongly monotone and Lipschitz continuous, (1.2) and (1.3) have strong convergence results under certain conditions of parameters. If F is inverse strongly monotone, (1.2) and (1.3) have weak convergence results under some suitable conditions.

In 1976, Korpelevich [5] proposed the following so-called extragradient method for solving VI (1.1) when F is monotone and Lipschitz continuous in the finite-dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{cases} x_1 = x \in C \quad \text{is chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda F x_n), \\ x_{n+1} = P_C(x_n - \lambda F y_n), \end{cases} \tag{1.4}$$

for each $n \in \mathbb{N}$. Under some suitable conditions, the sequences $\{x_n\}$ and $\{y_n\}$ converge to the same point $z \in VI(C,F)$. The recent variants of Korpelevich's method can be found in [6].

In 1997, He [7] proposed another method to solve VI with monotone mappings. His method is called projection and contraction method:

$$\begin{cases} x_1 = x \in C \quad \text{is chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda F x_n), \\ d(x_n, y_n) = (x_n - y_n) - \lambda(F x_n - F y_n), \\ x_{n+1} = x_n - \gamma \beta_n d(x_n, y_n), \end{cases} \tag{1.5}$$

for each $n \in \mathbb{N}$, where $\gamma \in (0,2)$,

$$\beta_n = \begin{cases} \frac{\varphi(x_n,y_n)}{\|d(x_n,y_n)\|}, & \text{if } d(x_n,y_n) \neq 0, \\ 0, & \text{if } d(x_n,y_n) = 0, \end{cases}$$

and

$$\varphi(x_n, y_n) = \langle x_n - y_n, d(x_n, y_n) \rangle.$$

This method has a convergence result under certain conditions.

In 2017, Dong et al. [8] proposed the following so-called inertial projection and contraction method:

$$\begin{cases} x_0, x_1 \in H \quad \text{are chosen arbitrarily,} \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F w_n), \\ d(w_n, y_n) = (w_n - y_n) - \lambda(F w_n - F y_n), \\ x_{n+1} = w_n - \gamma \beta_n d(x_n, y_n), \end{cases} \tag{1.6}$$

for each $n \in \mathbb{N}$, where $\gamma \in (0,2)$,

$$\beta_n = \begin{cases} \frac{\varphi(w_n, y_n)}{\|d(w_n, y_n)\|}, & \text{if } d(w_n, y_n) \neq 0, \\ 0, & \text{if } d(w_n, y_n) = 0, \end{cases}$$

and

$$\varphi(w_n, y_n) = \langle w_n - y_n, d(w_n, y_n) \rangle.$$

They proved that the sequence $\{x_n\}$ generated by (1.6) converges weakly to a point in $VI(C, F)$ under certain conditions.

Sometimes, a weak convergence result is not very good. We want to get a strong convergence result. Very recently, Dong et al. [9] used hybrid method to modify an inertial forward-backward algorithm for solving zero point problems in Hilbert spaces:

$$\begin{cases} x_0, x_1 \in H \quad \text{are chosen arbitrarily,} \\ y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = (I + r_n B)^{-1}(y_n - r_n A y_n), \\ C_n = \{u \in H : \|z_n - u\|^2 \leq \|x_n - u\|^2 - 2\alpha_n \langle x_n - u, x_{n-1} - x_n \rangle \\ \qquad + \alpha_n^2 \|x_{n-1} - x_n\|^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.7}$$

They proved that $\{x_n\}$ converges strongly to $P_{(A+B)^{-1}(0)} x_0$ under some suitable conditions.

Based on the work above, we propose an inertial hybrid method for finding a solution of a variational inequality problem with a monotone mapping. As applications, we use algorithm we proposed to solve other related problems in Hilbert spaces.

2 Preliminaries

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proofs of our main results.

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\{x_n\}$ be a sequence in H, we write "$x_n \rightharpoonup x$" to indicate that the sequence $\{x_n\}$ converges weakly to x and "$x_n \to x$" to indicate that the sequence $\{x_n\}$ converges strongly to x. z is called a weak cluster point of $\{x_n\}$ if there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to z. We write $\omega_w(x_n)$ to indicate the set of all weak cluster points of $\{x_n\}$. A fixed point of a mapping $T : H \to H$ is a point $x \in H$ such that $Tx = x$, and we denote the set of all fixed points of mapping T by $Fix(T)$.

We introduce definitions of some operators we will use in the following sections.

Definition 2.1 ([10–12]) Let $T : H \to H$ be the nonlinear operators.

(i) T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H.$$

(ii) T is firmly nonexpansive if

$$\langle Tx - Ty, x - y\rangle \ge \|Tx - Ty\|^2, \quad \forall x, y \in H.$$

We can easily show that a firmly nonexpansive mapping is always nonexpansive by using the Cauchy–Schwarz inequality.

(iii) T is α-averaged, with $0 < \alpha < 1$, if

$$T = (1 - \alpha)I + \alpha S,$$

where $S : H \to H$ is nonexpansive. The term "averaged mapping" was introduced in the early paper by Baillon, Bruck, and Reich [13]. It is obvious that $Fix(S) = Fix(T)$. We can easily show that a firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

(iv) T is L-Lipschitz continuous, with $L \ge 0$, if

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in H.$$

We call T a contractive mapping when $0 \le L < 1$.

