Xing et al. *Journal of Inequalities and Applications* (2020) 2020:171
https://doi.org/10.1186/s13660-020-02437-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH — Open Access

Some new integral inequalities of Wendorff type for discontinuous functions with integral jump conditions

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Abstract

In this paper, we investigate some new integral inequalities of Wendorff type for discontinuous functions with two independent variables and integral jump conditions. These integral inequalities with discontinuities are of non-Lipschitz type. New lower bounds are obtained, integral inequalities with retardation are also involved.

Keywords: Integral inequality; Impulsive differential inequality; Discontinuous function; Integro-sum inequality

1 Introduction

The differential equations with impulse perturbations lie in a special important position in the theory of differential equations. Among these theories, integral inequality method is an important tool to investigate the qualitative characteristics of solutions of different kinds of equations such as difference equations, differential equations, impulsive differential equations, and partial differential equations (see [1–11] for details). For some summary papers, the readers are refereed to [11–18]. In papers [19–24], the authors give qualitative analysis of some integro-differential equations using certain integral inequalities; in papers [2, 25–28], the authors give some integral inequalities with more than two independent variables; papers [24, 29–33] give integral inequalities with weak singular kernels and some qualitative properties of fractional differential equations; the dynamic integral inequalities on time scales are given in papers [34, 35], and the inequalities for essentially bounded functions of one or three variables are investigated by [36, 37].

Phoilakrit Thiramanus and Jessade Tariboon [1] investigated impulsive integral inequality of one independent variable

$$\varphi(t) \le C + \int_{t_0}^{t} b(s)\varphi(s)\,\mathrm{d}s + \sum_{t_0<t_i<t} \gamma_i \varphi(t_i) + \sum_{t_0<t_i<t} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i} \varphi(s)\,\mathrm{d}s, \tag{1}$$

where $0 \le t_0 < t_1 < \cdots$, $\gamma_i, \beta_i \ge 0$, $0 \le \sigma_i \le \tau_i \le t_i - t_{i-1}$, $C \ge 0$ is a constant, and the points t_i are of the first discontinuities.

In 1989, Borysenko [7] investigated impulsive integral inequality of two independent variables of the form

$$\varphi(t,x) \le a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi\bigl(t_i^-,x_i^-\bigr), \tag{2}$$

where $\varphi(t,x)$ is continuous in Ω, with the exception of the points t_i, x_i where there are finite jumps: $\varphi(t_i^-,x_i^-) \neq \varphi(t_i^+,x_i^+)$, $\forall i = 1,2,\ldots$.

In 2007, Borysenko and Iovane [10] investigated some integral inequalities of Wendorff type

$$\varphi(t,x) \le a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi^m\bigl(t_i^-,x_i^-\bigr), \tag{3}$$

$$\varphi(t,x) \le a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi^m\bigl(t_i^-,x_i^-\bigr), \tag{4}$$

$$\varphi(t,x) \le a(t,x) + g(t,x)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi^m\bigl(t_i^-,x_i^-\bigr), \tag{5}$$

$$\begin{aligned}\varphi(t,x) \le{}& a(t,x) + g(t,x)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi\bigl(\sigma(\xi),\sigma(\eta)\bigr)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi^m\bigl(t_i^-,x_i^-\bigr),\end{aligned} \tag{6}$$

$$\begin{aligned}\varphi(t,x) \le{}& a(t,x) + g(t,x)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi^m\bigl(\sigma(\xi),\sigma(\eta)\bigr)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi^m\bigl(t_i^-,x_i^-\bigr),\end{aligned} \tag{7}$$

where $a(t,x) > 0$ is nondecreasing with respect to (t,x), and $g(t,x) \ge 1$, $b(t,x) \ge 0$, $\gamma_i \ge 0$ are constants. The delay term $\sigma(t)$ is continuous and nondecreasing in $[t_0,+\infty)$, $\lim_{t\to\infty}\sigma(t) \le \infty$ for all $t \ge t_0$ and $\sigma(t) \le t$.

In this paper, in a similar way to [8–12] for the inequalities of the functions with one independent variable, we investigate a new Wendorff type inequality for discontinuous functions with two independent variables and give some integro-sum functional inequalities with delay.

2 Integral inequalities for discontinuous functions with discontinuities of non-Lipschitz type

For a given function a defined in a domain Ω with two variables, we say a is a nondecreasing function if, for all $(p,q),(P,Q) \in \Omega$ with $p \le P$, $q \le Q$, one always has $a(p,q) \le a(P,Q)$.

Theorem 2.1 *Let a nonnegative function $\varphi(t,x)$, determined in the domain*

$$\Omega = \bigcup_{k,j\ge 1}\Omega_{kj} = \bigcup_{k,j\ge 1}\bigl\{(t,x) : t \in [t_{k-1},t_k], x \in [x_{k-1},x_k]\bigr\},$$

be continuous in Ω, with the exception of the points (t_i, x_i) where there are finite jumps

$$\varphi(t_i^+, x_i^+) \neq \varphi(t_i^-, x_i^-), \quad \forall i = 1, 2, \dots,$$

and satisfy a certain integro-sum inequality in Ω

$$\begin{aligned} \varphi(t,x) \leq{}& a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \varphi^m(t_i^-, x_i^-) \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\sigma_i}^{t_i-\tau_i}\int_{x_i-\lambda_i}^{x_i-\delta_i} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned} \tag{8}$$

where $m > 0$, $t_0 \geq 0$, $x_0 \geq 0$, $\gamma_i = \mathrm{const} \geq 0$, $\beta_i = \mathrm{const} \geq 0$, *and* $a(t,x) > 0$ *is nondecreasing,* $b(t,x) > 0$ *and satisfies* $b(\xi,\eta) = 0$. *If* $(\xi,\eta) \in \Omega_{ij}$ *with* $i \neq j$, $\lim_{i\to\infty} t_i = \infty$, $\lim_{i\to\infty} x_i = \infty$, *then the function* $\varphi(t,x)$ *satisfies the following estimates*:

$$\begin{aligned} \varphi(t,x) &\leq a(t,x) \prod_{i=1}^{k-1} A_i \cdot \exp\left[\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right], \\ A_i &= \left(1 + \gamma_i a^{m-1}(t_i,x_i)\right) \cdot \exp\left[\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right] \\ &\quad + \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \exp\left[\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(s,t)\,\mathrm{d}s\,\mathrm{d}t\right]\mathrm{d}\xi\,\mathrm{d}\eta \end{aligned} \tag{9}$$

if $0 < m \leq 1$; *and*

$$\begin{aligned} \varphi(t,x) &\leq a(t,x) \prod_{i=1}^{k-1} B_i^{m^{k-i}} \cdot \exp\left[\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right], \\ B_i &= \left(1 + \gamma_i a^{m-1}(t_i,x_i)\right) \cdot \exp\left[m\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right] \\ &\quad + \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \exp\left(\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(s,t)\,\mathrm{d}s\,\mathrm{d}t\right)\mathrm{d}\xi\,\mathrm{d}\eta \end{aligned} \tag{10}$$

if $m > 1$.

