


RESEARCH

Open Access



# $(p, q)$ -Generalization of Szász–Mirakjan operators and their approximation properties

Mustafa Kara<sup>1\*</sup> and Nazim I. Mahmudov<sup>1</sup> 

\*Correspondence:

mustafa.kara@emu.edu.tr

<sup>1</sup>Department of Mathematics,  
Eastern Mediterranean University,  
Gazimagusa TRNC, Turkey

## Abstract

We introduce a new modification of  $(p, q)$ -analogue of Szász–Mirakjan operators. Firstly, we give a recurrence relation for the moments of  $(p, q)$ -analogue of Szász–Mirakjan operators and present some explicit formulae for the moments and central moments up to order 4. Next, we obtain quantitative estimates for the convergence in the polynomial weighted spaces. In addition, we give the Voronovskaya theorem for the new  $(p, q)$ -Szász–Mirakjan operators.

**Keywords:**  $(p, q)$ -Integers;  $(p, q)$ -Szász–Mirakjan operators; Weighted approximation

## 1 Introduction

In the last two decades, quantum calculus plays a significant role in the approximation of functions by a positive linear operator. In 1987, Lupaş [1] introduced the Bernstein (rational) polynomials based on the  $q$ -integers. In 1996, Phillips [2] introduced another generalization of Bernstein polynomials based on  $q$ -integers. In [3–23], in the case of  $0 < q < 1$ , many operators have been introduced and examined. Among the most important operators there are  $q$ -Szász operators. In [18–21] the authors constructed and studied different  $q$ -generalizations of Szász–Mirakjan operators in the case  $0 < q < 1$ . In 2012, Mahmudov [24] introduced the  $q$ -Szász operator in the case  $q > 1$  and studied quantitative estimates of convergence in polynomial weighted spaces and gave the Voronovskaya theorem.

In recent years, the rapid rise of  $(p, q)$ -calculus has led to the discovery of new generalizations of Bernstein polynomials containing  $(p, q)$ -integers. In 2015, Mursaleen [25] introduced  $(p, q)$ -Bernstein operators and studied approximation properties based on a Korovkin-type approximation theorem of  $(p, q)$ -Bernstein operators. Also, In 2017, Khan and Lobiyal [26] constructed a  $(p, q)$ -analogue of Lupaş-Bernstein functions. In [27–39], the authors constructed many operators by using  $(p, q)$ -integers and studied their approximation properties. Acar [40] introduced  $(p, q)$ -Szász–Mirakjan operators. In addition, Acar gave a recurrence relation for the moments of these operators. In the same year, H. Sharma and C. Gupta [41] introduced the generalization of the  $(p, q)$ -Szász–Mirakjan Kantorovich operators and examined their approximation properties. In 2017, Mursaleen, AAH Al-Abied, and Alotaibi [42] constructed new Szász–Mirakjan operators based on  $(p, q)$ -calculus and studied weighted approximation and a Voronovskaya-type

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

theorem. Also,  $(p, q)$ -analogues of Szász–Mirakjan–Baskakov operators [18] and Stancu-type Szász–Mirakjan–Baskakov operators [43] were defined, and their approximation properties were investigated. Acar, Agrawal, and Kumar [44] introduced a sequence of  $(p, q)$ -Szász–Mirakjan operators, and their weighted approximation properties were investigated.

## 2 Construction of $K_{l,p,q}$ and moment estimations

We give some basic notations and definitions of the  $(p, q)$ -calculus.

The  $(p, q)$ -integer and  $(p, q)$ -factorial are defined by

$$[l]_{p,q} := \begin{cases} \frac{p^l - q^l}{p - q} & \text{if } 0 < q < p \leq 1, \\ l & \text{if } p = q = 1 \end{cases} \quad \text{for } l \in \mathbb{N}, \text{ and } [0]_{p,q} = 0,$$

$$[l]_{p,q}! := [1]_{p,q}[2]_{p,q} \cdots [l]_{p,q} \quad \text{for } l \in \mathbb{N}, \text{ and } [0]_{p,q}! = 1.$$

For integers  $0 \leq k \leq l$ , the  $(p, q)$ -binomial is defined by

$$\begin{bmatrix} l \\ k \end{bmatrix}_{p,q} := \frac{[l]_{p,q}!}{[k]_{p,q}![l - k]_{p,q}!}.$$

The  $(p, q)$ -derivative  $D_{p,q}g$  of a function  $g(z)$  is defined by

$$(D_{p,q}g)(z) := \frac{g(pz) - g(qz)}{(p - q)z}, \quad z \neq 0, \quad (D_{p,q}g)(0) = g'(0)$$

The product and quotient formulae for the  $(p, q)$ -derivative are as follows:

$$D_{p,q}(f(z)g(z)) = g(pz)D_{p,q}(f(z)) + f(qz)D_{p,q}(g(z)), \tag{1}$$

$$D_{p,q}\left(\frac{f(z)}{g(z)}\right) = \frac{g(pz)D_{p,q}(f(z)) - f(pz)D_{p,q}(g(z))}{g(pz)g(qz)}. \tag{2}$$

It is known that

$$D_{p,q}z^l = [l]_{p,q}z^{l-1}. \tag{3}$$

The  $(p, q)$ -analogues of an exponential function, denoted by  $e_{p,q}(z)$  and  $E_{p,q}(z)$ , are defined by

$$e_{p,q}(z) := \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}} z^k}{[k]_{p,q}!}, \quad E_{p,q}(z) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} z^k}{[k]_{p,q}!},$$

and the  $(p, q)$ -derivatives of  $e_{p,q}(az)$  and  $E_{p,q}(z)$  are

$$D_{p,q}e_{p,q}(az) = ae_{p,q}(apz), \quad D_{p,q}E_{p,q}(az) = ae_{p,q}(aqz).$$

Further, the  $(p, q)$ -power is defined by

$$(z - y)_{p,q}^l = (z - y)(pz - qy)(p^2z - q^2y) \cdots (p^{l-1}z - q^{l-1}y). \tag{4}$$

For any integer  $l$ ,

$$D_{p,q}(z - y)_{p,q}^l = [l]_{p,q}!(pz - y)_{p,q}^{l-1}, \tag{5}$$

and  $D_{p,q}(z - y)_{p,q}^0 = 0$ .

The formula of the  $k$ th  $(p, q)$ -derivative of the polynomial  $(z - y)_{p,q}^l$  is

$$D_{p,q}^k(z - y)_{p,q}^l = p^{\binom{k}{2}} \frac{[l]_{p,q}!}{[l - k]_{p,q}!} (p^k z - y)_{p,q}^{l-k}, \tag{6}$$

where  $l \in \mathbb{Z}_+$  and  $0 \leq k \leq l$ .

The  $(p, q)$ -analogue of the Taylor formulas for any function  $g(z)$  is defined by

$$g(z) = \sum_{k=0}^l (-1)^k q^{-\binom{k}{2}} \frac{(D_{p,q}^k g)(zq^{-k})(z - t)_{p,q}^k}{[k]_{p,q}!}. \tag{7}$$

Let  $C_\beta$  denote the set of all real-valued continuous functions  $g$  on  $[0, \infty)$  such that  $w_\beta g$  is bounded and uniformly continuous on  $[0, \infty)$  endowed with the norm

$$\|g\|_m := \sup_{z \in [0, \infty)} w_\beta |g(z)|,$$

where  $w_0(z) = 1$ , and  $w_\beta(z) = \frac{1}{1+z^\beta}$  for  $\beta \in \mathbb{N}$ .