Definition 2.2 ([10, 11]) Let $F : H \to H$ be a nonlinear mapping.

(i) F is monotone if

$$\langle Fx - Fy, x - y\rangle \ge 0, \quad \forall x, y \in H.$$

(ii) F is η-strongly monotone, with $\eta > 0$, if

$$\langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in H.$$

(iii) F is ν-inverse strongly monotone (ν-ism), with $\nu > 0$, if

$$\langle Fx - Fy, x - y\rangle \ge \nu\|Fx - Fy\|^2, \quad \forall x, y \in H.$$

We can easily show that a ν-ism mapping is $\frac{1}{\nu}$-Lipschitz continuous by using the Cauchy–Schwarz inequality.

We introduce some definitions and propositions about projections.

Proposition 2.3 ([4]) *Let C be a nonempty closed convex subset of H. Then, for each $x \in H$, there exists a unique point $z \in C$ such that*

$$\|x - z\| \leq \|x - y\|, \quad \forall y \in C.$$

Definition 2.4 ([4]) Let C be a nonempty closed convex subset of H. Define

$$P_C x = \arg\min_{y \in C} \|y - x\|, \quad \forall x \in H.$$

P_C is called the metric projection on C. We can show that P_C is firmly nonexpansive.

Lemma 2.5 ([14, 15]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the inequality*

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.6 ([14–16]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the inequality*

$$\|x - y\|^2 \geq \|x - z\|^2 + \|y - z\|^2, \quad \forall y \in C.$$

More properties of metric projections can be found in [12].

Next, we introduce some definitions and propositions about set-valued mappings.

Definition 2.7 ([17]) Let H be a real Hilbert space. Let A be a set-valued mapping of H into 2^H. We denote the effective domain of A by $D(A)$, $D(A)$ is defined by

$$D(A) = \{x \in H : Ax \neq \emptyset\}.$$

The graph of A is defined by

$$G(A) = \big\{(x, u) \in H \times H : u \in Ax\big\}.$$

A set-valued mapping A is called monotone if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (x, u), (y, v) \in G(A).$$

A monotone mapping A is called maximal if its graph is not properly contained in the graph of any other monotone mappings on $D(A)$.

In fact, we cannot use the definition of the maximal monotone mapping conveniently, a property of the maximal monotone mapping is usually used: A monotone mapping B is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$ for each $(y, v) \in G(A)$ implies $(x, u) \in G(A)$. This property is just a reformulation of the definition of maximal monotone mappings.

Definition 2.8 ([17, 18]) Let $A : H \to 2^H$ be a mapping and $r > 0$. The resolvent of A is

$$J_r^A := (I + rA)^{-1}.$$

Lemma 2.9 ([17, 18]) *Let $A : H \to 2^H$ be a maximal monotone mapping and $r > 0$. Then $J_r^A : H \to D(A)$ is firmly nonexpansive.*

In particular, let C be a nonempty closed convex subset of a real Hilbert space H, recall the normal cone [19] to C at $x \in C$:

$$N_C x = \{z \in H : \langle z, y - x\rangle \leq 0, \forall y \in C\}.$$

We can easily show that N_C is a maximal monotone mapping and its resolvent is P_C. So we can consider the resolvent of a maximal monotone mapping as a generalization of metric projection operator.

Lemma 2.10 ([19]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a monotone and Lipschitz continuous mapping of C into H. Define*

$$Tv = \begin{cases} Fv + N_C v, & \forall v \in C, \\ \emptyset, & \forall v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, F)$.

3 Main result

In this section, we propose a strong convergence algorithm for finding a solution of a variational inequality problem. The algorithm we propose is based on the work in Sect. 1.

Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H. Let F be a mapping of H into H.

Algorithm 1 Choose $x_0, x_1 \in H$ arbitrarily. Calculate the $(n + 1)$th iterate x_{n+1} via the formula

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n F w_n), \\ d(w_n, y_n) = (w_n - y_n) - \lambda_n(F w_n - F y_n), \\ z_n = w_n - \gamma \beta_n d(w_n, y_n), \\ C_n = \{u \in H : \|z_n - u\|^2 \leq \|x_n - u\|^2 - 2\alpha_n \langle x_n - u, x_{n-1} - x_n\rangle \\ \qquad + \alpha_n^2 \|x_{n-1} - x_n\|^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_1 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \end{cases} \tag{3.1}$$

for each $n \geq 1$, where $\gamma \in (0, 2)$, $\lambda_n > 0$, and

$$\beta_n = \begin{cases} \frac{\varphi(w_n, y_n)}{\|d(w_n, y_n)\|}, & \text{if } d(w_n, y_n) \neq 0, \\ 0, & \text{if } d(w_n, y_n) = 0, \end{cases}$$

where

$$\varphi(w_n, y_n) = \langle w_n - y_n, d(w_n, y_n)\rangle.$$

If $y_n = w_n$ or $d(w_n, y_n) = 0$, then calculate x_{n+1} and the iterative process stops; otherwise, we set $n := n + 1$ and go on to (3.1) to calculate the next iterate x_{n+2}.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : H \to H$ be a monotone and L-Lipschitz continuous mapping with $L > 0$. Assume that $VI(C, F) \neq \emptyset$ and $0 < a \leq \lambda_n \leq b < \frac{1}{L}$. Let $\{x_n\}$ be a sequence generated by Algorithm 1. If $y_n = w_n$ or $d(w_n, y_n) = 0$, then $x_{n+1} \in VI(C, F)$.*