Proof Due to $a(t,x) > 0$, we can obtain that

$$\begin{aligned} \frac{\varphi(t,x)}{a(t,x)} \leq{}& 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)} \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{11}$$

Set

$$W(t,x) = 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)} + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{12}$$

with $W(t_0,x_0) = 1$, $\varphi(t,x) \le a(t,x)W(t,x)$, then

$$W(t,x) \le 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i a^{m-1}(t_i,x_i)\bigl[W\bigl(t_i^-,x_i^-\bigr)\bigr]^m + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{13}$$

We give the proof by induction. Firstly, we consider the domain $\Omega_{11} = \{(t,x) : t \in [t_0,t_1], x \in [x_0,x_1]\}$, then

$$\frac{\varphi(t,x)}{a(t,x)} \le 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{14}$$

set

$$V(t,x) = 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{15}$$

thus

$$\varphi(t,x) \le a(t,x)V(t,x),$$
$$V(t,x) \le 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)V(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Let

$$K(t,x) = 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)V(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{16}$$

Then

$$V(t,x) \le K(t,x), \quad K(t_0,x) = K(t,x_0) = 1.$$

Differentiating $K(t,x)$ with respect to t, the following equation holds:

$$K_t(t,x) = \int_{x_0}^{x} b(t,\eta)V(t,\eta)\,\mathrm{d}\eta,$$

because $b(t,x)$ and $V(t,x)$ are continuous in Ω_{11}. Besides, $V(t,x) > 0$, it means that $V(t,x)$ maintains the sign in Ω_{11}. So, on account of generalized mean value theorem of integrals, we can get that

$$K_t(t,x) = \int_{x_0}^{x} b(t,\eta)V(t,\eta)\,\mathrm{d}\eta \le \int_{x_0}^{x} b(t,\eta)\,\mathrm{d}\eta \cdot K(t,x),$$
$$\frac{K_t(t,x)}{K(t,x)} \le \int_{x_0}^{x} b(t,\eta)\,\mathrm{d}\eta.$$

Integrating this inequality from t_0 to t implies

$$\int_{t_0}^{t} \frac{K_t(\xi,x)}{K(\xi,x)}\,\mathrm{d}\xi \le \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$
$$\ln K(t,x)|_{t_0}^{t} = \ln K(t,x) - \ln K(t_0,x) = \ln K(t,x),$$
$$\ln K(t,x) \le \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

then

$$V(t,x) \le K(t,x) \le \exp\left[\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right],$$

so we can get that

$$\varphi(t,x) \le a(t,x)\exp\left[\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]. \tag{17}$$

This shows that the estimates are true in Ω_{11}. Secondly, suppose that (9) and (10) are true in the domain Ω_{kk}. If $0 < m \le 1$, then for $(t,x) \in \Omega_{k+1,k+1}$ the following inequality holds:

$$\begin{aligned}
W(t,x) \le{}& 1 + \sum_{i=1}^{k-1} \gamma_i a^{m-1}(t_i,x_i)\bigl[W\bigl(t_i^-,x_i^-\bigr)\bigr]^m + \sum_{i=1}^{k-1}\beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&+ \int_{t_0}^{t_k}\int_{x_0}^{x_k} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \gamma_k a^{m-1}(t_k,x_k)\bigl[W(t_k-0,x_k-0)\bigr]^m \\
&+ \beta_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k} W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
\le{}& \prod_{i=1}^{k-1} A_i \exp\left[\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right] \\
&+ \gamma_k a^{m-1}(t_k,x_k)\left\{\prod_{i=1}^{k-1} A_i \exp\left[\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]\right\}^m \\
&+ \beta_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k} \prod_{i=1}^{k-1} A_i \exp\left[\int_{t_{k-1}}^{\xi}\int_{x_{k-1}}^{\eta} b(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]\mathrm{d}\xi\,\mathrm{d}\eta \\
&+ \int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}$$

$$
\begin{aligned}
&\leq \prod_{i=1}^{k-1} A_i \biggl\{ \bigl(1+\gamma_k a^{m-1}(t_k,x_k)\bigr) \exp\biggl[\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr] \\
&\quad + \beta_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k} \exp\biggl[\int_{t_{k-1}}^{\xi}\int_{x_{k-1}}^{\eta} b(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\biggr]\mathrm{d}\xi\,\mathrm{d}\eta\biggr\} \\
&\quad + \int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\leq \prod_{i=1}^{k} A_i \exp\biggl[\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr],
\end{aligned}
$$

so when $0 < m \leq 1$, (9) stands.

If $m > 1$, then for $(t,x) \in \Omega_{k+1,k+1}$ the following inequality holds:

$$
\begin{aligned}
W(t,x) &\leq 1 + \sum_{i=1}^{k-1} \gamma_i a^{m-1}(t_i,x_i)\bigl[W\bigl(t_i^-,x_i^-\bigr)\bigr]^m + \sum_{i=1}^{k-1}\beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + \int_{t_0}^{t_k}\int_{x_0}^{x_k} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \gamma_k a^{m-1}(t_k,x_k)\bigl[W(t_k-0,x_k-0)\bigr]^m \\
&\quad + \beta_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k} W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\leq \prod_{i=1}^{k-1} B_i^{m^{k-i-1}} \exp\biggl[\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr] \\
&\quad + \gamma_k a^{m-1}(t_k,x_k)\biggl\{\prod_{i=1}^{k-1} B_i^{m^{k-i-1}} \exp\biggl[\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr]\biggr\}^m \\
&\quad + \beta_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k} \prod_{i=1}^{k-1} B_i^{m^{k-i-1}} \exp\biggl[\int_{t_{k-1}}^{\xi}\int_{x_{k-1}}^{\eta} b(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\biggr]\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + \int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\leq \prod_{i=1}^{k-1} B_i^{m^{k-i}} \exp\biggl[m\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr] \\
&\quad + \gamma_k a^{m-1}(t_k,x_k)\biggl\{\prod_{i=1}^{k-1} B_i^{m^{k-i}} \exp\biggl[m\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr]\biggr\} \\
&\quad + \beta_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k} \prod_{i=1}^{k-1} B_i^{m^{k-i}} \exp\biggl[\int_{t_{k-1}}^{\xi}\int_{x_{k-1}}^{\eta} b(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\biggr]\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + \int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\leq \prod_{i=1}^{k-1} B_i^{m^{k-i}} \biggl\{\bigl(1+\gamma_k a^{m-1}(t_k,x_k)\bigr)\exp\biggl[m\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr] \\
&\quad + \beta_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k} \exp\biggl[\int_{t_{k-1}}^{\xi}\int_{x_{k-1}}^{\eta} b(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\biggr]\mathrm{d}\xi\,\mathrm{d}\eta\biggr\}
\end{aligned}
$$

$$+\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$\le \prod_{i=1}^{k} B_i^{m^{k-i}} \exp\left[\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right],$$

hence when $m > 1$, (10) stands. Finally, by mathematical induction, we get (9) and (10) hold on Ω. This finishes the proof. □

Theorem 2.2 *Suppose that there exists a nonnegative piecewise continuous function $\varphi(t,x)$ determined in the domain Ω, with discontinuity of the first kind in the points (t_k,x_k) ($t_0 < t_1 < t_2 < \cdots$, $x_0 < x_1 < x_2 < \cdots$, $\lim_{i\to\infty} t_i = \infty$, $\lim_{i\to\infty} x_i = \infty$), and it satisfies the inequality*

$$\varphi(t,x) \le a(t,x) + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi^m\left(t_i^-,x_i^-\right)$$
$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\sigma_i}^{t_i-\tau_i}\int_{x_i-\lambda_i}^{x_i-\delta_i} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{18}$$

$m > 0$, $m \ne 1$, where a, b, γ_i, β_i satisfy the conditions of Theorem 2.1. Then, for $(t,x) \in \Omega$, $k = 1,2,\dots$, the following estimates hold:

$$\varphi(t,x) \le a(t,x)\prod_{i=1}^{k-1} C_i \cdot \left[1 + (1-m)\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}, \tag{19}$$
$$C_i = \left(1 + \gamma_i a^{m-1}(t_i,x_i)\right)\cdot\left[1 + (1-m)\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}$$
$$+ \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i}\left[1 + (1-m)\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)a^{m-1}(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{\frac{1}{1-m}}\mathrm{d}\xi\,\mathrm{d}\eta,$$
$$C_0 = 1,$$

if $0 < m < 1$; *and*

$$\varphi(t,x) \le a(t,x)\prod_{i=1}^{k} D_i^{m^{k-i}}$$
$$\cdot\left[1 - (m-1)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m-1}\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{1}{m-1}}, \tag{20}$$
$$D_i = \left(1 + \gamma_i a^{m-1}(t_i,x_i)\right)$$
$$\cdot\left[1 - (m-1)\left(\prod_{j=1}^{i} D_{j-1}^{m^{i-j}}\right)^{m-1}\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{m}{m-1}}$$
$$+ \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i}\left[1 - (m-1)\left(\prod_{j=1}^{i} D_{j-1}^{m^{i-j}}\right)^{m-1}\right.$$
$$\left.\times\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)a^{m-1}(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{-\frac{1}{m-1}}\mathrm{d}\xi\,\mathrm{d}\eta, \quad D_0 = 1,$$

if $m > 1$ *with* $\forall(t,x) \in \Omega$ *satisfying*

$$\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \le \frac{1}{(m-1)(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}})^{m-1}}.$$

Proof Because of $a(t,x) > 0$, we get

$$\begin{aligned}\frac{\varphi(t,x)}{a(t,x)} \le{}& 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi^m(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)} \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{21}$$

Set

$$\begin{aligned} W(t,x) ={}& 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi^m(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)} \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned} \tag{22}$$

then $W(t_0,x) = W(t,x_0) = 1$, $\varphi(t,x) \le a(t,x)W(t,x)$, and

$$\begin{aligned} W(t,x) \le{}& 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)\left[\frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\right]^m \mathrm{d}\xi\,\mathrm{d}\eta \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i a^{m-1}(t_i,x_i)\left[\frac{\varphi(t_i^-,x_i^-)}{a(t_i^-,x_i^-)}\right]^m \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ \le{}& 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i a^{m-1}(t_i,x_i)\left[W\left(t_i^-,x_i^-\right)\right]^m \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{23}$$

By mathematical induction, we consider the function in the domain $\Omega_{11} = \{(t,x) : t \in [t_0,t_1], x \in [x_0,x_1]\}$ firstly. We get

$$W(t,x) \le 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{24}$$

set

$$K(t,x) = 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{25}$$

then

$$\varphi(t,x) \le a(t,x)K(t,x), \quad K(t_0,x) = 1, K(t,x_0) = 1.$$

Differentiating $K(t,x)$ with respect to t, the following equation holds:

$$K_t(t,x) = \int_{x_0}^{x} b(t,\eta)a^{m-1}(\xi,\eta)W^m(t,\eta)\,\mathrm{d}\eta.$$

Since $b(t,x)$ and $W(t,x)$ are continuous in Ω_{11}, besides $W(t,x) > 0$, it means that $W(t,x)$ maintains the same sign in Ω_{11}. So, on account of the generalized first mean value theorem of integrals, we can get that

$$\begin{aligned} K_t(t,x) &= \int_{x_0}^{x} b(t,\eta)a^{m-1}(t,\eta)W^m(t,\eta)\,\mathrm{d}\eta \le \int_{x_0}^{x} b(t,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\eta \cdot K^m(t,x), \\ \frac{K_t(t,x)}{K^m(t,x)} &\le \int_{x_0}^{x} b(t,\eta)a^{m-1}(t,\eta)\,\mathrm{d}\eta. \end{aligned}$$