The corresponding Lipschitz class is given for  $0 < \alpha \leq 2$  by

$$\begin{aligned} \Delta_j^2 g(z) &:= g(z + 2j) - 2g(z + j) + g(z), \\ w_\beta^2(g; \gamma) &:= \sup_{0 < j \leq \gamma} \|\Delta_j^2 g\|, \quad \text{Lip}_\beta^2 \alpha := \{g \in C_\beta : w_\beta^2(g; \gamma) = O(\gamma^\alpha), \gamma \rightarrow 0^+\}. \end{aligned}$$

Now we introduce the  $(p, q)$ -Szász–Mirakjan operator.

**Definition 1** Let  $0 < q < p \leq 1$  and  $l \in \mathbb{N}$ . For  $g : [0, \infty) \rightarrow R$ , we define the  $(p, q)$ -Szász–Mirakjan operator as

$$K_{l,p,q}(g; z) := \sum_{k=0}^l g\left(\frac{p^{l-k}[k]_{p,q}}{[l]_{p,q}}\right) \frac{p^{k(k-l)}}{q^{k(k-1)/2}} \frac{[l]_{p,q}^k z^k}{[k]_{p,q}!} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k} z). \tag{8}$$

It is clear that the operator  $K_{l,p,q}$  is linear and positive. It is known that the moments  $K_{l,p,q}(t^m; z)$  play a fundamental role in the approximation theory of positive operators.

**Lemma 2** Let  $0 < q < p \leq 1$  and  $m \in \mathbb{N}$ . We have the following recurrence formula:

$$K_{l,p,q}(t^{m+1}; z) = \sum_{j=0}^m \binom{m}{j} \frac{zq^j p^{m(l-1)-lj}}{[l]_{p,q}^{m-j}} K_{l,p,q}(t^j; pq^{-1}z). \tag{9}$$

*Proof* According to the definition of  $K_{l,p,q}$  (8), we have

$$\begin{aligned} &K_{l,p,q}(t^{m+1}; z) \\ &= \sum_{k=0}^{\infty} \frac{p^{(l-k)(m+1)} [k]_{p,q}^{m+1} p^{k(k-l)} \frac{[l]_{p,q}^k z^k}{[k]_{p,q}!}}{[l]_{p,q}^{m+1} q^{k(k-1)/2}} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k} z) \\ &= \sum_{k=1}^{\infty} \frac{p^{(l-k)(m+1)} [k]_{p,q}^m p^{k(k-l)} \frac{[l]_{p,q}^{k-1} z^k}{[k-1]_{p,q}!}}{[l]_{p,q}^m q^{k(k-1)/2}} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k} z). \end{aligned}$$

Next, we use the identity  $q[k]_{p,q} + p^k = [k+1]_{p,q}$  to get the desired formula

$$\begin{aligned} &K_{l,p,q}(t^{m+1}; z) \\ &= \sum_{k=0}^{\infty} \frac{p^{(l-k-1)(m+1)} (q[k]_{p,q} + p^k)^m p^{(k+1)(k-l+1)} \frac{[l]_{p,q}^k z^{k+1}}{[k]_{p,q}!}}{[l]_{p,q}^m q^{k(k+1)/2}} e_{p,q}(-[l]_{p,q} p^{k-l+2} q^{-k-1} z) \\ &= \sum_{k=0}^{\infty} \frac{1}{[l]_{p,q}^m} \sum_{j=0}^m \binom{m}{j} q^j [k]_{p,q}^j p^{k(m-j)} p^{(l-k-1)(m+1)} \frac{p^{(k+1)(k-l+1)} [l]_{p,q}^k z^{k+1}}{q^{k(k+1)/2} [k]_{p,q}!} \\ &\quad \times e_{p,q}(-[l]_{p,q} p^{k-l+2} q^{-k-1} z) \\ &= \sum_{j=0}^m \binom{m}{j} \frac{z q^j p^{m(l-1)-lj}}{[l]_{p,q}^{m-j}} \sum_{k=0}^{\infty} \frac{p^{(l-k)j} [k]_{p,q}^j p^{k(k-l)} \frac{[l]_{p,q}^k p^k z^k}{[k]_{p,q}! q^k}}{[l]_{p,q}^j q^{k(k-1)/2}} e_q(-[l]_{p,q} p^{k-l+2} q^{-k-1} z) \\ &= \sum_{j=0}^m \binom{m}{j} \frac{z q^j p^{m(l-1)-lj}}{[l]_{p,q}^{m-j}} K_{l,p,q}(t^j; p q^{-1} z). \quad \square \end{aligned}$$

**Lemma 3** Let  $0 < q < p \leq 1$ ,  $z \in [0, \infty)$ ,  $l \in \mathbb{N}$ , and  $k \geq 0$ . We have the following identities related to the  $(p, q)$ -derivative:

$$\begin{aligned} zD_{p,q} s_{lk}(p, q; z) &= [l]_{p,q} \left( \frac{p^{-k} [k]_{p,q}}{[l]_{p,q}} - z p^{-(l-1)} \right) s_{lk}(p, q; pz), \\ K_{l,p,q}(t^{m+1}; z) &= \frac{z}{[l]_{p,q}^{-(l-1)}} D_{p,q} K_{l,p,q} \left( t^m; \frac{z}{p} \right) + z K_{l,p,q}(t^m; z), \end{aligned} \tag{10}$$

where  $s_{lk}(p, q; z) = \frac{p^{k(k-l)} [l]_{p,q}^k z^k}{q^{k(k-1)/2} [k]_{p,q}!} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k} z)$ .

*Proof* We take the  $(p, q)$ -derivative of  $s_{lk}(p, q; z)$ :

$$\begin{aligned} D_{p,q} s_{lk}(p, q; z) &= \frac{p^{k(k-l)}}{q^{k(k-1)/2} [k]_{p,q}!} D_{p,q} (z^k e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k} z)) \\ &= \frac{p^{k(k-l)}}{q^{k(k-1)/2} [k]_{p,q}!} ([k]_{p,q} z^{k-1} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k} (pz))) \\ &= \frac{p^{k(k-l)}}{q^{k(k-1)/2} [k]_{p,q}!} ((qz)^k [l]_{p,q} p^{k-l+1} q^{-k} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k} (pz))). \end{aligned}$$

Then

$$\begin{aligned} zD_{p,q} s_{lk}(p, q; z) &= p^{-k} [k]_{p,q} \frac{p^{k(k-l)}}{q^{k(k-1)/2}} \frac{(pz)^k}{[k]_{p,q}!} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k}(pz)) \\ &\quad - z[l]_{p,q} p^{-(l-1)} \frac{p^{k(k-l)}}{q^{k(k-1)/2}} \frac{(pz)^k}{[k]_{p,q}!} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k}(pz)) \\ &= p^{-k} [k]_{p,q} s_{lk}(p, q; pz) - z[l]_{p,q} p^{-(l-1)} s_{lk}(p, q; pz) \\ &= [l]_{p,q} \left( \frac{p^{-k} [k]_{p,q}}{[l]_{p,q}} - zp^{-(l-1)} \right) s_{lk}(p, q; pz). \end{aligned}$$