Proof From the expression of $d(w_n, y_n)$ and the condition imposed on F, we have

$$\begin{aligned}
&\|d(w_n, y_n)\| \\
&\quad = \|(w_n - y_n) - \lambda_n(Fw_n - Fy_n)\| \\
&\quad \geq \|w_n - y_n\| - \lambda_n\|Fw_n - Fy_n\| \\
&\quad \geq \|w_n - y_n\| - \lambda_n L\|w_n - y_n\| \\
&\quad \geq (1 - bL)\|w_n - y_n\|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|d(w_n, y_n)\| \\
&\quad = \|(w_n - y_n) - \lambda_n(Fw_n - Fy_n)\| \\
&\quad \leq \|w_n - y_n\| + \lambda_n\|Fw_n - Fy_n\| \\
&\quad \leq \|w_n - y_n\| + \lambda_n L\|w_n - y_n\| \\
&\quad \leq (1 + bL)\|w_n - y_n\|.
\end{aligned}$$

So we have

$$(1 - bL)\|w_n - y_n\| \leq \|d(w_n, y_n)\| \leq (1 + bL)\|w_n - y_n\|. \tag{3.2}$$

Hence $y_n = w_n$ and $d(w_n, y_n) = 0$ are equivalent. Using Lemma 2.5, we can get the desired result. □

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : H \to H$ be a monotone and L-Lipschitz continuous mapping with $L > 0$. Assume that $VI(C, F) \neq \emptyset$ and $0 < a \leq \lambda_n \leq b < \frac{1}{L}$. Let $\{x_n\}$ be a sequence generated by Algorithm 1. If $y_n \neq w_n$ for each $n \in N$, then $\{x_n\}$ converges strongly to $x^* = P_{VI(C,F)}x_1$.*

Proof We divide the proof into four steps.

Step 1. We show that $VI(C, F) \subset C_n \cap Q_n$ for each $n \in \mathbb{N}$.

It is obvious that C_n and Q_n are half-spaces for each $n \in \mathbb{N}$.

$$\begin{aligned} &\varphi(w_n, y_n) \\ &\quad = \langle w_n - y_n, d(w_n, y_n)\rangle \\ &\quad = \langle w_n - y_n, (w_n - y_n) - \lambda_n(Fw_n - Fy_n)\rangle \\ &\quad = \|w_n - y_n\|^2 - \lambda_n\langle w_n - y_n, Fw_n - Fy_n\rangle \\ &\quad \geq \|w_n - y_n\|^2 - \lambda_n\|w_n - y_n\|\|Fw_n - Fy_n\| \\ &\quad \geq \|w_n - y_n\|^2 - bL\|w_n - y_n\|^2 \\ &\quad = (1 - bL)\|w_n - y_n\|^2. \end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned} &\|d(w_n, y_n)\|^2 \\ &\quad = \|(w_n - y_n) - \lambda_n(Fw_n - Fy_n)\|^2 \\ &\quad = \|w_n - y_n\|^2 + \lambda_n^2\|Fw_n - Fy_n\|^2 - 2\lambda_n\langle w_n - y_n, Fw_n - Fy_n\rangle \\ &\quad \leq \|w_n - y_n\|^2 + \lambda_n^2\|Fw_n - Fy_n\|^2 \\ &\quad \leq \|w_n - y_n\|^2 + b^2L^2\|w_n - y_n\|^2 \\ &\quad = (1 + b^2L^2)\|w_n - y_n\|^2. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\beta_n = \frac{\varphi(w_n, y_n)}{\|d(w_n, y_n)\|^2} \geq \frac{1 - bL}{1 + b^2L^2}. \tag{3.5}$$

Let $u \in VI(C, F)$, we have

$$\begin{aligned} &\|z_n - u\|^2 \\ &\quad = \|w_n - \gamma\beta_n d(w_n, y_n) - u\|^2 \\ &\quad = \|w_n - u\|^2 - 2\gamma\beta_n\langle w_n - u, d(w_n, y_n)\rangle + \gamma^2\beta_n^2\|d(w_n, y_n)\|^2 \\ &\quad = \|w_n - u\|^2 - 2\gamma\beta_n\langle w_n - y_n, d(w_n, y_n)\rangle - 2\gamma\beta_n\langle y_n - u, d(w_n, y_n)\rangle \\ &\qquad + \gamma^2\beta_n^2\|d(w_n, y_n)\|^2. \end{aligned} \tag{3.6}$$

