New proofs on two recent inequalities for unitarily invariant norms

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Abstract

In this short note, we provide alternative proofs for several recent results due to Audenaert (Oper. Matrices 9:475–479, 2015) and Zou (J. Math. Inequal. 10:1119–1122, 2016; Linear Algebra Appl. 552:154–162, 2019).

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1 Introduction

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. For $A \in \mathbb{M}_n$, the singular values and eigenvalues of A are denoted by $\sigma_i(A)$ and $\lambda_i(A)$, respectively, $i = 1, \ldots, n$. The singular values $\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)$ of a matrix A are the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ arranged in decreasing order and repeated according to multiplicity. The Ky Fan k-norm, a particular unitarily invariant norm, is defined as $\| \cdot \|_{(k)} = \sum_{j=1}^{k} \sigma_j(A)$, $1 \le k \le n$. If A is Hermitian, then all eigenvalues of A are real and ordered as $\lambda_1(A) \ge \cdots \ge \lambda_n(A)$.

Let $A, B \in \mathbb{M}_n$. Bhatia and Kittaneh [8] proved an arithmetic–geometric mean inequality for unitarily invariant norms

$$\big\| A^*B \big\| \le \frac{1}{2} \big\| AA^* + BB^* \big\|. \tag{1}$$

As a generalization of (1), Bhatia and Davis [7] proved that

$$\big\| A^*XB \big\| \le \frac{1}{2} \big\| AA^*X + XBB^* \big\| \tag{2}$$

for $A, X, B \in \mathbb{M}_n$.

Albadawi [3] obtained a stronger version of the Hölder inequality for unitarily invariant norms. Let $A, X, B \in \mathbb{M}_n$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $r \ge 0$. Then

$$\big\| \big| A^*XB \big|^r \big\| \le \big\| \big| AA^*X \big|^{\frac{rp}{2}} \big\|^{\frac{1}{p}} \big\| \big| XBB^* \big|^{\frac{rq}{2}} \big\|^{\frac{1}{q}}, \tag{3}$$

which is a generalization of Horn and Zhan's result [10] (also called the Hölder inequality)

$$\left\| \left| A^*B \right|^r \right\| \le \left\| \left(AA^* \right)^{\frac{rp}{2}} \right\|^{\frac{1}{p}} \left\| \left(BB^* \right)^{\frac{rq}{2}} \right\|^{\frac{1}{q}}. \tag{4}$$

Recently, Audenaert [5] proved that if $A, B \in \mathbb{M}_n$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $r \ge 0$, $\alpha \in [0, 1]$, then

$$\left\| \left| A^*B \right|^r \right\| \le \left\| \left(\alpha AA^* + (1-\alpha)BB^* \right)^{rp/2} \right\|^{\frac{1}{p}} \left\| \left((1-\alpha)AA^* + \alpha BB^* \right)^{rp/2} \right\|^{\frac{1}{q}}, \tag{5}$$

which is a unification of inequalities (1) and (4). By setting $r = 1$ and $p = p' = 2$ in (5) we have

$$\left\| \left| A^*B \right| \right\| \le \left\| \alpha AA^* + (1-\alpha)BB^* \right\|^{\frac{1}{2}} \left\| (1-\alpha)AA^* + \alpha BB^* \right\|^{\frac{1}{2}}. \tag{6}$$

Lin [12] gave a new proof of inequality (6). Zou and Jiang [16] generalized it to the following inequality: Let $A, B, X \in \mathbb{M}_n$ and $q \in [0, 1]$. Then

$$\left\| AXB^* \right\|^2 \le \left\| qA^*AX + (1-q)XB^*B \right\| \left\| (1-q)A^*AX + qXB^*B \right\|. \tag{7}$$

Al-khlyleh and Kittaneh [2, Theorem 2.5] presented an inequality that refines inequality (7) for the particular unitarily invariant norm, Hilbert–Schmidt norm. For more results on interpolation between the arithmetic–geometric mean inequality and the Cauchy–Schwarz inequality for matrices, see [1].

In this paper, we provide alternative proofs of inequalities (5) and (7), which provide new perspectives to the elegant results.