Using the obtained formula and the definition of the operator  $K_{l,p,q}$ , we get the second desired formula:

$$\begin{aligned} zD_{p,q} K_{l,p,q}(t^m; z) &= [l] \sum_{k=0}^{\infty} \left( \frac{p^{(l-k)} [k]_{p,q}}{[l]_{p,q}} \right)^m p^{-(l-1)} \left( \frac{p^{-k} [k]_{p,q}}{p^{-(l-1)} [l]_{p,q}} - z \right) s_{lk}(p, q; pz) \\ &= [l] \sum_{k=0}^{\infty} \left( \frac{[k]_{p,q}}{[l]_{p,q}} \right)^{m+1} p^{(l-k)(m+1)} \frac{p^{-(l-1)}}{p} s_{lk}(p, q; pz) \\ &\quad - [l]_{p,q} p^{-(l-1)} z \sum_{k=0}^{\infty} \left( \frac{[k]_{p,q}}{[l]_{p,q}} \right)^m p^{(l-k)m} s_{lk}(p, q; pz) \\ &= p^{-l} [l]_{p,q} K_{l,p,q}(t^{m+1}; pz) - [l]_{p,q} p^{-(l-1)} z K_{l,p,q}(t^m; pz). \quad \square \end{aligned}$$

**Lemma 4** For  $0 < q < p \leq 1$  and  $l \in \mathbb{N}$ , we have

$$\begin{aligned} K_{l,p,q}(1; z) &= 1, \quad K_{l,p,q}(t; z) = z, \quad K_{l,p,q}(t^2; z) = z^2 + \frac{1}{p^{-(l-1)} [l]} z, \\ K_{l,p,q}(t^3; z) &= z^3 + \frac{2p+q}{p^{-(l-1)} p [l]_{p,q}} z^2 + \frac{1}{p^{-2(l-1)} [l]_{p,q}^2} z, \\ K_{l,p,q}(t^4; z) &= z^4 + \left( \frac{3p^2 + 2qp + q^2}{p^2} \right) \frac{z^3}{p^{-(l-1)} [l]_{p,q}} \\ &\quad + \left( \frac{3p^2 + 3pq + q^2}{p^2} \right) \frac{z^2}{p^{-2(l-1)} [l]_{p,q}^2} + \frac{1}{p^{-3(l-1)} [l]_{p,q}^3} z. \end{aligned}$$

*Proof* From the  $(p, q)$ -Taylor theorem [45] we have

$$\psi_l(t) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}}}{[k]_{p,q}!} (D_{p,q}^k \psi_l)(zq^{-k})(z \ominus t)^k_{p,q}.$$

For  $t = 0$ , having in mind the equalities

$$\begin{aligned} (z)_{p,q}^k &= z^k p^{k(k-1)/2}, \quad D_{p,q}^k e_{p,q}(-[l]_{p,q} p^{-(l-1)} z) \\ &= (-1)^k p^{k(k-1)/2} p^{-(l-1)k} [l]_{p,q}^k e_q(-[l]_{p,q} p^{-(l-1)} p^k z) \end{aligned}$$

for  $\psi_l(z) = e_{p,q}(-[l]_{p,q}p^{-(l-1)}z)$ , we get the formula

$$\begin{aligned} 1 &= \psi_l(0) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} (z)_{p,q}^k}{[k]_{p,q}!} (D_{p,q}^k \varphi_l)(zq^{-k}) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} z^k p^{k(k-l)}}{[k]_{p,q}! q^{k(k-1)/2}} (-1)^k [l]_{p,q}^k e_{p,q}(-[l]_{p,q}p^{-(l-1-k)}q^{-k}z) \\ &= \sum_{k=0}^{\infty} \frac{[l]_{p,q}^k z^k}{[k]_{p,q}! p^{-k(k-1)/2}} \left(\frac{p}{q}\right)^{k(k-1)/2} p^{-(l-1)k} e_{p,q}\left(-[l]_{p,q}p^{-(l-1)}\left(\frac{p}{q}\right)^k z\right), \end{aligned}$$

that is,  $K_{l,p,q}(1; z) = 1$ .

For  $i = 2, 3, 4$ , recurrence formula (10) gives us the following results:

$$\begin{aligned} K_{l,p,q}(t^2; z) &= \frac{z}{p^{-(l-1)}[l]_{p,q}} \left\{ D_{p,q} K_{l,p,q}\left(t; \frac{z}{p}\right) + \frac{[l]_{p,q}}{p^{(l-1)}} K_{l,p,q}(t; z) \right\} \\ &= \frac{z}{p^{-(l-1)}[l]_{p,q}} \left\{ 1 + \frac{[l]_{p,q}}{p^{(l-1)}} z \right\} = \frac{z}{p^{-(l-1)}[l]_{p,q}} + z^2, \\ K_{l,p,q}(t^3; z) &= \frac{z}{p^{-(l-1)}[l]_{p,q}} D_{p,q} K_{l,p,q}\left(t^2; \frac{z}{p}\right) + z K_{l,p,q}(t^2; z) \\ &= \frac{z}{p^{-(l-1)}[l]_{p,q}} \frac{1}{[l]_{p,q} p^{-(l-1)}} + [2]_{p,q} \frac{z}{p} + z \left\{ z^2 + \frac{z}{p^{-(l-1)}[l]_{p,q}} \right\} \\ &= z^3 + \frac{(2p+q)}{p^{-(l-2)}[l]_{p,q}} z^2 + \frac{z}{[l]_{p,q}^2 p^{-2(l-1)}}, \\ K_{l,p,q}(t^4; z) &= \frac{z}{p^{-(l-1)}[l]_{p,q}} D_{p,q} K_{l,p,q}\left(t^3; \frac{z}{p}\right) + z K_{l,p,q}(t^3; z) \\ &= \frac{z}{p^{-(l-1)}[l]_{p,q}} \left\{ \frac{[3]_{p,q} z^2}{p^2} + \frac{[2]_{p,q}(2p+q)}{p p^{-(l-2)}[l]_{p,q}} z + \frac{1}{[l]_{p,q}^2 p^{-2(l-1)}} \right\} \\ &\quad + z \left\{ z^3 + \frac{(2p+q)}{p^{-(l-2)}[l]_{p,q}} z^2 + \frac{z}{[l]_{p,q}^2 p^{-2(l-1)}} \right\} \\ &= z^4 + \left( + \frac{3p^2 + 2qp + q^2}{p^2} \right) \frac{z^3}{p^{-(l-1)}[l]_{p,q}} \\ &\quad + \left( \frac{3p^2 + 3pq + q^2}{p^2} \right) \frac{z^2}{p^{-2(l-1)}[l]_{p,q}^2} + \frac{1}{p^{-3(l-1)}[l]_{p,q}^3} z. \quad \square \end{aligned}$$