By the definition of y_n and Lemma 2.5,

$$\langle y_n - u, w_n - y_n - \lambda_n Fw_n\rangle \geq 0.$$

So we have

$$\begin{aligned} &\langle y_n - u, d(w_n, y_n)\rangle \\ &\quad = \langle y_n - u, w_n - y_n - \lambda_n(Fw_n - Fy_n)\rangle \end{aligned}$$

$$= \langle y_n - u, w_n - y_n - \lambda_n F w_n \rangle + \lambda_n \langle y_n - u, F y_n - F u \rangle + \lambda_n \langle y_n - u, F u \rangle \\ \geq 0. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\begin{aligned} &\|z_n - u\|^2 \\ &\leq \|w_n - u\|^2 - 2\gamma\beta_n \langle w_n - y_n, d(w_n, y_n) \rangle + \gamma^2 \beta_n^2 \|d(w_n, y_n)\|^2 \\ &= \|w_n - u\|^2 - 2\gamma\beta_n \varphi(w_n, y_n) + \gamma^2 \beta_n^2 \|d(w_n, y_n)\|^2 \\ &= \|w_n - u\|^2 - 2\gamma\beta_n^2 \|d(w_n, y_n)\|^2 + \gamma^2 \beta_n^2 \|d(w_n, y_n)\|^2 \\ &= \|w_n - u\|^2 - \gamma(2-\gamma)\beta_n^2 \|d(w_n, y_n)\|^2 \\ &= \|w_n - u\|^2 - \frac{2-\gamma}{\gamma} \|z_n - w_n\|^2 \\ &\leq \|w_n - u\|^2. \end{aligned} \tag{3.8}$$

By the expression of w_n, we have

$$\begin{aligned} &\|w_n - u\|^2 \\ &= \|(x_n - u) - \alpha_n (x_{n-1} - x_n)\|^2 \\ &= \|x_n - u\|^2 - 2\alpha_n \langle x_n - u, x_{n-1} - x_n \rangle + \alpha_n^2 \|x_{n-1} - x_n\|^2. \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} &\|z_n - u\|^2 \\ &\leq \|w_n - u\|^2 \\ &= \|x_n - u\|^2 - 2\alpha_n \langle x_n - u, x_{n-1} - x_n \rangle + \alpha_n^2 \|x_{n-1} - x_n\|^2. \end{aligned} \tag{3.10}$$

Therefore, $u \in C_n$ for each $n \in \mathbb{N}$. Hence, $VI(C, F) \subset C_n$ for each $n \in \mathbb{N}$.

For $n = 1$, we have $Q_1 = H$ and hence $VI(C, F) \subset C_1 \cap Q_1$.

Suppose that x_k is given and $VI(C, F) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. It follows from x_{k+1} and Lemma 2.5 that

$$\langle y - x_{k+1}, x_1 - x_{k+1} \rangle \leq 0, \quad \forall y \in VI(C, F).$$

It means that $VI(C, F) \subset Q_{k+1}$. Hence, $VI(C, F) \subset C_{k+1} \cap Q_{k+1}$.

By induction, we obtain $VI(C, F) \subset C_n \cap Q_n$ for each $n \in \mathbb{N}$.

Step 2. We show that $\{x_n\}$ is bounded.

From

$$\langle y - x_n, x_1 - x_n \rangle \leq 0, \quad \forall y \in Q_n$$

and Lemma 2.5, we have

$$x_n = P_{Q_n} x_1$$

and hence

$$\|x_n - x_1\| \le \|x_1 - y\|, \quad \forall y \in Q_n.$$

Since $VI(C,F) \subset Q_n$, we have

$$\|x_n - x_1\| \le \|x_1 - y\|, \quad \forall y \in VI(C,F). \tag{3.11}$$

In particular, since $x_{n+1} \in Q_n$, we obtain

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\|. \tag{3.12}$$

Therefore, there exists

$$c = \lim_{n\to\infty} \|x_n - x_1\|. \tag{3.13}$$

It means that $\{x_n\}$ is bounded.

Step 3. We show that $\omega_w(x_n) \subset VI(C,F)$.

Since $x_n = P_{Q_n}x_1$, $x_{n+1} \in Q_n$ and Lemma 2.6, we obtain

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2,$$

and hence

$$x_{n+1} - x_n \to 0, \quad n \to \infty. \tag{3.14}$$

From

$$\begin{aligned} \|w_n - x_n\| &= \bigl\|x_n - \alpha_n(x_n - x_{n-1}) - x_n\bigr\| \\ &= \alpha_n\|x_n - x_{n-1}\| \end{aligned}$$

and that $\{x_n\}$ is bounded, we have

$$w_n - x_n \to 0, \quad n \to \infty. \tag{3.15}$$

Since $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|z_n - x_{n+1}\|^2 &\le \|x_n - x_{n+1}\|^2 - 2\alpha_n\langle x_n - x_{n+1}, x_{n-1} - x_n\rangle + \alpha_n^2\|x_{n-1} - x_n\|^2 \\ &\le \|x_n - x_{n+1}\|^2 + 2\alpha_n\|x_n - x_{n+1}\|\|x_{n-1} - x_n\| + \alpha_n^2\|x_{n-1} - x_n\|^2 \end{aligned}$$

and hence

$$z_n - x_{n+1} \to 0, \quad n \to \infty. \tag{3.16}$$

Combining (3.14), (3.15), and (3.16), we obtain

$$z_n - w_n \to 0, \quad n \to \infty. \tag{3.17}$$

From (3.1), (3.2), (3.5), and (3.17), we have

$$w_n - y_n \to 0, \quad n \to \infty. \tag{3.18}$$

Since $\{x_n\}$ is bounded, we can take a suitable subsequence $\{x_{n_i}\}$ such that $x_{n_i} \rightharpoonup z$. So we have $w_{n_i} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z$. Let