2 Main results
For presenting the new proofs, we need the following several lemmas.

Lemma 2.1 (see [6, Proposition IX.1.2]) *Let $A, B \in \mathbb{M}_n$ be any two matrices such that the product AB is Hermitian. Then, for every unitarily invariant norm, we have*

$$\|AB\| \le \left\| \Re(BA) \right\|. \tag{8}$$

Lemma 2.2 (see [6, p. 41]) *Let $A, B \in \mathbb{M}_n$ and suppose that f is convex and increasing on $[0, \infty)$. If*

$$\sum_{j=1}^{k} \sigma_j(A) \le \sum_{j=1}^{k} \sigma_j(B), \quad k = 1, \dots, n,$$

then

$$\sum_{j=1}^{k} f\left(\sigma_j(A) \right) \le \sum_{j=1}^{k} f\left(\sigma_j(B) \right), \quad k = 1, \dots, n.$$

Lemma 2.3 (see [6, p. 35]) *Let $A, B \in \mathbb{M}_n$. Then*

$$\sum_{j=1}^{k} \sigma_j(A + B) \le \sum_{j=1}^{k} \sigma_j(A) + \sigma_j(B), \quad k = 1, \dots, n.$$

Lemma 2.4 (see [14, p. 63]) *If $A \in \mathbb{M}_n$, then*

$$\lambda_j(\mathfrak{R}A) \le \sigma_j(A), \quad j = 1, \dots, n. \tag{9}$$

Lemma 2.5 (see [4] and [13, p. 228]) *Let $A, B \in \mathbb{M}_n$ be positive semidefinite and $0 \le q \le 1$. Then*

$$\sigma_j\big(A^q B^{1-q}\big) \le \sigma_j\big(qA + (1-q)B\big), \quad j = 1, \dots, n. \tag{10}$$

Audenaert [5] proved the following theorem. We give a different proof of the result.

Theorem 2.6 *Let $A, B \in \mathbb{M}_n$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $r \ge 0$, $\alpha \in [0, 1]$. Then*

$$\left\| \left| A^* B \right|^r \right\| \le \left\| \big(\alpha AA^* + (1-\alpha)BB^*\big)^{\frac{rp}{2}} \right\|^{\frac{1}{p}} \left\| \big((1-\alpha)AA^* + \alpha BB^*\big)^{\frac{rq}{2}} \right\|^{\frac{1}{q}}. \tag{11}$$

Proof By Fan's dominance theorem (see [11, Theorem 1.4]) (11) is equivalent to

$$\left\| \left| A^* B \right|^r \right\|_{(k)} \le \left\| \big(\alpha AA^* + (1-\alpha)BB^*\big)^{\frac{rp}{2}} \right\|_{(k)}^{\frac{1}{p}} \left\| \big((1-\alpha)AA^* + \alpha BB^*\big)^{\frac{rq}{2}} \right\|_{(k)}^{\frac{1}{q}} \tag{12}$$

for $k = 1, \dots, n$.

First, let us show this inequality for the Ky Fan 1-norm, that is, the spectral norm:

$$\begin{aligned}
\left\| \left| A^* B \right|^r \right\|_{(1)}^2 &= \sigma_1^2\big(\left| A^* B \right|^r\big) \\
&= \lambda_{\max}\big(\left| A^* B \right|^{2r}\big) \\
&= \lambda_{\max}^r\big(BB^* AA^*\big) \\
&= \lambda_{\max}^r\big[\big(BB^*\big)^\alpha AA^* \big(BB^*\big)^{1-\alpha}\big] \\
&\le \sigma_1^r\big(\big(BB^*\big)^\alpha AA^* \big(BB^*\big)^{1-\alpha}\big) \\
&= \left\| \big(BB^*\big)^\alpha AA^* \big(BB^*\big)^{1-\alpha} \right\|_{(1)}^r \\
&\le \left\| \big(BB^*\big)^\alpha \big(AA^*\big)^{1-\alpha} \right\|_{(1)}^r \left\| \big(AA^*\big)^\alpha \big(BB^*\big)^{1-\alpha} \right\|_{(1)}^r,
\end{aligned}$$

which means that

$$\sigma_1^2\big(\left| A^* B \right|^r\big) \le \sigma_1^r\big(\big(BB^*\big)^\alpha \big(AA^*\big)^{1-\alpha}\big) \sigma_1^r\big(\big(AA^*\big)^\alpha \big(BB^*\big)^{1-\alpha}\big). \tag{13}$$