**Lemma 5** For every  $z \in [0, \infty)$ , we have

$$K_{l,p,q}(t-z; z) = 0, \tag{11}$$

$$K_{l,p,q}((t-z)^2; z) = \frac{z}{p^{-(l-1)}[l]_{p,q}},$$

$$K_{l,p,q}((t-z)^3; z) = \frac{1}{[l]_{p,q}^2 p^{-2(l-1)}} z + \frac{z^2}{p^{-(l-1)}[l]_{p,q}} \left(\frac{q}{p} - 1\right),$$

$$K_{l,p,q}((t-z)^4; z) = \frac{1}{[l]_{p,q}^3 p^{-3(l-1)}} z + \left(\frac{3pq + q^2 - p^2}{p^2}\right) \frac{z^2}{[l]_{p,q}^2 p^{-2(l-1)}} + \frac{(p-q)^2 z^3}{p^2 p^{-(l-1)}[l]_{p,q}}. \tag{12}$$

*Proof* In fact, we may easily calculate third- and fourth-order central moments as follows:

$$\begin{aligned}
 K_{l,p,q}((t-z)^3; z) &= K_{l,p,q}(t^3; z) - 3zK_{l,p,q}(t^2; z) + 3z^2K_{l,p,q}(t; z) - z^3 \\
 &= \left( z^3 + \frac{2p+q}{[l]_{p,q} p p^{-(l-1)}} z^2 + \frac{1}{[l]_{p,q}^2 p^{-2(l-1)}} z \right) \\
 &\quad - 3z \left( z^2 + \frac{z}{[l]_{p,q} p^{-(l-1)}} \right) + 3z^3 - z^3 \\
 &= \frac{1}{[l]_{p,q}^2 p^{-2(l-1)}} z + \frac{z^2(q-p)}{p p^{-(l-1)} [l]_{p,q}}, \\
 K_{l,p,q}((t-z)^4; z) &= K_{l,p,q}(t^4; z) - 4zK_{l,p,q}(t^3; z) + 6z^2K_{l,p,q}(t^2; z) - 4z^3K_{l,p,q}(t; z) + z^4 \\
 &= z^4 + \left( \frac{3p^2+2qp+q^2}{p^2} \right) \frac{z^3}{p^{-(l-1)} [l]_{p,q}} + \left( \frac{3p^2+3pq+q^2}{p^2} \right) \frac{z^2}{p^{-2(l-1)} [l]^2} + \frac{1}{p^{-3(l-1)} [l]^3} z \\
 &\quad - 4z \left( z^3 + \frac{(2p+q)}{p^{-(l-2)} [l]_{p,q}} z^2 + \frac{z}{[l]_{p,q}^2 p^{-2(l-1)}} \right) + 6z^2 \left( \frac{z}{p^{-(l-1)} [l]_{p,q}} + z^2 \right) - 4z^4 + z^4 \\
 &= \frac{1}{[l]_{p,q}^3 p^{-3(l-1)}} z + \left( \frac{3pq+q^2-p^2}{p^2} \right) \frac{z^2}{[l]_{p,q}^2 p^{-2(l-1)}} + \frac{(p-q)^2 z^3}{p^2 p^{-(l-1)} [l]_{p,q}}. \quad \square
 \end{aligned}$$

*Remark 6* For  $0 < q < p \leq 1$ ,

$$\lim_{l \rightarrow \infty} [l]_{p,q} = 0 \quad \text{or} \quad \frac{1}{p-q}.$$

In our study, we assume that  $q = q_l \in (0, 1)$  and  $p = p_l \in (q, 1]$  are such that

$$\lim_{l \rightarrow \infty} q_l = 1, \quad \lim_{l \rightarrow \infty} p_l = 1,$$

and

$$\lim_{l \rightarrow \infty} q_l^l = 1, \quad \lim_{l \rightarrow \infty} p_l^l = 1.$$

Therefore

$$\lim_{l \rightarrow \infty} [l]_{p_l, q_l} = \infty.$$

For all  $0 < q < p \leq 1$  and  $j \geq 0$ , the  $(p, q)$ -difference operators are defined as

$$\Delta_{p,q}^0 g(z_j) = 0, \quad \Delta_{p,q}^1 g(z_j) = \Delta_{p,q} g(z_j),$$

and

$$\Delta_{p,q}^{k+1} g(z_j) = p^k \Delta_{p,q}^k g(z_{j+1}) - q^k \Delta_{p,q}^k g(z_{j+1}),$$

where  $z_j = \frac{p^{l-j} [j]_{p,q}}{[l]_{p,q}}$ . Using this definition, we can prove the following lemmas.

**Lemma 7** For all  $0 < q < p \leq 1$  and  $j, k \in \mathbb{N} \cup \{0\}$ , we have

$$g[z_j, z_{j+1}, \dots, z_{j+k}] = \frac{p^{-(l-1)k} [l]_{p,q}^k p^{k(k-1)/2} \Delta_{p,q}^k g(z_j)}{p^{-k(2j+k-1)/2} q^{k(2j+k-1)/2} [k]_{p,q}!}.$$

**Lemma 8** For all  $0 < q < p \leq 1$ , we have

$$\Delta_{p,q}^k g(z_0) = g \left[ 0, \frac{1}{p^{-(l-1)} [l]_{p,q}}, \dots, \frac{[k]_{p,q}}{p^{-(l-k)} [l]_{p,q}} \right] \frac{q^{k(k-1)/2} [k]_{p,q}!}{p^{k(k-l)} [l]_{p,q}^k}.$$

**Lemma 9** We have

$$\Delta_{p,q}^k g(z_j) = \sum_{i=0}^k \frac{(-1)^i q^{\frac{i(i-1)}{2}} p^{-i(k-i)}}{p^{\frac{i(i-1)}{2}}} \begin{bmatrix} k \\ i \end{bmatrix}_{p,q} g(z_{j+k-i}).$$

**Lemma 10** The  $(p, q)$ -Szász–Mirakjan operator can be represented as

$$K_{l,p,q}(g; z) = \sum_{k=0}^{\infty} g \left[ 0, \frac{1}{p^{-(l-1)} [l]_{p,q}}, \dots, \frac{[k]_{p,q}}{p^{-(l-k)} [l]_{p,q}} \right] z^k.$$

*Proof* Indeed,

$$\begin{aligned} K_{l,p,q}(g; z) &= \sum_{k=0}^{\infty} g \left( \frac{p^{l-k} [k]_{p,q}}{[l]_{p,q}} \right) \frac{p^{k(k-l)} [l]_{p,q}^k z^k}{q^{k(k-1)/2} [k]_{p,q}!} e_{p,q}(-[l]_{p,q} p^{k-l+1} q^{-k} z) \\ &= \sum_{k=0}^{\infty} g \left( \frac{p^{l-k} [k]_{p,q}}{[l]_{p,q}} \right) \frac{p^{k(k-l)} [l]_{p,q}^k z^k}{q^{k(k-1)/2} [k]_{p,q}!} \sum_{j=0}^{\infty} \frac{(-1)^j [l]_{p,q}^j p^{(k-l+1)j} z^j}{p^{-j(j-1)/2} q^{kj} [j]_{p,q}!} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^k g \left( \frac{p^{l-k+j} [k-j]_{p,q}}{[l]_{p,q}} \right) \frac{p^{(k-j)(k-j-l)}}{q^{(k-j)(k-j-1)/2}} \\ &\quad \times \frac{[l]_{p,q}^{k-j} z^{k-j}}{[k-j]_{p,q}!} \frac{(-1)^j [l]_{p,q}^j p^{(k-j-l+1)j} z^j}{p^{-j(j-1)/2} q^{(k-j)j} [j]_{p,q}!} \\ &= \sum_{k=0}^{\infty} \Delta_{p,q}^k g(z_0) \frac{p^{k(k-l)} [l]_{p,q}^k z^k}{q^{\frac{k(k-1)}{2}} [k]_{p,q}!}. \quad \square \end{aligned}$$

The next result gives an explicit formula for the moments  $K_{l,p,q}(t^m; z)$  in terms of Stirling numbers, which is a  $(p, q)$ -analogue of Becker’s formula; see [46].

