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Some identities of type 2 q -Bernoulli polynomials

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Abstract

Recently, symmetric properties of some special polynomials arising from p -adic q -integrals on \mathbb{Z}_p have been investigated extensively by many researchers.

In this paper, we find some symmetric identities for type 2 q -Bernoulli polynomials under the symmetry group S_n of degree n by using the bosonic p -adic q -integral on \mathbb{Z}_p .

Keywords: Bosonic p -adic q -integral; Type 2 q -Bernoulli numbers; Symmetry group of degree n

1 Introduction

In the book *Ars Conjectandi*, Jakob Bernoulli introduced the Bernoulli numbers for studying of the sums of powers of consecutive integers $1^k + 2^k + \dots + n^2$. As a generalization of these numbers, we have the Bernoulli polynomials, which are defined by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{where } |t| < 2\pi \text{ (see [2, 4, 5, 7, 12, 15, 16, 28, 34, 35, 41]).}$$

In the special case $x = 0$, $B_n(0) = B_n$ are the *Bernoulli numbers*. It is well known that

$$0^k + 1^k + 2^k + \dots + n^k = \frac{1}{k+1} (B_{k+1}(n+1) - B_{k+1}),$$

for each nonnegative integer k and positive integer n (see [2, 15, 21]).

The Bernoulli numbers and polynomials have very important roles in the pure and applied mathematics. In [14], Kim and Kim represented several trigonometric functions as a formal power series involving either Bernoulli or Euler numbers, and relationships between the Bernoulli numbers and zeta functions are investigated by Arakawa et al. in [2]. In [1], authors found some applications of the Bernoulli numbers and polynomials corresponding to zeta functions, and Kim and Kim studied some special polynomials in connection with Bernoulli numbers and polynomials (see [27]).

In addition, the Bernoulli numbers and polynomials have been generalized by many researchers. For example, Carlitz degenerated the Bernoulli polynomials and investigated

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the properties of those polynomials in [5]. Kim represented the q -Bernoulli polynomials and numbers by using q -Volkenborn integration (see [19, 20]). In [40], authors constructed a family of modified p -adic twisted functions and gave some applications and examples related to these polynomials, and Simsek defined some new sequences containing the Bernoulli numbers and the Euler numbers and found some computation formulas and identities for those sequences (see [38]). Gaboury et al. defined some new classes of Bernoulli polynomials and found explicit formulas of those polynomials in [7], and Park and Rim defined modified q -Bernoulli polynomials with weight (see [34]). Moreover, Simsek gave the generating functions of the twisted Bernoulli polynomials and obtained some equations which are general versions of Eq. (1) in [36], and Kim et al. defined type 2 q -Bernoulli polynomials and investigated the properties of those polynomials (see [15]).

As a new generalization of the Bernoulli numbers, Jang and Kim introduced the type 2 degenerate Bernoulli polynomials in [9] and investigated relationships between some special functions and those polynomials. In addition, they found some interesting identities for those numbers and polynomials.

Carlitz generalized the Bernoulli polynomials to q -Bernoulli numbers in [3, 4, 6, 15, 21, 28, 34, 35] as follows:

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convection about replacing β_q^n by $\beta_{n,q}$, and Kim defined the modified q -Bernoulli numbers as follows:

$$B_{0,q} = \frac{q-1}{\log q}, \quad (qB_q + 1)^n - B_{n,q} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{1}$$

with the usual convection about replacing B_q^n by $B_{n,q}$ (see [18, 21, 22]).

For a given prime number p , \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p denote the ring of p -adic integers, the field of p -adic rational numbers, and the completions of algebraic closure of \mathbb{Q}_p , respectively. The p -adic norm is normalized as $|p|_p = \frac{1}{p}$.

Let $q \in \mathbb{C}_p$ be an indeterminate with $|q-1|_p < p^{-\frac{1}{p-1}}$. Then the q -analogue of number x are defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \rightarrow 1} [x]_q = x$ for each $x \in \mathbb{Z}_p$.

By (1), we know that the power sums of consecutive nonnegative q -integers are given by

$$[0]_q^k + q[1]_q^k + q^2[2]_q^k + \dots + q^{n-1}[n-1]_q^k = \frac{1}{k+1} (B_{k+1,q}(n) - B_{k+1,q}) \quad (n \geq 1, k \geq 0)$$

(see [21, 23, 31, 39]).