$$Tv = \begin{cases} Fv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then from Lemma 2.10, we know that T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,F)$. For each $(v,w) \in G(T)$, we have

$$w \in Tv = Fv + N_C v$$

and hence

$$w - Fv \in N_C v.$$

By the definition of N_C, we obtain

$$\langle v - p, w - Fv\rangle \geq 0, \quad \forall p \in C. \tag{3.19}$$

On the other hand, from $v \in C$ and the expression of y_n, we have

$$\langle w_n - \lambda_n F w_n - y_n, y_n - v\rangle \geq 0$$

and hence

$$\left\langle v - y_n, \frac{y_n - w_n}{\lambda_n} + F w_n\right\rangle \geq 0. \tag{3.20}$$

Therefore, from (3.19) and (3.20), we obtain

$$\begin{aligned} &\langle v - y_{n_i}, w\rangle \\ &\quad \geq \langle v - y_{n_i}, Fv\rangle \\ &\quad \geq \langle v - y_{n_i}, Fv\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - w_{n_i}}{\lambda_{n_i}} + F w_{n_i}\right\rangle \\ &\quad = \langle v - y_{n_i}, Fv - F w_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - w_{n_i}}{\lambda_{n_i}}\right\rangle \\ &\quad = \langle v - y_{n_i}, Fv - F y_{n_i}\rangle + \langle v - y_{n_i}, F y_{n_i} - F w_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - w_{n_i}}{\lambda_{n_i}}\right\rangle \\ &\quad \geq + \langle v - y_{n_i}, F y_{n_i} - F w_{n_i}\rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - w_{n_i}}{\lambda_{n_i}}\right\rangle. \end{aligned} \tag{3.21}$$

As $i \to \infty$, we have

$$\langle v - z, w \rangle \geq 0. \tag{3.22}$$

Since T is maximal monotone, we have $0 \in Tz$ and hence $z \in VI(C,F)$. So we obtain $\omega_w(x_n) \subset VI(C,F)$.

Step 4. We show that $x_n \to x^*$ as $n \to \infty$.

Since the norm is convex and lower continuous and $z \in VI(C,F)$, it follows from (3.11) that

$$\|x_1 - x^*\| \leq \|x_1 - z\| \leq \liminf_{i\to\infty} \|x_{n_i} - x_1\| \leq \limsup_{i\to\infty} \|x_{n_i} - x_1\| \leq \|x_1 - x^*\|. \tag{3.23}$$

So we have

$$\lim_{i\to\infty} \|x_{n_i} - x_1\| = \|x_1 - z\| = \|x_1 - x^*\|. \tag{3.24}$$

From $x^* = P_{VI(C,F)}x_1$, we obtain $z = x^*$, i.e., $\omega_w(x_n) = \{x^*\}$. So we have

$$\lim_{n\to\infty} \|x_n - x_1\| = \|x_1 - x^*\| \tag{3.25}$$

and

$$x_n \rightharpoonup x^*, \quad n \to \infty. \tag{3.26}$$

Hence $x_n - x_1 \rightharpoonup x^* - x_1$. Since H satisfies the K-K property, we can obtain $x_n - x_1 \to x^* - x_1$, i.e., $x_n \to x^*$. □

Remark 3.3 If we set $\alpha_n = 0$ for each $n \in \mathbb{N}$, we can get the following algorithm:

$$\begin{cases} y_n = P_C(x_n - \lambda_n F x_n), \\ d(x_n, y_n) = (x_n - y_n) - \lambda_n(Fx_n - Fy_n), \\ z_n = x_n - \gamma\beta_n d(x_n, y_n), \\ C_n = \{u \in H : \|z_n - u\|^2 \leq \|x_n - u\|^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_1 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1. \end{cases}$$

4 Applications

In this section, we introduce some applications which are useful in nonlinear analysis and optimization problems in Hilbert spaces.

4.1 Constrained convex minimization problem

Let C be a nonempty closed convex subset of a real Hilbert space H. The constrained convex minimization problem [14] is to find a point $x^* \in C$ such that

$$f(x^*) = \min_{x\in C} f(x), \tag{4.1}$$

where f is a real-valued convex function. We denote the solution set of problem (4.1) by Ω.

We need the following lemma.

Lemma 4.1 ([11, 20]) *Let H be real Hilbert space, and let C be a nonempty closed convex subset of H. Let f be a convex function of H into $\mathbb{R}$. If f is differentiable, then $z \in \Omega$ if and only if $z \in VI(C, \nabla f)$.*

Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H. Let f be a real-valued convex function of H. Assume that f is differentiable.