**Lemma 11** For  $0 < q < p \leq 1$  and  $m \in \mathbb{N}$ , we have

$$K_{l,p,q}(t^m; z) = \sum_{k=1}^m S_{p,q}(m, k) \frac{z^k}{p^{-(l-1)(m-k)} [l]_{p,q}^{m-k}}, \tag{13}$$

where

$$S_{p,q}(m, k) = \frac{1}{q^{\frac{k(k-1)}{2}} p^{-\frac{k(k-1)}{2}} [k]_{p,q}!} \sum_{j=0}^k \frac{(-1)^j q^{\frac{j(j-1)}{2}}}{p^{-\frac{(k-j)(k-j-1)}{2}}} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} p^{-(k-j-1)m} [k-j]_{p,q}^m$$



are the second-type Stirling polynomials satisfying the equalities

$$\begin{aligned} \mathbb{S}_{p,q}(m+1, j) &= p^{-(j-1)} [j]_{p,q} \mathbb{S}_{p,q}(m, j) + \mathbb{S}_{p,q}(m, j-1), \quad m \geq 0, j \geq 1, \\ \mathbb{S}_{p,q}(0, 0) &= 1, \quad \mathbb{S}_{p,q}(m, 0) = 0, \quad m > 0, \quad \mathbb{S}_{p,q}(m, j) = 0, \quad m < j. \end{aligned} \tag{14}$$

Clearly,  $K_{l,p,q}(t^m; z)$  are polynomials of degree  $m$  without a constant term.

*Proof* Because of  $K_{l,p,q}(t; z) = z$  and  $K_{l,p,q}(t^2; z) = z^2 + \frac{z}{p^{-(l-1)} [l]_{p,q}}$ , representation (13) holds for  $m = 1, 2$  with  $\mathbb{S}_{p,q}(2, 1) = 1, \mathbb{S}_{p,q}(1, 1) = 1$ .

Using mathematical induction, assume (13) to be valued for  $m$ . Then from Lemma 3 we get

$$\begin{aligned} K_{l,p,q}(t^m; z) &= \sum_{k=1}^m \mathbb{S}_{p,q}(m, k) \frac{z^k}{p^{-(l-1)(m-k)} [l]_{p,q}^{m-k}}, \\ K_{l,p,q}(t^{m+1}; pz) &= \frac{zp^l}{[l]_{p,q}} D_{p,q} K_{l,p,q}(t^m; z) + zp K_{l,p,q}(t^m; pz) \\ &= \frac{zp^l}{[l]_{p,q}} \sum_{j=1}^m [j]_{p,q} \mathbb{S}_{p,q}(m, j) \frac{z^{j-1}}{p^{-(l-1)(m-j)} [l]_{p,q}^{m-j}} + zp \sum_{j=1}^m \mathbb{S}_{p,q}(m, j) \frac{(pz)^j}{p^{-(l-1)(m-j)} [l]_{p,q}^{m-j}} \\ &= \frac{1}{p^{-l}} \sum_{j=1}^m [j]_{p,q} \mathbb{S}_{p,q}(m, j) \frac{z^j}{p^{-(l-1)(m-j)} [l]_{p,q}^{m-j+1}} + \sum_{j=1}^m \mathbb{S}_{p,q}(m, j) \frac{(pz)^{j+1}}{p^{-(l-1)(m-j)} [l]_{p,q}^{m-j}} \\ &= \sum_{j=1}^m p^{-(j-1)} [j]_{p,q} \mathbb{S}_{p,q}(m, j) \frac{(zp)^j}{p^{-(l-1)(m-j+1)} [l]_{p,q}^{m-j+1}} + \sum_{j=1}^m \mathbb{S}_{p,q}(m, j) \frac{(pz)^{j+1}}{p^{-(l-1)(m-j)} [l]_{p,q}^{m-j}} \\ &= \frac{zp}{p^{-(l-1)m} [l]_{p,q}^m} \mathbb{S}_{p,q}(m, 1) + (pz)^{m+1} \mathbb{S}_{p,q}(m, m) \\ &\quad + \sum_{j=2}^m (p^{-(j-1)} [j]_{p,q} \mathbb{S}_{p,q}(m, j) + \mathbb{S}_{p,q}(m, j-1)) \frac{(zp)^j}{p^{-(l-1)(m-j+1)} [l]_{p,q}^{m-j+1}}. \quad \square \end{aligned}$$

*Remark 12* For  $p = q = 1$ , formulae (14) become recurrence formulas satisfied by the second-type Stirling numbers from [8].

### 3 $(p, q)$ -Szász–Mirakjan operators in a polynomial weighted space

**Lemma 13** For given any fixed  $\beta \in \mathbb{N} \cup \{0\}$  and  $0 < q < p \leq 1$ , we have

$$\|K_{l,p,q}(1/w_\beta; z)\|_\beta \leq K_1(p, q, \beta), \quad l \in \mathbb{N}, \tag{15}$$

where  $K_1(p, q, \beta)$  are positive constants. Moreover, for every  $g \in C_\beta$ , we have

$$\|K_{l,p,q}(g)\|_\beta \leq K_1(p, q, \beta) \|g\|_\beta, \quad l \in \mathbb{N}. \tag{16}$$

Thus  $K_{l,p,q}$  is a linear positive operator from  $C_\beta$  into  $C_\beta$ .

*Proof* Inequality (15) is obvious for  $\beta = 0$ . Let  $\beta \geq 1$ . Then by (13) we have

$$\begin{aligned} w_\beta(z)M_{l,q}(1/w_\beta; z) &= w_\beta(z)M_{l,q}(1 + z^\beta; z) \\ &= w_\beta(z)M_{l,q}(1; z) + w_\beta(z)M_{l,q}(z^\beta; z) \\ &= w_\beta(z) + w_\beta(z) \sum_{j=1}^\beta S_{p,q}(\beta, j) \frac{z^j}{p^{-(l-1)(\beta-j)} [l]_{p,q}^{\beta-j}} \leq K_1(p, q, \beta), \end{aligned}$$

where  $K_1(p, q, \beta) > 0$  is a constant depending on  $\beta, p$ , and  $q$ . From this (15) follows. Moreover, for every  $g \in C_\beta$ ,

$$\|K_{l,p,q}(g)\|_\beta \leq \|g\|_\beta \|K_{l,p,q}(1/w_\beta)\|_\beta.$$

By applying (15) we obtain

$$\|K_{l,p,q}(g)\|_\beta \leq K_1(p, q, \beta) \|g\|. \quad \square$$

**Lemma 14** *For given any fixed  $\beta \in \mathbb{N} \cup \{0\}$  and  $0 < q < p \leq 1$ , we have*

$$\left\| K_{l,p,q} \left( \frac{(t - \cdot)^2}{w_\beta(t)}; \cdot \right) \right\|_\beta \leq \frac{K_2(p, q, \beta)}{p^{-(l-1)} [l]_{p,q}}, \quad l \in \mathbb{N}, \tag{17}$$

where  $K_2(p, q, \beta)$  are positive constants.