For the set of all uniformly differentiable on \mathbb{Z}_p denoted by $UD(\mathbb{Z}_p)$, the bosonic p -adic q -integrals on \mathbb{Z}_p are defined by the Kim as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \quad (\text{see [18-20]}). \tag{2}$$

In [9], the authors defined *type 2 Bernoulli polynomials* as follows:

$$\sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} = \frac{t}{e^t - e^{-t}} e^{xt},$$

and found relations between some special functions or numbers and those polynomials (see [9, 12, 15, 27]). Note that a Witt-type formula of the type 2 Bernoulli polynomials is

$$b_n(x) = \int_{\mathbb{Z}_p} (2y + x + 1)^n d\mu_0(y) \quad (\text{see [9, 12]}),$$

for each nonnegative integer n where

$$\int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)$$

is the bosonic p -adic integral on \mathbb{Z}_p .

As a natural generalization of the type 2 Bernoulli polynomials, Kim et al. defined the *type 2 q -Bernoulli polynomials* by using the bosonic p -adic q -integral on \mathbb{Z}_p as follows:

$$\sum_{n=0}^{\infty} b_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-y} e^{[2y+x+1]_q t} d\mu_q(y) \quad (\text{see [15, 28]}),$$

and found the power sums of consecutive odd positive q -integers are represented by the type 2 q -Bernoulli polynomials as follows:

$$q[1]_q^k + q^3[3]_q^k + \dots + q^{2n-1}[2n-1]_q^k = \frac{1}{2(k+1)} (b_{k+1,q}(2n) - b_{k+1,q}),$$

for each positive integer n and each nonnegative integer k (see [12, Eq. (2.13), Theorem 2.2]).

In the past decade, many symmetric identities of special functions have been found by many authors (see [6, 8, 10–13, 17, 24–26, 28–30, 32, 33, 37]). In particular, Kim et al. [28] obtained some symmetric identities for type 2 q -Bernoulli polynomials under symmetry group S_3 , and Duran et al. [6] investigated some symmetric identities of Carlitz’s generalized twisted q -Bernoulli polynomials under the symmetric group of order 3. He et al. [8] studied some symmetric identities on a sequence of polynomials and derived interesting identities involving Bernoulli numbers and polynomials as particular cases of those identities.

In this paper, we derive symmetric identities for type 2 q -Bernoulli polynomials under the symmetry group S_n of degree n by using the bosonic p -adic q -integral on \mathbb{Z}_p , and the proof methods, which are found by Kim, are also used as good tools in this paper (see [13, 25, 29, 30]). In particular, those identities are generalizations of the results of Kim et al. [28].

2 Symmetric identities for the type 2 q -Bernoulli polynomials

Let $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, and let S_n be the symmetry group of degree n . For each positive integers w_1, \dots, w_n and each nonnegative integers k_1, \dots, k_{n-1} , we consider the following

integral equation for the bosonic p -adic q -integral on \mathbb{Z}_p ;

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} q^{-w_1 w_2 \cdots w_{n-1} y} e^{[2(\prod_{i=1}^{n-1} w_i)y + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{j=1}^{n-1} w_j)k_i]qt} d\mu_{q^{w_1 w_2 \cdots w_{n-1}}}(y) \\
 &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{q^{w_1 \cdots w_{n-1}}}} \sum_{y=0}^{p^N-1} e^{[2(\prod_{i=1}^{n-1} w_i)y + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{j=1}^{n-1} w_j)k_i]qt} \\
 &= \lim_{N \rightarrow \infty} \frac{1}{[w_n p^N]_{q^{w_1 \cdots w_{n-1}}}} \sum_{y=0}^{w_n p^N-1} e^{[2(\prod_{i=1}^{n-1} w_i)y + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{j=1}^{n-1} w_j)k_i]qt} \\
 &= \lim_{N \rightarrow \infty} \frac{1}{[w_n p^N]_{q^{w_1 \cdots w_{n-1}}}} \sum_{l=0}^{w_{n-1} p^{N-1}} \sum_{y=0}^{[2(\prod_{i=1}^{n-1} w_i)(l+w_n y) + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{j=1}^{n-1} w_j)k_i]qt} e \\
 &= [w_1 \cdots w_{n-1}]_q \lim_{N \rightarrow \infty} \frac{1}{[w_1 \cdots w_n p^N]_q} \\
 &\quad \times \sum_{l=0}^{w_{n-1} p^{N-1}} \sum_{y=0}^{[2(\prod_{i=1}^{n-1} w_i)(l+w_n y) + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{j=1}^{n-1} w_j)k_i]qt} e. \tag{3}
 \end{aligned}$$