Algorithm 2 Choose $x_0, x_1 \in H$ arbitrarily. Calculate the $(n+1)$th iterate x_{n+1} via the formula

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n \nabla f(w_n)), \\ d(w_n, y_n) = (w_n - y_n) - \lambda_n(\nabla f(w_n) - \nabla f(y_n)), \\ z_n = w_n - \gamma \beta_n d(w_n, y_n), \\ C_n = \{u \in H : \|z_n - u\|^2 \leq \|x_n - u\|^2 - 2\alpha_n \langle x_n - u, x_{n-1} - x_n \rangle \\ \qquad + \alpha_n^2 \|x_{n-1} - x_n\|^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_1 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \end{cases} \tag{4.2}$$

for each $n \geq 1$, where $\gamma \in (0, 2)$, $\lambda_n > 0$ and

$$\beta_n = \begin{cases} \frac{\varphi(w_n, y_n)}{\|d(w_n, y_n)\|}, & \text{if } d(w_n, y_n) \neq 0, \\ 0, & \text{if } d(w_n, y_n) = 0, \end{cases}$$

where

$$\varphi(w_n, y_n) = \langle w_n - y_n, d(w_n, y_n) \rangle.$$

If $y_n = w_n$ or $d(w_n, y_n) = 0$, then calculate x_{n+1} and the iterative process stops; otherwise, we set $n := n + 1$ and go on to (4.2) to calculate the next iterate x_{n+2}.

Theorem 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be real-valued convex function of H. Assume that f is differentiable and ∇f is L-Lipschitz continuous with $L > 0$. Assume that $\Omega \neq \emptyset$ and $0 < a \leq \lambda_n \leq b < \frac{1}{L}$. Let $\{x_n\}$ be a sequence generated by Algorithm 2. If $y_n = w_n$ or $d(w_n, y_n) = 0$, then $x_{n+1} \in \Omega$.*

Proof Since f is convex, we conclude that ∇f is monotone. Putting $F = \nabla f$ in Theorem 3.1, we get the desired result by Lemma 4.1. □

Theorem 4.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : H \to H$ be a monotone and L-Lipschitz continuous mapping with $L > 0$. Assume that $VI(C, F) \neq \emptyset$ and $0 < a \leq \lambda_n \leq b < \frac{1}{L}$. Let $\{x_n\}$ be a sequence generated by Algorithm 2. If $y_n \neq w_n$ for each $n \in N$, then $\{x_n\}$ converges strongly to $x^* = P_\Omega x_1$.*

Proof Since f is convex, we conclude that ∇f is monotone. Putting $F = \nabla f$ in Theorem 3.2, we get the desired result by Lemma 4.1. □

4.2 Split feasibility problem

Next, we consider the split feasibility problem.

The split feasibility problem (SFP) was proposed by Censor and Elfving [21] in 1994. The SFP is to find a point x^* such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{4.3}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, A is a bounded linear operator of H_1 and H_2 with $A \neq 0$.

In 2004, Byrne [22] proposed the following algorithm for solving (4.3):

$$x_{n+1} = P_C\bigl(x_n - \gamma_n A^*(I - P_Q)Ax_n\bigr). \tag{4.4}$$

In this section, we introduce a new algorithm to solve (4.3). We need the following lemmas.

Lemma 4.4 ([20]) *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let A be a bounded linear operator of H_1 into H_2 with $A \neq 0$. Assume that $C \cap A^{-1}Q$ is nonempty. Let $\lambda \geq 0$. Then $z \in C \cap A^{-1}Q$ if and only if $z \in VI(C, A^*(I - P_Q)A)$, where A^* is the adjoint operator of A.*

Lemma 4.5 ([20]) *Let H_1 and H_2 be real Hilbert spaces. Let A be a bounded linear operator of H_1 into H_2 such that $A \neq 0$. Let Q be a nonempty closed convex subset of H_2. Then $A^*(I - P_Q)A$ is monotone and $\|A\|^2$-Lipschitz continuous.*

We propose the following algorithm for solving SFP (4.3).

Algorithm 3 Choose $x_0, x_1 \in H_1$ arbitrarily. Calculate the $(n + 1)$th iterate x_{n+1} via the formula

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A^*(I - P_Q)Aw_n), \\ d(w_n, y_n) = (w_n - y_n) - \lambda_n(A^*(I - P_Q)Aw_n - A^*(I - P_Q)Ay_n), \\ z_n = w_n - \gamma\beta_n d(w_n, y_n), \\ C_n = \{u \in H : \|z_n - u\|^2 \leq \|x_n - u\|^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle \\ \qquad + \alpha_n^2\|x_{n-1} - x_n\|^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_1 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \end{cases} \tag{4.5}$$

for each $n \geq 1$, where $\gamma \in (0, 2)$, $\lambda_n > 0$, and

$$\beta_n = \begin{cases} \frac{\varphi(w_n, y_n)}{\|d(w_n, y_n)\|}, & \text{if } d(w_n, y_n) \neq 0, \\ 0, & \text{if } d(w_n, y_n) = 0, \end{cases}$$

where

$$\varphi(w_n, y_n) = \langle w_n - y_n, d(w_n, y_n) \rangle.$$

If $y_n = w_n$ or $d(w_n, y_n) = 0$, then calculate x_{n+1} and the iterative process stops; otherwise, we set $n := n + 1$ and go on to (4.5) to calculate the next iterate x_{n+2}.