If $0 < m < 1$,

$$(1-m)\frac{K_t(t,x)}{K^m(t,x)} \le (1-m)\int_{x_0}^{x} b(t,\eta)a^{m-1}(t,\eta)\,\mathrm{d}\eta,$$

then

$$\frac{d}{dt}K^{1-m}(t,x) \le (1-m)\int_{x_0}^{x} b(t,\eta)a^{m-1}(t,\eta)\,\mathrm{d}\eta.$$

Integrating this inequality from t_0 to t, we get

$$\begin{aligned} K^{1-m}(t,x) &\le K^{1-m}(t_0,x) + (1-m)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= 1 + (1-m)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned}$$

then

$$W(t,x) \le K(t,x) \le \left[1 + (1-m)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}.$$

For the case of $m > 1$,

$$(1-m)\frac{K_t(t,x)}{K^m(t,x)} \ge (1-m)\int_{x_0}^{x} b(t,\eta)a^{m-1}(t,\eta)\,\mathrm{d}\eta,$$

thus

$$\frac{d}{dt}K^{1-m}(t,x) \ge (1-m)\int_{x_0}^{x} b(t,\eta)a^{m-1}(t,\eta)\,\mathrm{d}\eta.$$

Integrating this inequality from t_0 to t, we get

$$\begin{aligned}K^{1-m}(t,x) &\geq K^{1-m}(t_0,x)-(m-1)\int_{t_0}^{t}\int_{x_0}^{x}b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &=1-(m-1)\int_{t_0}^{t}\int_{x_0}^{x}b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned}$$

then $\forall(t,x)\in\Omega_{11}$ satisfying

$$\int_{t_0}^{t}\int_{x_0}^{x}b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta<\frac{1}{m-1},$$

we get

$$W(t,x)\leq K(t,x)\leq\left[1-(m-1)\int_{t_0}^{t}\int_{x_0}^{x}b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{1}{m-1}}.$$

Now, we firstly consider the case of $0<m<1$. Suppose that (19) is justified in the domain Ω_{kk}, then for $(t,x)\in\Omega_{k+1,k+1}$ the following inequality holds:

$$\begin{aligned}&W(t,x)\\ &\quad\leq 1+\sum_{i=1}^{k-1}\gamma_i a^{m-1}(t_i,x_i)\bigl[W\bigl(t_i^-,x_i^-\bigr)\bigr]^m+\sum_{i=1}^{k-1}\beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i}W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &\qquad+\int_{t_0}^{t_k}\int_{x_0}^{x_k}b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta+\gamma_k a^{m-1}(t_k,x_k)\bigl[W(t_k-0,x_k-0)\bigr]^m\\ &\qquad+\beta_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k}W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta+\int_{t_k}^{t}\int_{x_k}^{x}b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad\leq\prod_{i=1}^{k-1}C_i\exp\left[1+(1-m)\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k}b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}\\ &\qquad+\gamma_k a^{m-1}(t_k,x_k)\left(\prod_{i=1}^{k-1}C_i\right)^m\left[1+(1-m)\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k}b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{m}{1-m}}\\ &\qquad+\beta_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k}\prod_{i=1}^{k-1}C_i\left[1+(1-m)\int_{t_{k-1}}^{\xi}\int_{x_{k-1}}^{\eta}b(\tau,s)a^{m-1}(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{\frac{1}{1-m}}\mathrm{d}\xi\,\mathrm{d}\eta\\ &\qquad+\int_{t_k}^{t}\int_{x_k}^{x}b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad\leq\prod_{i=1}^{k-1}C_i\exp\left[1+(1-m)\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k}b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}\\ &\qquad+\gamma_k a^{m-1}(t_k,x_k)\prod_{i=1}^{k-1}C_i\left[1+(1-m)\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k}b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}\end{aligned}$$

$$
\begin{aligned}
&+ \beta_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{x_k-\delta_k}^{x_k-\lambda_k} \prod_{i=1}^{k-1} C_i \left[1 + (1-m) \int_{t_{k-1}}^{\xi} \int_{x_{k-1}}^{\eta} b(\tau,s) a^{m-1}(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{\frac{1}{1-m}} \mathrm{d}\xi\,\mathrm{d}\eta \\
&+ \int_{t_k}^{t} \int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta) W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
\le{}& \prod_{i=1}^{k-1} C_i \biggl\{ \bigl(1 + \gamma_k a^{m-1}(t_k,x_k)\bigr) \left[1 + (1-m) \int_{t_{k-1}}^{t_k} \int_{x_{k-1}}^{x_k} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}} \\
&+ \beta_k \left[1 + (1-m) \int_{t_{k-1}}^{\xi} \int_{x_{k-1}}^{\eta} b(\tau,s) a^{m-1}(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{\frac{1}{1-m}} \mathrm{d}\xi\,\mathrm{d}\eta \biggr\} \\
&+ \int_{t_k}^{t} \int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta) W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
\le{}& \prod_{i=1}^{k} C_i + \int_{t_k}^{t} \int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta) W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}
$$

The right-hand side of this inequality is defined as $V(t,x)$, then $V(t_k,x) = V(t,x_k) = \prod_{i=1}^{k} C_i$ and $W(t,x) \le V(t,x)$. Differentiating $V(t,x)$ with respect to t, and on account of the generalized first mean value theorem of integrals, we can get that

$$
\begin{aligned}
V_t(t,x) &= \int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta) W^m(t,\eta)\,\mathrm{d}\eta \\
&\le \int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta)\,\mathrm{d}\eta V^m(t,\eta), \\
\frac{V_t(t,x)}{V^m(t,x)} &\le \int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta)\,\mathrm{d}\eta. \\
(1-m)\frac{V_t(t,x)}{V^m(t,x)} &\le (1-m) \int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta)\,\mathrm{d}\eta,
\end{aligned}
$$

thus

$$
\frac{d}{dt} V^{1-m}(t,x) \le (1-m) \int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta)\,\mathrm{d}\eta.
$$

Integrating the above inequality from t_k to t, we get

$$
\begin{aligned}
V^{1-m}(t,x) &\le V^{1-m}(t_k,x) + (1-m) \int_{t_k}^{t} \int_{x_1}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \left(\prod_{i=1}^{k}\right) C_i^{1-m} + (1-m) \int_{t_k}^{t} \int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned}
$$

then we can get that

$$
\begin{aligned}
V(t,x) &\le \left[\prod_{i=1}^{k} C_i^{1-m} + (1-m) \int_{t_k}^{t} \int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}} \\
&\le \prod_{i=1}^{k} C_i \left[1 + (1-m)\left(\prod_{i=1}^{k}\right) C_i^{m-1} \int_{t_k}^{t} \int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}
\end{aligned}
$$

$$\le \prod_{i=1}^{k} C_i \left[1 + (1-m)\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}},$$

so when $0 < m < 1$, (19) stands.