*Proof* Formula (11) imply (17) for  $\beta = 0$ . We have

$$K_{l,p,q} \left( \frac{(t - z)^2}{w_\beta(t)}; z \right) = K_{l,p,q}((t - z)^2; z) + K_{l,p,q}((t - z)^2 t^\beta; z)$$

for  $\beta, l \in \mathbb{N}$ . If  $\beta = 1$ , then we get

$$\begin{aligned} K_{l,p,q}((t - z)^2(1 + t); z) &= K_{l,p,q}((t - z)^2; z) + K_{l,p,q}((t - z)^2 t; z) \\ &= K_{l,p,q}((t - z)^3; z) + (1 + z)K_{l,p,q}((t - z)^2; z), \end{aligned}$$

which by Lemma 5 yields (17) for  $\beta = 1$ .

Let  $\beta \geq 2$ . By applying (13) we get

$$\begin{aligned} &w_\beta(z)K_{l,p,q}((t - z)^2 t^\beta; z) \\ &= w_\beta(z)(K_{l,p,q}(t^{\beta+2}; z) - 2zK_{l,p,q}(t^{\beta+1}; z) + z^2K_{l,p,q}(t^\beta; z)) \\ &= w_\beta(z) \left( z^{\beta+2} + \sum_{j=1}^{\beta+1} S_{p,q}(\beta + 2, j) \frac{z^j}{p^{-(l-1)(\beta+2-j)} [l]_{p,q}^{\beta+2-j}} \right. \\ &\quad \left. - 2z^{\beta+2} - 2 \sum_{j=1}^\beta S_{p,q}(\beta + 1, j) \frac{z^{j+1}}{p^{-(l-1)(\beta+1-j)} [l]_{p,q}^{\beta+1-j}} \right. \\ &\quad \left. + z^{\beta+2} + \sum_{j=1}^{\beta-1} S_{p,q}(\beta, j) \frac{z^{j+2}}{p^{-(l-1)(\beta-j)} [l]_{p,q}^{\beta-j}} \right) \end{aligned}$$

$$\begin{aligned}
 &= w_\beta(z) \left( \sum_{j=2}^\beta \mathbb{S}_{p,q}(\beta + 2, j) - 2\mathbb{S}_{p,q}(\beta + 1, j) + \mathbb{S}_{p,q}(\beta, j - 1) \right) \frac{z^{j+1}}{p^{-(l-1)(\beta+1-j)} [l]_{p,q}^{\beta+1-j}} \\
 &\quad + \mathbb{S}_{p,q}(\beta + 2, 1) \frac{z}{p^{-(l-1)(\beta+1)} [l]_{p,q}^{\beta+1}} + (\mathbb{S}_{p,q}(\beta + 2, 2) - 2\mathbb{S}_{p,q}(\beta + 2, 1)) \frac{z^2}{p^{-(l-1)\beta} [l]_{p,q}^\beta} \\
 &= w_\beta(z) \frac{z}{p^{-(l-1)} [l]_{p,q}} \wp_\beta(p, q; z),
 \end{aligned}$$

where  $\wp_\beta(p, q; z)$  is a polynomial of degree  $\beta$ . Therefore we have

$$w_\beta(z) K_{l,p,q}((t - z)^2 t^\beta; z) \leq K_2(p, q, \beta) \frac{z}{p^{-(l-1)} [l]_{p,q}}. \quad \square$$

In the next theorem, we give an approximation property of  $K_{l,p,q}$ .

**Theorem 15** *Let  $g \in C_p^2$ ,  $0 < q < p \leq 1$ , and  $z \in [0, \infty)$ . There exist positive constants  $K_3(p, q, \beta) > 0$  such that*

$$w_\beta(z) |K_{l,p,q}(g; z) - g(z)| \leq K_3(p, q, \beta) \|g''\| \frac{z}{p^{-(l-1)} [l]_{p,q}}.$$

*Proof* By the Taylor formula

$$g(t) = g(z) + g'(z)(t - z) + \int_z^t \int_z^s g''(u) du ds, \quad g \in C_p^2,$$

we obtain that

$$\begin{aligned}
 &w_\beta(z) |K_{l,p,q}(g; z) - g(z)| \\
 &= w_\beta(z) \left| K_{l,p,q} \left( \int_z^t \int_z^s g''(u) du ds; z \right) \right| \\
 &\leq w_\beta(z) K_{l,p,q} \left( \left| \int_z^t \int_z^s g''(u) du ds \right|; z \right) \\
 &\leq w_\beta(z) K_{l,p,q} \left( \|g''\|_\beta \left| \int_z^t \int_z^s (1 + u^m) du ds \right|; z \right) \\
 &\leq w_\beta(z) \frac{1}{2} \|g''\|_\beta K_{l,p,q}((t - z)^2 (1/w_\beta(z) + 1/w_\beta(t)); z) \\
 &\leq \frac{1}{2} \|g''\|_\beta (K_{l,p,q}((t - z)^2; z) + w_\beta(z) M_{l,q}((t - z)^2 w_\beta(t); z)) \\
 &\leq K_3(p, q, z) \|g''\|_\beta \frac{z}{p^{-(l-1)} [l]_{p,q}}. \quad \square
 \end{aligned}$$

We consider the modified Steklov means

$$g_h(z) := \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2g(z + s + t) - g(z + 2(s + t))] ds dt,$$

which have the following properties:

$$g(z) - g_h(z) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \Delta_{s+t}^2 g(z) ds dt,$$

$$g_h''(z) = h^{-2} (8\Delta_{\frac{h}{2}}^2 g(z) - \Delta_h^2 g(z)),$$

and therefore

$$\|g - g_h\|_\beta \leq \omega_\beta^2(g; h),$$

$$\|g_h''\|_\beta \leq \frac{1}{9h^2} \omega_\beta^2(g; h).$$

We may prove the following so-called direct approximation theorem.