Theorem 4.6 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A be a bounded linear operator with $A \neq 0$. Set $\Gamma = C \cap A^{-1}Q$. Assume that $\Gamma \neq \emptyset$ and $0 < a \leq \lambda_n \leq b < \frac{1}{L}$. Let $\{x_n\}$ be a sequence generated by Algorithm 3. If $y_n = w_n$ or $d(w_n, y_n) = 0$, then $x_{n+1} \in \Gamma$.*

Proof Putting $F = A^*(I - P_Q)A$ in Theorem 3.1, we get the desired result by Lemmas 4.4 and 4.5. □

Theorem 4.7 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let A be a bounded linear operator with $A \neq 0$. Set $\Gamma = C \cap A^{-1}Q$. Assume that $\Gamma \neq \emptyset$ and $0 < a \leq \lambda_n \leq b < \frac{1}{L}$. Let $\{x_n\}$ be a sequence generated by Algorithm 3. If $y_n \neq w_n$ for each $n \in N$, then $\{x_n\}$ converges strongly to $x^* = P_\Gamma x_1$.*

Proof Putting $F = A^*(I - P_Q)A$ in Theorem 3.2, we get the desired result by Lemmas 4.4 and 4.5. □

5 Numerical experiments

In this section, we give some numerical results to illustrate the effectiveness of our iterative scheme in Sect. 3 and compare with extragradient method [5] and iterative scheme (1.2). All the programs are written in Matlab 7.10 and performed on a PC Desktop Intel® Core™ i5-2450M CPU @ 2.50 GHz 2.50 GHz, RAM 4.00 GB. All the projections over C and $C_n \cap Q_n$ are computed effectively by the function *quadprog* in Matlab 7.10 Optimization Toolbox.

Example 1 Let $H = \mathbb{R}$ and $C = [-2, 5]$. Let F be a function given by

$$Fx := x + \sin x$$

for each $x \in \mathbb{R}$. For all $x, y \in H$, we have

$$\|Fx - Fy\| = \|x + \sin x - y - \sin y\| \leq \|x - y\| + \|\sin x - \sin y\| \leq 2\|x - y\|,$$

$$\langle Fx - Fy, x - y \rangle = (x + \sin x - y - \sin y)(x - y) = (x - y)^2 + (\sin x - \sin y)(x - y) \geq 0.$$

Therefore, F is monotone and 2-Lipschitz continuous.

Choose $x_0 = 2$, $\lambda_n = \lambda$, $\alpha_n = 2$, and $\gamma = 1$ for our iterative scheme (3.1). It is easy to find that $VI(C, F) = \{0\}$. We denote $x^* = 0$ and use $\|x_n - x^*\| \leq 10^{-5}$ for stopping criterion. The numerical results for this example are described in Table 1.

Table 1 Numerical results as regards Example 1

x_1	λ	Alg. (3.1)		Extragradient method		Alg. (1.2)	
		Iter.	Time [s]	Iter.	Time [s]	Iter.	Time [s]
1	0.05	82	4.89	124	8.52	111	4.32
	0.02	178	9.42	297	19.45	285	9.87
	0.01	177	8.60	585	39.08	574	19.35
2	0.05	97	5.53	132	8.95	119	4.47
	0.02	81	4.81	317	20.57	305	12.14
	0.01	138	7.11	627	41.91	615	20.62
3	0.05	94	5.36	139	9.40	126	4.64
	0.02	234	11.63	334	21.41	321	12.92
	0.01	190	9.15	659	42.03	647	22.89

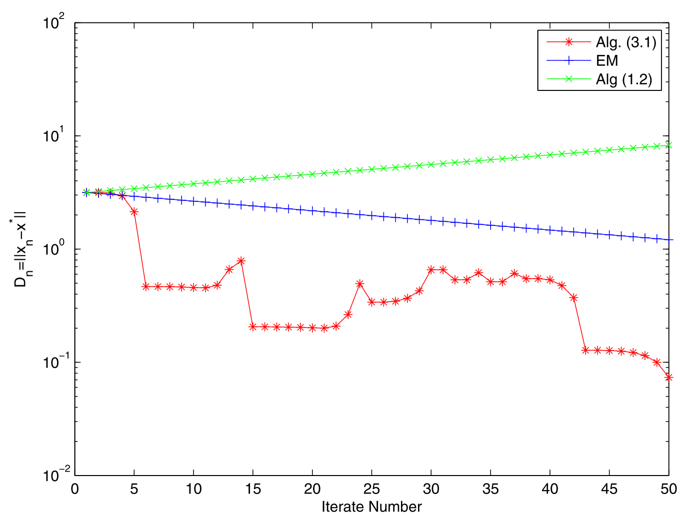

Figure 1 Numerical results as regards Example 2

Example 2 Let $H = \mathbb{R}^m$. We consider a classical problem [23, 24]. The feasible set is $C = \mathbb{R}^m$ and $F : \mathbb{R}^m \to \mathbb{R}^m$ is a linear operator in the form

$$F(x) := Ax$$

for each $x \in \mathbb{R}^m$, where $A = (a_{i,j})_{1\leq i,j\leq m}$ is a matrix in $\mathbb{R}^{m\times m}$ whose terms are given by

$$a_{i,j} = \begin{cases} -1, & \text{if } j = m+1-i \text{ and } j > i, \\ 1, & \text{if } j = m+1-i \text{ and } j < i, \\ 0, & \text{otherwise.} \end{cases}$$

Then F is monotone and $\|A\|$-Lipschitz continuous. This is a classical example of a problem where the usual gradient method does not converge. We can easily see that $VI(C,F) = F^{-1}(0)$ and the zero vector is the unique element in $VI(C,F)$. We denote $x^* = (0,0,\dots,0)^T$.