Next, we prove the case of $m > 1$. Assume that (20) is fulfilled in the domain Ω_{kk}, then for $(t,x) \in \Omega_{k+1,k+1}$ the following inequality holds:

$$\begin{aligned}
&W(t,x)\\
&\quad\le 1 + \sum_{i=1}^{k-1}\gamma_i a^{m-1}(t_i,x_i)\left[W\left(t_i^-,x_i^-\right)\right]^m + \sum_{i=1}^{k-1}\beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\qquad+\int_{t_0}^{t_k}\int_{x_0}^{x_k} b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \gamma_k a^{m-1}(t_k,x_k)\left[W(t_k-0,x_k-0)\right]^m\\
&\qquad+\beta_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k} W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad\le \prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\exp\left[1-(m-1)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m-1}\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{1}{m-1}}\\
&\qquad+\gamma_k a^{m-1}(t_k,x_k)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m}\\
&\qquad\times\left[1-(m-1)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m-1}\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{m}{m-1}}\\
&\qquad+\beta_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k}\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\\
&\qquad\times\left[1-(m-1)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m-1}\int_{t_{k-1}}^{\xi}\int_{x_{k-1}}^{\eta} b(\tau,s)a^{m-1}(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{-\frac{1}{m-1}}\mathrm{d}\xi\,\mathrm{d}\eta\\
&\qquad+\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad\le \prod_{i=1}^{k} D_{i-1}^{m^{k-i+1}}\Bigg\{\left(1+\gamma_k a^{m-1}(t_k,x_k)\right)\\
&\qquad\times\left[1-(m-1)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m-1}\int_{t_{k-1}}^{t_k}\int_{x_{k-1}}^{x_k} b(\xi,\eta)a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{m}{m-1}}\\
&\qquad+\beta_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{x_k-\delta_k}^{x_k-\lambda_k}\Bigg[1-(m-1)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m-1}\\
&\qquad\times\int_{t_{k-1}}^{\xi}\int_{x_{k-1}}^{\eta} b(\tau,s)a^{m-1}(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg]^{-\frac{1}{m-1}}\mathrm{d}\xi\,\mathrm{d}\eta\Bigg\}\\
&\qquad+\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}$$

$$= \prod_{i=1}^{k} D_{i-1}^{m^{k-i+1}} \cdot D_k + \int_{t_k}^{t} \int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta) W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$= \prod_{i=1}^{k+1} D_{i-1}^{m^{k-i+1}} + \int_{t_k}^{t} \int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta) W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

The right-hand side of the last inequality is defined as $U(t,x)$, then $U(t_k,x) = U(t,x_k) = \prod_{i=1}^{k+1} D_{i-1}^{m^{k-i+1}}$ and $W(t,x) \le U(t,x)$. Differentiating $U(t,x)$ with respect to t, and on account of the generalized first mean value theorem of integrals, we can get that

$$\begin{aligned} U_t(t,x) &= \int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta) W^m(t,\eta)\,\mathrm{d}\eta \\ &\le \int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta)\,\mathrm{d}\eta U^m(t,\eta), \\ \frac{U_t(t,x)}{U^m(t,x)} &\le \int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta)\,\mathrm{d}\eta, \\ (1-m)\frac{U_t(t,x)}{U^m(t,x)} &\le (1-m)\int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta)\,\mathrm{d}\eta, \end{aligned}$$

thus

$$\frac{d}{dt} U^{1-m}(t,x) \le (1-m)\int_{x_k}^{x} b(t,\eta) a^{m-1}(t,\eta)\,\mathrm{d}\eta.$$

Integrating this inequality from t_k to t, we have

$$\begin{aligned} U^{1-m}(t,x) &\ge U^{1-m}(t_k,x) - (m-1)\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \left(\prod_{i=1}^{k+1} D_{i-1}^{m^{k-i+1}}\right)^{1-m} - (m-1)\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned}$$

Then we can get that $\forall (t,x) \in \Omega_{k+1,k+1}$ satisfying that

$$\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta < \frac{1}{(m-1)(\prod_{i=1}^{k+1} D_{i-1}^{m^{k-i+1}})^{m-1}},$$
$$\begin{aligned} U(t,x) &\le \left[\left(\prod_{i=1}^{k+1} D_{i-1}^{m^{k-i+1}}\right)^{1-m} - (m-1)\int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{1}{m-1}} \\ &\le \prod_{i=1}^{k+1} D_{i-1}^{m^{k-i+1}} \left[1 - (m-1)\left(\prod_{i=1}^{k+1} D_{i-1}^{m^{k-i+1}}\right)^{m-1} \right. \\ &\quad \left. \times \int_{t_k}^{t}\int_{x_k}^{x} b(\xi,\eta) a^{m-1}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{1}{m-1}}. \end{aligned}$$

So when $m > 1$, (20) stands. By mathematical induction, this completes the proof. □

Theorem 2.3 *Suppose that there exists a nonnegative piecewise continuous function $\varphi(t,x)$ determined in the domain Ω, with discontinuity of the first kind in the points (t_k,x_k) $(t_0 < t_1 < t_2 < \cdots, x_0 < x_1 < x_2 < \cdots, \lim_{i\to\infty} t_i = \infty, \lim_{i\to\infty} x_i = \infty)$, and it satisfies the inequality*

$$\begin{aligned}\varphi(t,x) \le{}& a(t,x) + g(t,x)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi^m\bigl(t_i^-,x_i^-\bigr) \\ &+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i\int_{t_i-\sigma_i}^{t_i-\tau_i}\int_{x_i-\lambda_i}^{x_i-\delta_i} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned} \tag{26}$$