By (3) we have

$$\begin{aligned}
 & \frac{1}{[w_1 \cdots w_{n-1}]_q} \\
 & \times \prod_{m=1}^{n-1} \sum_{k_m=0}^{w_{m-1}} \int_{\mathbb{Z}_p} q^{-w_1 \cdots w_{n-1} y} e^{[2(\prod_{i=1}^{n-1} w_i)y + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{j=1}^{n-1} w_j)k_i]qt} d\mu_{q^{w_1 \cdots w_{n-1}}}(y) \\
 &= \lim_{N \rightarrow \infty} \frac{1}{[w_1 \cdots w_n p^N]_q} \\
 & \times \prod_{m=1}^n \sum_{k_m=0}^{w_{m-1} p^{N-1}} \sum_{y=0}^{[2(\prod_{i=1}^{n-1} w_i)(k_m+w_n y) + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{j=1}^{n-1} w_j)k_i]qt} e. \tag{4}
 \end{aligned}$$

If we put

$$\begin{aligned}
 & F(w_1, \dots, w_n) \\
 &= \frac{1}{[w_1 \cdots w_{n-1}]_q} \\
 & \times \prod_{m=1}^{n-1} \sum_{k_m=0}^{w_{m-1}} \int_{\mathbb{Z}_p} q^{-w_1 \cdots w_{n-1} y} e^{[2(\prod_{i=1}^{n-1} w_i)y + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{j=1}^{n-1} w_j)k_i]qt} d\mu_{q^{w_1 \cdots w_{n-1}}}(y), \tag{5}
 \end{aligned}$$

then, by (5), we obtain the following theorem.

Theorem 2.1 *Let w_1, w_2, \dots, w_n be positive integers. For each $\sigma \in S_n$, the $F(w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(n)})$ have the same value.*

Note that, by the definition of the q -analogue of the number x ,

$$\begin{aligned} & \left[2 \left(\prod_{i=1}^{n-1} w_i \right) y + \left(\prod_{i=1}^n w_i \right) (x + 1) + 2w_n \sum_{i=1}^{n-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j \right) k_i \right]_q \\ &= \left[\prod_{i=1}^{n-1} w_i \right]_q \left[2y + w_n(x + 1) + 2 \frac{w_n}{w_1} k_1 + \dots + 2 \frac{w_n}{w_{n-1}} k_{n-1} \right]_{q^{w_1 \dots w_{n-1}}} \\ &= \left[\prod_{i=1}^{n-1} w_i \right]_q \left[2y + w_n(x + 1) + 2 \sum_{i=1}^{n-1} \frac{w_n}{w_i} k_i \right]_{q^{w_1 \dots w_{n-1}}} . \end{aligned} \tag{6}$$

By the definition of type 2 q -Bernoulli polynomials and (6), we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{-w_1 w_2 \dots w_{n-1} y} e^{[2(\prod_{i=1}^{n-1} w_i)y + (\prod_{i=1}^n w_i)(x+1) + 2w_n \sum_{i=1}^{n-1} (\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j)k_i]_q t} d\mu_{q^{w_1 w_2 \dots w_{n-1}}}(y) \\ &= \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n-1} w_i \right]_q^m \int_{\mathbb{Z}_p} q^{-w_1 w_2 \dots w_{n-1} y} \left[2y + w_n(x + 1) \right. \\ & \quad \left. + 2 \sum_{i=1}^{n-1} \frac{w_n}{w_i} k_i - 1 + 1 \right]_{q^{w_1 \dots w_{n-1}}}^m d\mu_{q^{w_1 w_2 \dots w_{n-1}}}(y) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left[\prod_{i=1}^{n-1} w_i \right]_q^m b_{m,q^{w_1 \dots w_{n-1}}} \left(w_n(x + 1) + 2 \sum_{i=1}^{n-1} \frac{w_n}{w_i} k_i - 1 \right) \frac{t^m}{m!}, \end{aligned} \tag{7}$$