Second, using a standard argument via the antisymmetric product (see [5, p. 18]), (13) yields

$$\prod_{j=1}^{k} \sigma_j\big(\left| A^* B \right|^r\big) \le \prod_{j=1}^{k} \sigma_j^{\frac{r}{2}}\big(\big(BB^*\big)^\alpha \big(AA^*\big)^{1-\alpha}\big) \prod_{j=1}^{k} \sigma_j^{\frac{r}{2}}\big(\big(AA^*\big)^\alpha \big(BB^*\big)^{1-\alpha}\big)$$

for $k = 1, \ldots, n$. Since weak log-majorization implies weak majorization (see, [9, p. 174]), by (10) we have

$$\sum_{j=1}^{k} \sigma_j \left(\left| A^* B \right|^r \right) \leq \sum_{j=1}^{k} \sigma_j^{\frac{r}{2}} \left(\left(BB^* \right)^\alpha \left(AA^* \right)^{1-\alpha} \right) \sigma_j^{\frac{r}{2}} \left(\left(AA^* \right)^\alpha \left(BB^* \right)^{1-\alpha} \right)$$

$$\leq \sum_{j=1}^{k} \sigma_j^{\frac{r}{2}} \left((1-\alpha)AA^* + \alpha BB^* \right) \sigma_j^{\frac{r}{2}} \left(\alpha AA^* + (1-\alpha)BB^* \right)$$

for $k = 1, \ldots, n$. The left-hand side is $\| |A^* B|^r \|_{(k)}$. By the Hölder inequality the right-hand side is bounded from above by

$$\left[\sum_{j=1}^{k} \sigma_j^{\frac{rp}{2}} \left((1-\alpha)AA^* + \alpha BB^* \right) \right]^{\frac{1}{p}} \left[\sum_{j=1}^{k} \sigma_j^{\frac{rq}{2}} \left(\alpha AA^* + (1-\alpha)BB^* \right) \right]^{\frac{1}{q}}$$

$$= \left\| \left((1-\alpha)AA^* + \alpha BB^* \right)^{\frac{rp}{2}} \right\|_{(k)}^{\frac{1}{p}} \left\| \left(\alpha AA^* + (1-\alpha)BB^* \right)^{\frac{rq}{2}} \right\|_{(k)}^{\frac{1}{q}}.$$

Thus (12) holds, and so does the conclusion. This completes the proof. $\qquad \square$

In fact, by a similar technique used in the theorem, we may present a new proof of the following result due to Zou [10], which is a unified version of inequalities (2) and (3).

Theorem 2.7 *Let* $A, B, X \in \mathbb{M}_n$ *and* $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1, r \geq \max\{\frac{1}{p}, \frac{1}{q}\}, \alpha \in [0, 1]$. *Then*

$$\left\| \left| A^* XB \right|^{2r} \right\| \leq \left\| \left(\alpha AA^* X + (1-\alpha)XBB^* \right)^{rp} \right\|^{\frac{1}{p}}$$

$$\times \left\| \left((1-\alpha)AA^* X + \alpha XBB^* \right)^{rq} \right\|^{\frac{1}{q}}. \tag{14}$$

Proof There is a subtle difference between the proof of (14) and that of the previous theorem although most techniques are similar. For the readers' convenience, we present the proof simply.

By Fan's dominance theorem (14) is equivalent to

$$\left\| \left| A^* XB \right|^{2r} \right\|_{(k)} \leq \left\| \left(\alpha AA^* X + (1-\alpha)XBB^* \right)^{rp} \right\|_{(k)}^{\frac{1}{p}}$$

$$\times \left\| \left((1-\alpha)AA^* X + \alpha XBB^* \right)^{rq} \right\|_{(k)}^{\frac{1}{q}}$$

for all $k = 1, \ldots, n$.