Choose $x_1 = (1,1,\dots,1)^T$ and $\lambda_n = \lambda = 0.2/\|A\|$ in each iterative scheme. Take $x_0 = (2,2,\dots,2)^T$, $\alpha_n = 2$, and $\gamma = 1$ in our iterative scheme (3.1). We show the numerical results for the cases $m = 10, 20, 30, 40$ respectively in Fig. 1, Fig. 2, Fig. 3, and Fig. 4.

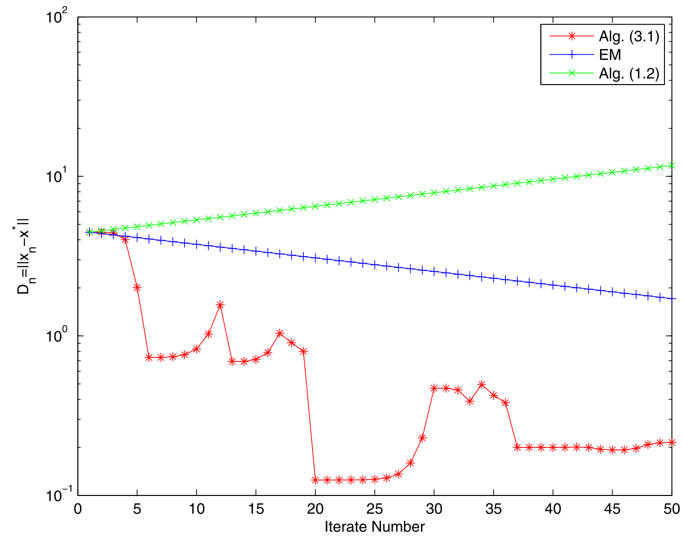

Figure 2 Numerical results as regards Example 2

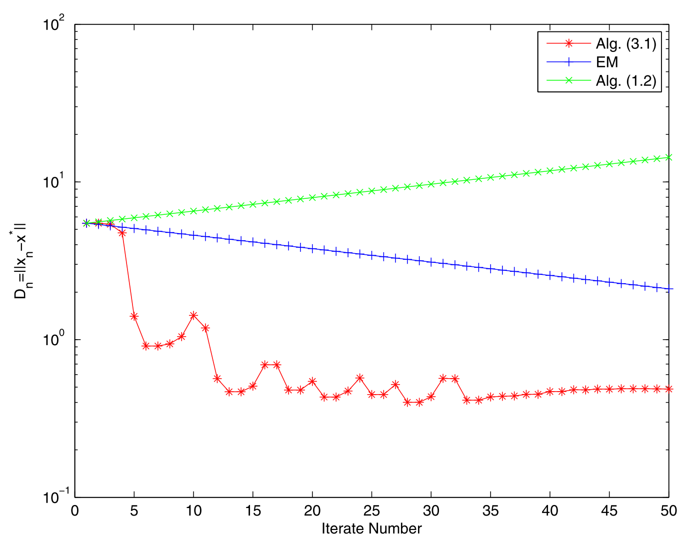

Figure 3 Numerical results as regards Example 2

6 Conclusion

For a variational inequality problem, Algorithms (1.2) and (1.3) have been studied. Considering that sometimes the conditions of operators are not strong enough, He proposed the projection and contraction algorithm. In 2017, Dong et al. proposed the inertial projection and contraction algorithm originated from the second-order dynamical systems. Recently, Dong et al. proposed a strong convergence method for solving zero point problems by using hybrid method. Motivated by their work, we propose an inertial hybrid algorithm for solving variational inequality problems in Hilbert spaces and obtain strong convergence theorems.

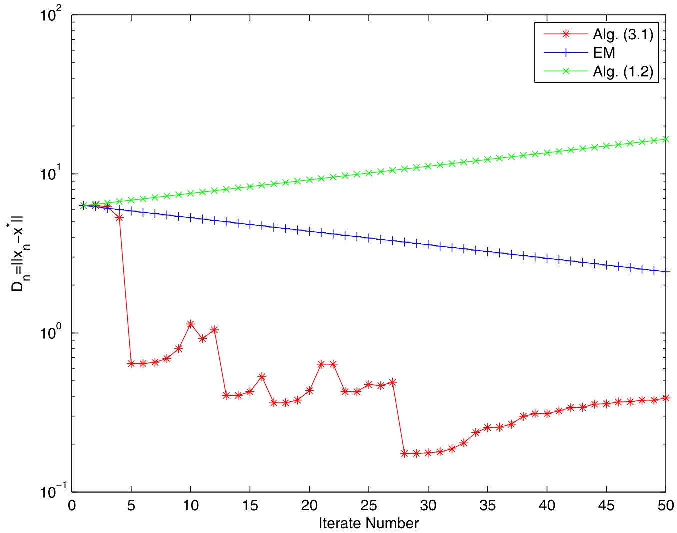

Figure 4 Numerical results as regards Example 2

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