$m > 0$, $m \ne 1$, *where a, b, γ_i, β_i satisfy the conditions of Theorem 2.1. Then, for $(t,x) \in \Omega$, $k = 1,2,\ldots$, the following estimates hold*:

$$\begin{aligned}\varphi(t,x) \le{}& a(t,x)g(t,x)\prod_{i=1}^{k-1} C_i \\ &\cdot\left[1 + (1-m)\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}},\end{aligned} \tag{27}$$

$$\begin{aligned}C_i ={}& \bigl(1 + \gamma_i a^{m-1}(t_i,x_i)\bigr)g^m(t_i,x_i) \\ &\cdot\left[1 + (1-m)\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}} \\ &+ \beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i}\left[1 + (1-m)\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)a^{m-1}(\tau,s)g^m(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{\frac{1}{1-m}}\mathrm{d}\xi\,\mathrm{d}\eta, \\ C_0 ={}& 1,\end{aligned}$$

if $0 < m < 1$; *and*

$$\begin{aligned}\varphi(t,x) \le{}& a(t,x)g(t,x)\sum_{i=1}^{k-1}\left[\bigl(1 + \gamma_i g(t_i,x_i)\bigr)\exp\left[\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)g(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]\right. \\ &\left.+ \beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i}\exp\left[\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)g(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]\mathrm{d}\xi\,\mathrm{d}\eta\right],\end{aligned} \tag{28}$$

if $m = 1$; *and*

$$\begin{aligned}\varphi(t,x) \le{}& a(t,x)g(t,x)\prod_{i=1}^{k} D_i^{m^{k-i}}\left[1 - (m-1)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m-1}\right. \\ &\left.\times\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{-1}{m-1}},\end{aligned} \tag{29}$$

$$\begin{aligned}
D_i = {} & \bigl(1+\gamma_i\bigl(a(t_i,x_i)g(t_i,x_i)\bigr)^{m-1} \\
& \cdot \left[1-(m-1)\left(\prod_{j=1}^{i} D_{j-1}^{m^{i-j}}\right)^{m-1} \int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{m}{m-1}} \\
& + \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i}\left[1-(m-1)\left(\prod_{j=1}^{i} D_{j-1}^{m^{i-j}}\right)^{m-1}\right. \\
& \times \left.\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)a^{m-1}(\tau,s)g^m(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{-\frac{1}{m-1}} \mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned}$$

if $m>1$ *with* $D_0=1$ *and* $(t,x)\in\Omega$ *satisfying*

$$\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \le \frac{1}{(m-1)(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}})^{m-1}}.$$

Proof Since $a(t,x)>0$, $g(t,x)>0$, we get

$$\begin{aligned}
\frac{\varphi(t,x)}{a(t,x)} \le g(t,x)\Biggl[1 & + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi^m(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)} \\
& + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\Biggr].
\end{aligned} \tag{30}$$

Set

$$\begin{aligned}
W(t,x) = 1 & + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi^m(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)} \\
& + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned} \tag{31}$$

$W(t_0,x)=W(t,x_0)=1$, $\varphi(t,x)\le a(t,x)g(t,x)W(t,x)$, then

$$\begin{aligned}
W(t,x) \le 1 & + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\left[\frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\right]^m \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i a^{m-1}(t_i,x_i)g^m(t_i,x_i)\left[\frac{\varphi(t_i^-,x_i^-)}{a(t_i^-,x_i^-)}\right]^m \\
& + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
\le 1 & + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)W^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
& + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i a^{m-1}(t_i,x_i)g^m(t_i,x_i)\bigl[W\bigl(t_i^-,x_i^-\bigr)\bigr]^m \\
& + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} g(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned} \tag{32}$$

Using the procedure for $W(t,x)$ in Theorem 2.2, it is possible to obtain for $W(t,x)$ the following estimates:

$$W(t,x) \leq \prod_{i=1}^{k-1} C_i \cdot \left[1 + (1-m)\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}, \tag{33}$$

$$\begin{aligned} C_i &= \bigl(1+\gamma_i a^{m-1}(t_i,x_i)\bigr)g^m(t_i,x_i) \\ &\quad\cdot\left[1+(1-m)\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}} \\ &\quad+\beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} g(\tau,s) \\ &\quad\times\left[1+(1-m)\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)a^{m-1}(\tau,s)g^m(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{\frac{1}{1-m}}\mathrm{d}\xi\,\mathrm{d}\eta, \quad C_0 = 1, \end{aligned}$$

if $0 < m \leq 1$;

$$\begin{aligned} W(t,x) \leq \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \Biggl[&\bigl(1+\gamma_i g(t_i,x_i)\bigr)\exp\left[\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)g(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right] \\ &+\beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} g(\tau,s)\exp\left[\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)g(\tau,s)\,\mathrm{d}\tau\,\mathrm{d}s\right]\mathrm{d}\xi\,\mathrm{d}\eta\Biggr], \end{aligned} \tag{34}$$

if $m = 1$;

$$\begin{aligned} W(t,x) \leq \prod_{i=1}^{k} D_i^{m^{k-i}} \Biggl[1 - (m-1)\left(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}}\right)^{m-1} \\ \times\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\Biggr]^{-\frac{1}{m-1}}, \end{aligned} \tag{35}$$

$$\begin{aligned} D_i &= \bigl(1+\gamma_i\bigl(a(t_i,x_i)g(t_i,x_i)\bigr)\bigr)^{m-1} \\ &\quad\cdot\left[1-(m-1)\left(\prod_{j=1}^{i} D_{j-1}^{m^{i-j}}\right)^{m-1}\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{m}{m-1}} \\ &\quad+\beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} g(\tau,s)\Biggl[1-(m-1)\left(\prod_{j=1}^{i} D_{j-1}^{m^{i-j}}\right)^{m-1} \\ &\quad\times\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)a^{m-1}(\tau,s)g^m(\xi,\eta)\,\mathrm{d}\tau\,\mathrm{d}s\Biggr]^{\frac{-1}{m-1}}\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned}$$

if $m > 1$ with $D_0 = 1$ and $\forall(t,x) \in \Omega$:

$$\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} b(\xi,\eta)a^{m-1}(\xi,\eta)g^m(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \leq \frac{1}{(m-1)(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}})^{m-1}}.$$

From (33)–(35) estimates (28)–(29) for the function φ will follow. □

3 Inequalities with retardation

Let us define a class of functions in the $\Im$-class of continuous functions $\sigma(t)$ as retardation, and for $\sigma(t)$ the following estimates hold:

(a_1) $\sigma(t) \le t$, $\forall t \in R_+$, $R_+ := [0, +\infty)$;

(a_2) $\lim_{t\to+\infty} \sigma(t) = +\infty$;

(a_3) $\sigma(t)$ is nondecreasing.