If X is a positive semidefinite matrix, then for Ky Fan 1-norm, we have

$$\left\| \left| A^* XB \right|^{2r} \right\|_{(1)} = \sigma_1 \left(\left| A^* XB \right|^{2r} \right)$$

$$= \lambda_{\max} \left(\left| A^* XB \right|^{2r} \right)$$

$$= \lambda_{\max}^r \left(B^* X^{\frac{1}{2}} X^{\frac{1}{2}} AA^* X^{\frac{1}{2}} X^{\frac{1}{2}} B \right)$$

$$= \lambda_{\max}^r \left(X^{\frac{1}{2}} BB^* X^{\frac{1}{2}} X^{\frac{1}{2}} AA^* X^{\frac{1}{2}} \right)$$

$$= \lambda_{\max}^r \left(\left(X^{\frac{1}{2}} BB^* X^{\frac{1}{2}} \right)^\alpha X^{\frac{1}{2}} AA^* X^{\frac{1}{2}} \left(X^{\frac{1}{2}} BB^* X^{\frac{1}{2}} \right)^{1-\alpha} \right)$$

$$\le \sigma_1^r \left(\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^\alpha X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{1-\alpha}\right)$$

$$= \left\|\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^\alpha X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{1-\alpha}\right\|_{(1)}^r$$

$$\le \left\|\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^\alpha \left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)^{1-\alpha}\right\|_{(1)}^r$$

$$\times \left\|\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)^\alpha \left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{1-\alpha}\right\|_{(1)}^r,$$

which means that

$$\sigma_1^2\left(\left|A^*XB\right|^r\right) \le \sigma_1^r\left(\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^\alpha \left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)^{1-\alpha}\right)$$

$$\times \sigma_1^r\left(\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)^\alpha \left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{1-\alpha}\right). \tag{15}$$

Using a standard argument via the antisymmetric product (see [5, p. 18]), (15) yields

$$\prod_{j=1}^k \sigma_j^2\left(\left|A^*XB\right|^r\right) \le \prod_{j=1}^k \sigma_j^r\left(\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^\alpha \left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)^{1-\alpha}\right)$$

$$\times \prod_{j=1}^k \sigma_j^r\left(\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)^\alpha \left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{1-\alpha}\right)$$

for $k = 1, \dots, n$. Since weak log-majorization implies weak majorization (see, [9, p. 174]), we have

$$\sum_{j=1}^k \sigma_j^2\left(\left|A^*XB\right|^r\right) \le \sum_{j=1}^k \sigma_j^r\left(\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^\alpha \left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)^{1-\alpha}\right)$$

$$\times \sigma_j^r\left(\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)^\alpha \left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{1-\alpha}\right)$$

$$\le \sum_{j=1}^k \sigma_j^r\left((1-\alpha)\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right) + \alpha\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)\right)$$

$$\times \sigma_j^r\left(\alpha\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right) + (1-\alpha)\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)\right)$$

for $k = 1, \dots, n$. The left-hand side is $\||A^*XB|^{2r}\|_{(k)}$. By the Hölder inequality the right-hand side is bounded from above by

$$\left[\sum_{j=1}^k \sigma_j^{rp}\left((1-\alpha)\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right) + \alpha\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)\right)\right]^{\frac{1}{p}}$$

$$\times \left[\sum_{j=1}^k \sigma_j^{rq}\left(\alpha\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right) + (1-\alpha)\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)\right)\right]^{\frac{1}{q}}$$

$$= \left\|\left((1-\alpha)X^{\frac{1}{2}}AA^*X^{\frac{1}{2}} + \alpha X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{rp}\right\|_{(k)}^{\frac{1}{p}}$$

$$\times \left\|\left(\alpha X^{\frac{1}{2}}AA^*X^{\frac{1}{2}} + (1-\alpha)X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{rq}\right\|_{(k)}^{\frac{1}{q}}.$$

Thus

$$
\left\| \left| A^*XB \right|^{2r} \right\|_{(k)} \le \left\| \left((1-\alpha)X^{\frac{1}{2}}AA^*X^{\frac{1}{2}} + \alpha X^{\frac{1}{2}}BB^*X^{\frac{1}{2}} \right)^{rp} \right\|_{(k)}^{\frac{1}{p}}
$$

$$
\times \left\| \left(\alpha X^{\frac{1}{2}}AA^*X^{\frac{1}{2}} + (1-\alpha)X^{\frac{1}{2}}BB^*X^{\frac{1}{2}} \right)^{rq} \right\|_{(k)}^{\frac{1}{q}}.
$$