**Theorem 16** For given any  $\beta \in \mathbb{N} \cup \{0\}$ ,  $g \in C_\beta$ ,  $z \in [0, \infty)$ , and  $0 < q < p \leq 1$ , we have

$$w_\beta(z) |K_{l,p,q}(g; z) - g(z)| \leq M_\beta \omega_\beta^2 \left( g; \sqrt{\frac{z}{p^{-(l-1)} [l]_{p,q}}} \right) = M_p \omega_p^2 \left( g; \sqrt{\frac{p^{l-1}(q-p)z}{(q^l - p^l)}} \right).$$

Particularly, if  $\text{Lip}_\beta^\alpha$  for some  $\alpha \in (0, 2]$ , then

$$w_\beta(z) |K_{l,p,q}(g; z) - g(z)| \leq M_\beta \left( \frac{z}{p^{-(l-1)} [l]_{p,q}} \right)^{\frac{\alpha}{2}}.$$

*Proof* For  $g \in C_\beta$  and  $h > 0$ ,

$$|K_{l,p,q}(g; z) - g(z)| \leq |K_{l,p,q}((g - g_h); z) - (g - g_h)(z)| + |K_{l,p,q}(g_h; z) - g_h(z)|,$$

and therefore

$$w_\beta(z) |K_{l,p,q}(g; z) - g(z)| \leq \|g - g_h\|_\beta \left( w_\beta(z) K_{l,p,q} \left( \frac{1}{w_\beta(t)}; z \right) + 1 \right) + K_3(p, q, \beta) \|g_h''\|_\beta \frac{z}{p^{-(l-1)} [l]_{p,q}}.$$

Since  $w_\beta(z) K_{l,p,q} \left( \frac{1}{w_\beta(t)}; z \right) \leq K_1(p, q, \beta)$ , we get that

$$w_\beta(z) |K_{l,p,q}(g; z) - g(z)| \leq l(p, q, \beta) w_\beta^2(g, h) \left( 1 + \frac{z}{h^2 p^{-(l-1)} [l]_{p,q}} \right)$$

Thus choosing  $h = \sqrt{\frac{z}{p^{-(l-1)} [l]_{p,q}}}$ , we complete the proof. □

**Corollary 17** If  $\beta \in \mathbb{N} \cup \{0\}$ ,  $g \in C_\beta$ ,  $0 < q < p \leq 1$ , and  $z \in [0, \infty)$ , then

$$\lim_{l \rightarrow \infty} K_{l,p,q}(g; z) = g(z)$$

uniformly on every  $[c, d]$ ,  $0 \leq c < d$ .

### 4 Convergence of $(p, q)$ -Szász–Mirakjan operators

In [47, Theorem 1] and [48, Theorem 1], Totik and de la Cal investigated the class problem of all continuous functions  $g$  such that  $K_{l,p,q}(g)$  converges to  $g$  uniformly on the whole interval  $[0, \infty)$  as  $l \rightarrow \infty$ . The following theorem is a  $(p, q)$ -analogue of Theorem 1 in [48].

**Theorem 18** *Assume that  $g : [0, \infty) \rightarrow \mathbb{R}$  is either bounded or uniformly continuous. Let*

$$g^*(z) = g(z^2), \quad z \in [0, \infty).$$

*Then, for all  $t > 0$  and  $z \geq 0$ ,*

$$|K_{l,p,q}(g; z) - g(z)| \leq 2\omega\left(g^*; \sqrt{\frac{1}{p^{-(l-1)}[l]_{p,q}}}\right). \tag{18}$$

*Therefore  $K_{l,p,q}(g; z)$  converges to  $g$  uniformly on  $[0, \infty)$  as  $l \rightarrow \infty$  whenever  $g^*$  is uniformly continuous.*

*Proof* By the definition of  $g^*$  we have

$$K_{l,p,q}(g; z) = K_{l,p,q}(g^*(\sqrt{\cdot}); z).$$

Thus we can write

$$\begin{aligned} |K_{l,p,q}(g; z) - g(z)| &= |K_{l,p,q}(g^*(\sqrt{\cdot}); z) - g^*(\sqrt{z})| \\ &= \left| \sum_{k=0}^{\infty} \left( g^*\left(\sqrt{\frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}}}\right) - g^*(\sqrt{z}) \right) s_{l,k}(p, q; z) \right| \\ &\leq \sum_{k=0}^{\infty} \left| \left( g^*\left(\sqrt{\frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}}}\right) - g^*(\sqrt{z}) \right) \right| s_{l,k}(p, q; z) \\ &\leq \sum_{k=0}^{\infty} \omega\left(g^*; \left| \sqrt{\frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}}} - \sqrt{z} \right| \right) s_{l,k}(p, q; z) \\ &\leq \sum_{k=0}^{\infty} \omega\left(g^*; \frac{\left| \sqrt{\frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}}} - \sqrt{z} \right|}{K_{l,p,q}(|\sqrt{\cdot} - \sqrt{z}|; z)} K_{l,p,q}(|\sqrt{\cdot} - \sqrt{z}|; z) \right) s_{l,k}(p, q; z). \end{aligned}$$

Finally, from the inequality

$$\omega(g^*; \alpha\delta) \leq (1 + \alpha)\omega(g^*; \delta), \quad \alpha, \delta \geq 0,$$

we obtain

$$\begin{aligned} |K_{l,p,q}(g; z) - g(z)| &\leq \omega\left(g^*; K_{l,p,q}(|\sqrt{\cdot} - \sqrt{z}|; z)\right) \sum_{k=0}^{\infty} \left( 1 + \frac{\left| \sqrt{\frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}}} - \sqrt{z} \right|}{K_{l,p,q}(|\sqrt{\cdot} - \sqrt{z}|; z)} \right) s_{l,k}(p, q; z) \\ &= 2\omega\left(g^*; K_{l,p,q}(|\sqrt{\cdot} - \sqrt{z}|; z)\right). \end{aligned}$$

To complete the proof, we need to show that for all  $t > 0$  and  $z > 0$ , we have

$$M_{l,q}(|\sqrt{\cdot} - \sqrt{z}|; z) \leq \sqrt{\frac{1}{p^{-(l-1)}[l]}}.$$

Indeed, from the Cauchy–Schwarz inequality it follows that

$$\begin{aligned} &K_{l,p,q}(|\sqrt{\cdot} - \sqrt{z}|; z) \\ &= \sum_{k=0}^{\infty} \left| \sqrt{\frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}}} - \sqrt{z} \right| s_{l,k}(p, q; z) \\ &= \sum_{k=0}^{\infty} \frac{|\frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}} - z|}{\sqrt{\frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}} + \sqrt{z}}} s_{l,k}(p, q; z) \leq \frac{1}{\sqrt{z}} \sum_{k=0}^{\infty} \left| \frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}} - z \right| s_{l,k}(p, q; z) \\ &\leq \frac{1}{\sqrt{z}} \sqrt{\sum_{k=0}^{\infty} \left| \frac{[k]_{p,q}}{p^{-(l-k)}[l]_{p,q}} - z \right|^2 s_{l,k}(q; z)} = \frac{1}{\sqrt{z}} \sqrt{K_{l,p,q}((\cdot - z)^2; z)} \\ &= \frac{1}{\sqrt{z}} \sqrt{\frac{1}{p^{-(l-1)}[l]_{p,q}}} z = \sqrt{\frac{1}{p^{-(l-1)}[l]_{p,q}}}. \quad \square \end{aligned}$$

Our next results is a Voronovskaya-type theorem for  $(p, q)$ -Szász–Mirakjan operators.

**Theorem 19** *Let  $0 < q < p \leq 1$ . For any  $g \in C^2_{\beta}[0, \infty)$ , we have the equality*

$$\lim_{l \rightarrow \infty} [l]_{q_l} (K_{l,p,q}(g; z) - g(z)) = \frac{z}{2} g''(z)$$

for every  $z \in [0, \infty)$ .