and so by (7), we know that

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{-w_1 \dots w_{n-1} y} \left[2 \left(\prod_{i=1}^{n-1} w_i \right) y + \left(\prod_{i=1}^n w_i \right) (x + 1) + 2w_n \sum_{i=1}^{n-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j \right) k_i \right]_q^m d\mu_{q^{w_1 \dots w_{n-1}}}(y) \\ &= \left[\prod_{i=1}^{n-1} w_i \right]_q^m b_{m,q^{w_1 \dots w_{n-1}}} \left(w_n(x + 1) + 2 \sum_{i=1}^{n-1} \frac{w_n}{w_i} k_i - 1 \right). \end{aligned} \tag{8}$$

By Theorem 2.1 and (8), we obtain the following theorem.

Theorem 2.2 For each nonnegative integer m and each set of positive integers w_1, w_2, \dots, w_n ,

$$\prod_{m=1}^{n-1} \sum_{k_{\sigma(m)}=0}^{w_{\sigma(m)}-1} \left[\prod_{i=1}^{n-1} w_{\sigma(i)} \right]_q^{m-1} b_{m,q^{w_{\sigma(1)} \dots w_{\sigma(n-1)}}} \left(w_{\sigma(n)}(x + 1) + 2 \sum_{i=1}^{n-1} \frac{w_{\sigma(n)}}{w_{\sigma(i)}} k_{\sigma(i)} - 1 \right)$$

have the same value for any $\sigma \in S_n$.

By the definition of q -analogue of number x , we know that

$$\begin{aligned} & \left[2y + w_n(x + 1) + 2w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} \right]_{q^{w_1 w_2 \dots w_{n-1}}} \\ &= \frac{[w_n]_q}{[\prod_{i=1}^{n-1} w_i]_q} \left[2 \sum_{i=1}^{n-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j \right) k_i \right]_{q^{w_n}} \\ & \quad + q^{2w_n \sum_{i=1}^{n-1} (\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j) k_i} [2y + w_n(x + 1)]_{q^{w_1 w_2 \dots w_{n-1}}}, \end{aligned} \tag{9}$$

and by (9), we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{-w_1 \dots w_{n-1} y} \left[2y + w_n(x + 1) + 2w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} \right]_{q^{w_1 w_2 \dots w_{n-1}}}^m d\mu_{q^{w_1 \dots w_{n-1}}}(y) \\ &= \sum_{l=0}^m \binom{m}{l} \frac{[w_n]_q^{m-l}}{[\prod_{i=1}^{n-1} w_i]_q^{m-l}} \left[2 \sum_{i=1}^{n-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j \right) k_i \right]_{q^{w_n}}^{m-l} q^{2w_n l \sum_{i=1}^{n-1} (\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j) k_i} \\ & \quad \times \int_{\mathbb{Z}_p} q^{-w_1 \dots w_{n-1} y} [2y + w_n(x + 1) - 1 + 1]_{q^{w_1 w_2 \dots w_{n-1}}}^l d\mu_{q^{w_1 \dots w_{n-1}}}(y) \\ &= \sum_{l=0}^m \binom{m}{l} \frac{[w_n]_q^{m-l}}{[\prod_{i=1}^{n-1} w_i]_q^{m-l}} \left[2 \sum_{i=1}^{n-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j \right) k_i \right]_{q^{w_n}}^{m-l} \\ & \quad \times q^{2w_n l \sum_{i=1}^{n-1} (\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j) k_i} b_{l,q^{w_1 \dots w_{n-1}}}(w_n x + w_n - 1). \end{aligned} \tag{10}$$