Since $(1-\alpha)X^{\frac{1}{2}}AA^*X^{\frac{1}{2}} + \alpha X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}$ and $\alpha X^{\frac{1}{2}}AA^*X^{\frac{1}{2}} + (1-\alpha)X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}$ are Hermitian, since $r \ge \max\{\frac{1}{p}, \frac{1}{q}\}$, the previous inequalities become

$$
\left\| \left| A^*XB \right|^{2r} \right\|_{(k)}
$$

$$
\le \left\| \left(\frac{(1-\alpha)AA^*X + \alpha BB^*X + (1-\alpha)XAA^* + \alpha XBB^*}{2} \right)^{rp} \right\|_{(k)}^{\frac{1}{p}}
$$

$$
\times \left\| \left(\frac{\alpha AA^*X + (1-\alpha)BB^*X + \alpha XAA^* + \alpha XBB^*}{2} \right)^{rq} \right\|_{(k)}^{\frac{1}{q}}
$$

(by (8) and Lemma 2.2)

$$
\le \left\| \left((1-\alpha)AA^*X + \alpha XBB^* \right)^{rp} \right\|_{(k)}^{\frac{1}{p}} \times \left\| \left(\alpha AA^*X + (1-\alpha)XBB^* \right)^{rq} \right\|_{(k)}^{\frac{1}{q}}.
$$

(by Lemmas 2.2 and 2.3)   (16)

Next, we consider the case where X is any matrix. By the singular value decomposition we know that there exist unitary matrices U and V such that $X = UDV^*$, and then by (16) we have

$$
\left\| \left| A^*XB \right|^{2r} \right\|_{(k)} = \left\| \left| A^*UDV^*B \right|^{2r} \right\|_{(k)}
$$

$$
\le \left\| \left(\frac{(1-\alpha)(A^*U)^*(A^*U)D + \alpha(V^*B)(V^*B)^*D}{2} \right. \right.
$$

$$
\left. + \frac{(1-\alpha)D(A^*U)^*(A^*U) + \alpha D(V^*B)(V^*B)^*}{2} \right)^{rp} \right\|
$$

$$
\times \left\| \left(\frac{\alpha(A^*U)^*(A^*U)D + (1-\alpha)(V^*B)(V^*B)^*D}{2} \right. \right.
$$

$$
\left. + \frac{\alpha D(A^*U)^*(A^*U) + (1-\alpha)D(V^*B)(V^*B)^*}{2} \right)^{rq} \right\|_{(k)}^{\frac{1}{q}}
$$

$$
\le \left\| \left((1-\alpha)\left(A^*U\right)^*\left(A^*U\right)D + \alpha D\left(V^*B\right)\left(V^*B\right)^* \right)^{rp} \right\|_{(k)}^{\frac{1}{p}}
$$

$$
\times \left\| \left(\alpha\left(A^*U\right)^*\left(A^*U\right)D + (1-\alpha)D\left(V^*B\right)\left(V^*B\right)^* \right)^{rq} \right\|_{(k)}^{\frac{1}{q}}
$$

$$
\le \left\| \left[U^*\left((1-\alpha)AA^*X + \alpha XBB^* \right)V \right]^{rp} \right\|_{(k)}^{\frac{1}{p}}
$$

$$
\times \left\| \left[U^*\left(\alpha AA^*X + (1-\alpha)XBB^* \right)V \right]^{rq} \right\|_{(k)}^{\frac{1}{q}}
$$

$$
= \left\| \left((1-\alpha)AA^*X + \alpha XBB^* \right)^{rp} \right\|_{(k)}^{\frac{1}{p}}
$$

$$
\times \left\| U^*\left(\alpha AA^*X + (1-\alpha)XBB^* \right)^{rq} \right\|_{(k)}^{\frac{1}{q}},
$$

where the last equality is due to the fact that $\||U_1^* P U_2|^r\| = \||P|^r\|$ for any $P \in \mathbb{M}_n$ and unitary matrices U_1, U_2. This completes the proof. □

Finally, we give an alterative proof of (7) due to Zou and Jiang [16, Theorem 2.1].