Theorem 3.1 *Let $\sigma \in \Im$ and $\varphi(t,x)$ satisfy certain inequality*

$$\varphi(t,x) \le a(t,x) + g(t,x)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi\bigl(\sigma(\xi),\sigma(\eta)\bigr)\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\varphi^m\bigl(t_i^-,x_i^-\bigr)$$
$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \cdot \int_{t_i-\sigma_i}^{t_i-\tau_i}\int_{x_i-\lambda_i}^{x_i-\delta_i} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{36}$$

with $m > 0$, and the functions φ, a, g, b satisfy the conditions of Theorem 2.3, $\gamma_i, \beta_i = \mathrm{const} \ge 0$. Then, for $k = 1, 2, \ldots$, $\varphi(t,x)$, the following estimates are valid:

$$\varphi(t,x) \le a(t,x)g(t,x)\prod_{i=1}^{k-1} S_i \exp\biggl[\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} \mathcal{F}\bigl(\sigma(\xi),\sigma(\eta)\bigr)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr], \tag{37}$$
$$S_i = \bigl(1 + \gamma_i a^{m-1}(t_i,x_i)g^m(t_i,x_i)\bigr)\exp\biggl[\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} \mathcal{F}\bigl(\sigma(\xi),\sigma(\eta)\bigr)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr]$$
$$+ \beta_i \cdot \int_{t_i-\sigma_i}^{t_i-\tau_i}\int_{x_i-\lambda_i}^{x_i-\delta_i} g(\tau,s)\exp\biggl[\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} \mathcal{F}\bigl(\sigma(\tau),\sigma(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\biggr]\mathrm{d}\xi\,\mathrm{d}\eta, \quad S_0 = 1,$$

if $0 < m \le 1$;

$$\varphi(t,x) \le a(t,x)g(t,x)\prod_{i=1}^{k-1} T_i^{m^{k-i-1}} \exp\biggl[\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} \mathcal{F}\bigl(\sigma(\xi),\sigma(\eta)\bigr)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr], \tag{38}$$
$$T_i = \bigl(1 + \gamma_i a^{m-1}(t_i,x_i)g^m(t_i,x_i)\bigr)\exp\biggl[m\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} \mathcal{F}\bigl(\sigma(\xi),\sigma(\eta)\bigr)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr]$$
$$+ \beta_i \cdot \int_{t_i-\sigma_i}^{t_i-\tau_i}\int_{x_i-\lambda_i}^{x_i-\delta_i} g(\tau,s)\exp\biggl[\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} \mathcal{F}\bigl(\sigma(\tau),\sigma(s)\bigr)\,\mathrm{d}\tau\,\mathrm{d}s\biggr]\mathrm{d}\xi\,\mathrm{d}\eta, \quad T_0 = 1,$$

if $m \ge 1$. Here, the function $\mathcal{F}(\sigma(\xi),\sigma(\eta))$ is defined by

$$\mathcal{F}\bigl(\sigma(\xi),\sigma(\eta)\bigr) = \frac{b(\xi,\eta)a(\sigma(\xi),\sigma(\eta))g(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)}.$$

Proof Because of $a(t,x) > 0$, we get

$$\frac{\varphi(t,x)}{a(t,x)} \le g(t,x)\biggl[1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i\frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)}$$
$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr]. \tag{39}$$

Set

$$W(t,x) = 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)}$$
$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{40}$$

$W(t_0,x) = W(t,x_0) = 1$, $\varphi(t,x) \le a(t,x)g(t,x)W(t,x)$, then

$$W(t,x) \le 1 + \int_{t_0}^{t}\int_{x_0}^{x} \frac{b(\xi,\eta)a(\sigma(\xi),\sigma(\eta))g(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)} \cdot W\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i a^{m-1}(t_i,x_i) g^m(t_i,x_i)\big[W\big(t_i^-,x_i^-\big)\big]^m$$
$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} g(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{41}$$

Using the result of Theorem 2.1 for (41), $k = 1,2,\dots$, we could obtain certain estimates:

$$W(t,x) \le \prod_{i=1}^{k-1} S_i \exp\biggl[\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} \mathcal{F}\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr], \tag{42}$$
$$S_i = \big(1 + \gamma_i a^{m-1}(t_i,x_i) g^m(t_i,x_i)\big)\exp\biggl[\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} \mathcal{F}\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr]$$
$$+ \beta_i \cdot \int_{t_i-\sigma_i}^{t_i-\tau_i}\int_{x_i-\lambda_i}^{x_i-\delta_i} g(\tau,s)\exp\biggl[\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} \mathcal{F}\big(\sigma(\tau),\sigma(s)\big)\,\mathrm{d}\tau\,\mathrm{d}s\biggr]\mathrm{d}\xi\,\mathrm{d}\eta, \quad S_0 = 1,$$

if $0 < m \le 1$;

$$W(t,x) \le \prod_{i=1}^{k-1} T_i^{m^{k-i-1}} \exp\biggl[\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} \mathcal{F}\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr], \tag{43}$$
$$T_i = \big(1 + \gamma_i a^{m-1}(t_i,x_i) g^m(t_i,x_i)\big)\exp\biggl[m\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} \mathcal{F}\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\biggr]$$
$$+ \beta_i \cdot \int_{t_i-\sigma_i}^{t_i-\tau_i}\int_{x_i-\lambda_i}^{x_i-\delta_i} g(\tau,s)\exp\biggl[\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} \mathcal{F}\big(\sigma(\tau),\sigma(s)\big)\,\mathrm{d}\tau\,\mathrm{d}s\biggr]\mathrm{d}\xi\,\mathrm{d}\eta, \quad T_0 = 1,$$

if $m \ge 1$. From (42)–(43) and the inequality $\varphi(t,x) \le a(t,x)g(t,x)W(t,x)$, the result of Theorem 3.1 follows. □

Theorem 3.2 *Let us suppose that all the conditions of Theorem 3.1 are fulfilled and the function $\varphi(t,x)$ satisfies a certain inequality*

$$\varphi(t,x) \le a(t,x) + g(t,x)\int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\varphi^m\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \varphi^m\big(t_i^-,x_i^-\big)$$

$$+ \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \cdot \int_{t_i-\sigma_i}^{t_i-\tau_i} \int_{x_i-\lambda_i}^{x_i-\delta_i} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{44}$$

with $m > 0$.