By Theorem 2.1 and (10), we have

$$\begin{aligned} & \prod_{m=1}^{n-1} \sum_{k_m=0}^{w_m-1} \left[\prod_{i=1}^{n-1} w_i \right]_q^{m-1} \int_{\mathbb{Z}_p} q^{-w_1 \dots w_{n-1} y} \left[2y + w_n(x + 1) \right. \\ & \quad \left. + 2w_n \sum_{i=1}^{n-1} \frac{k_i}{w_i} \right]_{q^{w_1 w_2 \dots w_{n-1}}}^m d\mu_{q^{w_1 \dots w_{n-1}}}(y) \\ &= \prod_{m=1}^{n-1} \sum_{k_m=0}^{w_m-1} \sum_{l=0}^m \left[\prod_{r=1}^{n-1} w_r \right]_q^{l-1} \binom{m}{l} [w_n]_q^{m-l} \left[2 \sum_{i=1}^{n-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j \right) k_i \right]_{q^{w_n}}^{m-l} q^{2w_n l \sum_{i=1}^{n-1} (\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j) k_i} \\ & \quad \times b_{l,q^{w_1 \dots w_{n-1}}}(w_n x + w_n - 1) \\ &= \sum_{l=0}^m \left[\prod_{r=1}^{n-1} w_r \right]_q^{l-1} \binom{m}{l} [w_n]_q^{m-l} R_{m,q^{w_n}}(w_1, \dots, w_{n-1} | 2w_n l) b_{l,q^{w_1 \dots w_{n-1}}}(w_n x + w_n - 1), \end{aligned} \tag{11}$$

where

$$R_{m,q}(w_1, \dots, w_{n-1} | l) = \prod_{m=1}^{n-1} \sum_{k_m=0}^{w_m-1} \left[2 \sum_{i=1}^{n-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j \right) k_i \right]_q^{m-l} q^{l \sum_{i=1}^{n-1} (\prod_{\substack{j=1 \\ j \neq i}}^{n-1} w_j) k_i}.$$

Hence, by (11), we obtain the following theorem.

Theorem 2.3 *For each set of positive integers w_1, \dots, w_n and each nonnegative integer m ,*

$$\sum_{l=0}^m \left[\prod_{r=1}^{n-1} w_{\sigma(r)} \right]_q^{l-1} \binom{m}{l} [w_{\sigma(n)}]_q^{m-l} R_{m,q}^{w_{\sigma(n)}}(w_{\sigma(1)}, \dots, w_{\sigma(n-1)} | 2w_{\sigma(n)} l) \times b_{l,q}^{w_{\sigma(1)} \dots w_{\sigma(n-1)}}(w_{\sigma(n)} x + w_{\sigma(n)} - 1)$$

have the same value for any $\sigma \in S_n$.

3 Conclusions

The Bernoulli numbers and polynomials are very important things in the pure and applied mathematics, and closely connected with the Stirling numbers of the first and second kind, harmonic numbers, the Riemann zeta function, and so on. In addition, the Bernoulli numbers and polynomials can represent the power sums of consecutive integers.

In [15], Kim et al. showed that power sums of consecutive positive odd q -integers can be expressed by means of type 2 q -Bernoulli polynomials as follows:

$$\sum_{l=0}^{n-1} q^{2l+1} [2l+1]_q^m = \frac{1}{m+1} (b_{m+1,q}(2n) - b_{m+1,q}).$$

In this paper, we study that the function $F(w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(n)})$ about the type 2 q -Bernoulli polynomials is invariant for any $\sigma \in S_n$ where S_n is the symmetry group of degree n . By the those invariance, we derive some symmetric identities for the type 2 q -Bernoulli polynomials arising from the bosonic p -adic q -integral on \mathbb{Z}_p .

If we put $n = 3$ or $w_4 = w_5 = \dots = w_n = 1$, then Theorems 2.2 and 2.3 are the results of Kim et al. [28]. In addition, if we put $w_2 = w_3 = \dots = w_n = 1$, then we obtain the equation

$$b_{n,q}(w_1(x+1) - 1) = [w_1]_q^{n-1} \sum_{j=0}^{w_1-1} b_{n,q}^{w_1} \left(x + \frac{1}{w_1} k_1 \right)$$

which is the same result as in [28].

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Authors' contributions

All authors contributed equally to this work; All authors read and approved the final manuscript.

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