Theorem 2.8 *Let* $A, B, X \in \mathbb{M}_n$ *and* $q \in [0, 1]$. *Then*

$$\left\|AXB^*\right\|^2 \leq \left\|qA^*AX + (1-q)XB^*B\right\| \times \left\|(1-q)A^*AX + qXB^*B\right\|.$$

Proof First, consider the special case where A, B, X are Hermitian and $A = B$. Then

$$\begin{aligned}
\left\|AXB^*\right\| = \left\|AXA^*\right\| &= \|AXA\| \\
&\leq \frac{1}{2}\left\|A^2X + XA^2\right\| \quad \left(\text{by (2)}\right) \\
&= \left\|\mathfrak{R}\left(qA^2X + (1-q)XA^2\right)\right\| \\
&\leq \left\|qA^2X + (1-q)XA^2\right\| \quad \left(\text{by (9)}\right) \\
&= \left\|qA^*AX + (1-q)XB^*B\right\|.
\end{aligned} \tag{17}$$

Similarly,

$$\left\|AXB^*\right\| \leq \left\|(1-q)A^*AX + qXB^*B\right\|. \tag{18}$$

Thus by (17) and (18)

$$\left\|AXB^*\right\|^2 \leq \left\|qA^*AX + (1-q)XB^*B\right\| \times \left\|(1-q)A^*AX + qXB^*B\right\|,$$

which is just the desired inequality in this particular case.

Next, consider the more general situation where A and B are Hermitian and X is any matrix. Let

$$T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}.$$

Then by the particular case considered before

$$\begin{aligned}
\left\|TYT^*\right\| = \|TYT\| &\leq \frac{1}{2}\left\|T^2Y + YT^2\right\| \\
&\leq \left\|qT^2Y + (1-q)YT^2\right\|.
\end{aligned} \tag{19}$$

Multiplying out the block-matrices, we have

$$TYT = \begin{pmatrix} 0 & AXB \\ BX^*A & 0 \end{pmatrix},$$

$$\frac{1}{2}T^2Y + \frac{1}{2}YT^2 = \frac{1}{2}\begin{pmatrix} 0 & A^2X + XB^2 \\ B^2X^* + X^*A^2 & 0 \end{pmatrix}.$$

Hence we obtain the following inequality from (19):

$$\left\| \begin{pmatrix} 0 & AXB \\ BX^*A & 0 \end{pmatrix} \right\| \le \frac{1}{2} \left\| \begin{pmatrix} 0 & A^2X + XB^2 \\ B^2X^* + X^*A^2 & 0 \end{pmatrix} \right\|,$$

which means that

$$\|AXB\| \le \frac{1}{2} \left\| A^2X + XB^2 \right\|.$$

So by (17) we have

$$\begin{aligned}
\left\| AXB^* \right\| &= \|AXB\| \\
&\le \frac{1}{2} \left\| A^2X + XB^2 \right\| \\
&= \left\| qA^*AX + (1-q)XB^*B \right\|.
\end{aligned} \tag{20}$$

The following inequality can be proved in exactly the same way:

$$\begin{aligned}
\left\| AXB^* \right\| &= \|AXB\| \\
&\le \left\| (1-q)A^2X + qXB^2 \right\| \\
&= \left\| (1-q)A^*AX + qXB^*B \right\|.
\end{aligned} \tag{21}$$

In this case, from (20) and (21) we have

$$\left\| AXB^* \right\|^2 \le \left\| qA^*AX + (1-q)XB^*B \right\| \times \left\| (1-q)A^*AX + qXB^*B \right\|. \tag{22}$$

Finally, Let $A = UA_1$ and $B = VB_1$ be polar decompositions of A and B. Then

$$A^*AX + XB^*B = A_1^*U^*UA_1X + XB_1^*V^*VB_1 = A_1^2X + XB_1^2,$$

whereas

$$\left\| AXB^* \right\| = \left\| UA_1XB_1^*V^* \right\| = \left\| A_1XB_1^* \right\|.$$

So the theorem follows from inequality (22). This completes the proof. □

Authors' contributions

Both authors contributed almost the same amount of work to the manuscript. Both authors read and approved the final manuscript.

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