Then $\forall (t,x) \in \Omega$, *the following estimates hold*:

$$\varphi(t,x) \le a(t,x)g(t,x)\prod_{i=1}^{k-1} X_i\left[1 + (1-m)\int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} \mathcal{K}\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\right], \tag{45}$$

$$X_i = \big(1 + \gamma_i a^{m-1}(t_i,x_i)g^m(t_i,x_i)\big)\cdot\left[1 + (1-m)\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} \mathcal{K}\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{\frac{1}{1-m}}$$

$$+ \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} g(\xi,\eta)\left[1 + (1-m)\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} \mathcal{K}\big(\sigma(\tau),\sigma(s)\big)\,\mathrm{d}\tau\,\mathrm{d}s\right]^{\frac{1}{1-m}}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$X_0 = 1,$$

if $0 < m \le 1$;

$$\begin{aligned}
\varphi(t,x) \le a(t,x)g(t,x)\prod_{i=1}^{k-1}\Bigg\{&\big(1 + \gamma_i g(t_i,x_i)\big) \\
&\times \exp\left[\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} b(\xi,\eta)g\big(\sigma(\xi),\sigma(\eta)\big)\frac{a(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\right] \\
&+ \beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i}\exp\bigg[\int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} b(\tau,s)g\big(\sigma(\tau),\sigma(s)\big) \\
&\times \frac{a(\sigma(\tau),\sigma(s))}{a(\tau,s)}\,\mathrm{d}\tau\,\mathrm{d}s\bigg]\mathrm{d}\xi\,\mathrm{d}\eta\Bigg\},
\end{aligned} \tag{46}$$

if $m = 1$;

$$\begin{aligned}
\varphi(t,x) \le a(t,x)g(t,x)\prod_{i=1}^{k} Y_i^{m^{k-i}}\Bigg[&1 - (m-1)\left(\prod_{i=1}^{k} Y_{i-1}^{m^{k-i}}\right)^{m-1} \\
&\times \int_{t_{k-1}}^{t}\int_{x_{k-1}}^{x} \mathcal{K}\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\Bigg]^{-\frac{1}{m-1}},
\end{aligned} \tag{47}$$

$$\begin{aligned}
Y_i = {}&\big(1 + \gamma_i a^{m-1}(t_i,x_i)g^{m-1}(t_i,x_i)\big) \\
&\cdot\left[1 - (m-1)\left(\prod_{j=1}^{i} Y_{j-1}^{m^{i-j}}\right)^{m-1}\int_{t_{i-1}}^{t_i}\int_{x_{i-1}}^{x_i} \mathcal{K}\big(\sigma(\xi),\sigma(\eta)\big)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]^{-\frac{m}{m-1}} \\
&+ \beta_i\int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i}\Bigg[1 - (m-1)\left(\prod_{j=1}^{i} Y_{j-1}^{m^{i-j}}\right)^{m-1} \\
&\times \int_{t_{i-1}}^{\xi}\int_{x_{i-1}}^{\eta} \mathcal{K}\big(\sigma(\tau),\sigma(s)\big)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg]^{-\frac{1}{m-1}}\mathrm{d}\xi\,\mathrm{d}\eta, \quad Y_0 = 1,
\end{aligned}$$

if $m > 1$, $\forall(t,x) \in \Omega$:

$$\int_{t_{k-1}}^{t} \int_{x_{k-1}}^{x} b(\xi,\eta)\mathcal{K}\bigl(\sigma(\xi),\sigma(\eta)\bigr)\,\mathrm{d}\xi\,\mathrm{d}\eta \le \frac{1}{(m-1)(\prod_{i=1}^{k} D_{i-1}^{m^{k-i}})^{m-1}}.$$

Here, the function $\mathcal{K}(\sigma(\xi),\sigma(\eta))$ *is defined by*

$$\mathcal{K}\bigl(\sigma(\xi),\sigma(\eta)\bigr) = \frac{b(\xi,\eta)a^m(\sigma(\xi),\sigma(\eta))g^m(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)}.$$

Proof Due to $g(t,x) \ge 1$, the following inequality is valid:

$$\begin{aligned} \frac{\varphi(t,x)}{a(t,x)} \le g(t,x)\Biggl[1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi^m(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)} \\ + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\Biggr]. \end{aligned} \tag{48}$$

Set

$$\begin{aligned} W(t,x) = 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{\varphi^m(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i \frac{\varphi^m(t_i^-,x_i^-)}{a(t_i^-,x_i^-)} \\ + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} \frac{\varphi(\xi,\eta)}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta, \end{aligned} \tag{49}$$

thus

$$\varphi(t,x) \le a(t,x)g(t,x)W(t,x), \tag{50}$$

and $W(t_0,x) = W(t,x_0) = 1$, then

$$\begin{aligned} W(t,x) \le 1 + \int_{t_0}^{t}\int_{x_0}^{x} b(\xi,\eta)\frac{a^m(\sigma(\xi),\sigma(\eta))g^m(\sigma(\xi),\sigma(\eta))W^m(\sigma(\xi),\sigma(\eta))}{a(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \gamma_i a^{m-1}(t_i,x_i)g^m(t_i,x_i)W^m\bigl(t_i^-,x_i^-\bigr) \\ + \sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i \int_{t_i-\tau_i}^{t_i-\sigma_i}\int_{x_i-\delta_i}^{x_i-\lambda_i} g(\xi,\eta)W(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{51}$$

Using the result of Theorem 2.3 for inequality (51) and taking into account estimate (50), we obtain estimates (45)–(47). □

Acknowledgements

The authors sincerely thank the referees for a number of constructive suggestions and corrections which have significantly improved the contents and the exposition of the paper.

Funding

This research was partially supported by the NSF of China (Grant 11671227), NSF of Shandong Province (Grant ZR2019MA034), and Project of Qufu Normal University (201602028012, 18jg06).

Availability of data and materials
The data and material used to support the findings of this study are available within the paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Publisher's Note

Received: 9 August 2019 Accepted: 10 June 2020 Published online: 18 June 2020

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