*Proof* Let  $z \in [0, \infty)$  be fixed. By the Taylor formula we may write

$$g(t) = g(z) + g'(z)(t - z) + \frac{1}{2} g''(z)(t - z)^2 + r(t; z)(t - z)^2, \tag{19}$$

where  $r(t; z)$  is the Peano form of the remainder,  $r(\cdot; z) \in C_{\beta}$ , and  $\lim_{t \rightarrow z} r(t; z) = 0$ . Applying  $K_{l,p,q}$  to (19), we obtain

$$\begin{aligned} p,q(K_{l,p,q}(g; z) - g(z)) &= g'(z)[l]_{p,q} K_{l,p,q}(t - z; z) \\ &\quad + \frac{1}{2} g''(z)[l]_{p,q} K_{l,p,q}((t - z)^2; z) + [l]_{p,q} K_{l,p,q}(r(t; z)(t - z)^2; z). \end{aligned}$$

Applying the Cauchy–Schwarz inequality, we have

$$K_{l,p,q}(r(t; z)(t - z)^2; z) \leq \sqrt{K_{l,p,q}(r^2(t; z); z)} \sqrt{K_{l,p,q}((t - z)^4; z)}. \tag{20}$$

Obviously,  $r^2(z; z) = 0$ . Then it follows from Corollary 17 that

$$\lim_{l \rightarrow \infty} K_{l,p,q}(r^2(t; z); z) = r^2(z; z) = 0. \tag{21}$$

Now from (20), (21), and Lemma 5 we immediately get

$$\lim_{l \rightarrow \infty} [l]_{p,q} K_{l,p,q}(r(t; z)(t - z)^2; z) = 0.$$

The proof is completed.  $\square$

#### Acknowledgements

The author would like to thank the anonymous referees and the editor for their constructive suggestions on improving the presentation of the paper.

#### Funding

No funding is available.

#### Availability of data and materials

Not applicable.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

#### Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 February 2020 Accepted: 22 April 2020 Published online: 29 April 2020

#### References

1. Lupaş, A.: A  $q$ -analogue of the Bernstein operator. In: Seminar on Numerical and Statistical Calculus. University of Cluj-Napoca, vol. 9, pp. 85–92 (1987)
2. Phillips, G.M.: Bernstein polynomials based on the  $q$ -integers. *Ann. Numer. Math.* **4**, 511–518 (1997)
3. Il'inskiĭ, A., Ostrovska, S.: Convergence of generalized Bernstein polynomials. *J. Approx. Theory* **116**(1), 100–112 (2002)
4. Ostrovska, S.:  $q$ -Bernstein polynomials and their iterates. *J. Approx. Theory* **123**(2), 232–255 (2003)
5. Ostrovska, S.: On the limit  $q$ -Bernstein operators. *Math. Balk.* **18**, 165–172 (2004)
6. Ostrovska, S.: On the Lupaş  $q$ -analogue of the Bernstein operator. *Rocky Mt. J. Math.* **36**, 1615–1629 (2006)
7. Ostrovska, S.: Positive linear operators generated by analytic functions. *Proc. Indian Acad. Sci. Math. Sci.* **117**(4), 485–493 (2007)
8. Goodman, T.N.T., Oruç, H., Phillips, G.M.: Convexity and generalized Bernstein polynomials. *Proc. Edinb. Math. Soc.* **42**(1), 179–190 (1999)
9. Boehme, T.K., Powell, R.E.: Positive linear operators generated by analytic functions. *SIAM J. Appl. Math.* **16**, 510–519 (1968)
10. Mahmudov, N.I.: Korovkin-type theorems and applications. *Cent. Eur. J. Math.* **7**(2), 348–356 (2009)
11. Videnskii, V.S.: On some classes of  $q$ -parametric positive operators. In: *Operator Theory: Advances and Applications*, vol. 158, pp. 213–222. Birkhäuser, Basel (2005)
12. Trif, T.: Meyer–König and Zeller operators based on the  $q$ -integers. *Rev. Anal. Numér. Théor. Approx.* **29**, 221–229 (2000)
13. Doğru, O., Duman, O.: Statistical approximation of Meyer–König and Zeller operators based on  $q$ -integers. *Publ. Math. (Debr.)* **68**, 199–214 (2006)
14. Heping, W.: Properties of convergence for the  $q$ -Meyer–König and Zeller operators. *J. Math. Anal. Appl.* **335**(2), 1360–1373 (2007)
15. Aral, A., Doğru, O.: Bleimann, Butzer, and Hahn operators based on the  $q$ -integers. *J. Inequal. Appl.* **2007**, Article ID 79410 (2007)
16. Mahmudov, N.I., Sabancıgil, P.:  $q$ -Parametric Bleimann Butzer and Hahn operators. *J. Inequal. Appl.* **2008**, Article ID 816367 (2008)
17. Mahmudov, N.I., Sabancıgil, P.: Some approximation properties of  $q$ -parametric BBH operators. *J. Comput. Anal. Appl.* **12**(3), 111–123 (2010)
18. Aral, A., Gupta, V.: The  $q$ -derivative and applications to  $q$ -Szász Mirakjan operators. *Calcolo* **43**(3), 151–170 (2006)
19. Aral, A.: A generalization of Szász–Mirakjan operators based on  $q$ -integers. *Math. Comput. Model.* **47**(9–10), 1052–1062 (2008)
20. Radu, C.: On statistical approximation of a general class of positive linear operators extended in  $q$ -calculus. *Appl. Math. Comput.* (2009). <https://doi.org/10.1016/j.amc.2009.08.023>
21. Mahmudov, N.I.: On  $q$ -parametric Szász–Mirakjan operators. *Mediterr. J. Math.* **7**(3), 297–311 (2010)
22. Agratini, O., Radu, C.: On  $q$ -Baskakov–Mastroianni operators. *Rocky Mt. J. Math.* **42**(3), 773–790 (2012)
23. Mahmudov, N.I.: Statistical approximation of Baskakov and Baskakov–Kantorovich operators based on the  $q$ -integers. *Cent. Eur. J. Math.* **8**(4), 816–826 (2010)
24. Mahmudov, N.I.: Approximation by the  $q$ -Szász–Mirakjan operators. *Abstr. Appl. Anal.* **2012**, Article ID 754217 (2012)

25. Mursaleen, M., Ansari, K.J., Khan, A.: On  $(p, q)$ -analogue of Bernstein operators. *Appl. Math. Comput.* **266**, 874–882 (2015). Erratum: *Appl. Math. Comput.* **278**, 70–71 (2016)
26. Khan, K., Lobiyal, D.K.: Bézier curves based on Lupaş  $(p, q)$ -analogue of Bernstein functions in CAGD. *J. Comput. Appl. Math.* **317**, 458–477 (2017)
27. Mursaleen, M., Ansari, K.J., Khan, A.: Some approximation results by  $(p, q)$ -analogue of Bernstein–Stancu operators. *Appl. Math. Comput.* **264**, 392–402 (2015). Corrigendum: *Appl. Math. Comput.* **269**, 744–746 (2015)
28. Mursaleen, M., Nasiruzzaman, M., Khan, A., Ansari, K.J.: Some approximation results on Bleimann–Butzer–Hahn operators defined by  $(p, q)$ -integers. *Filomat* **30**(3), 639–648 (2016)
29. Mursaleen, M., Nasiuzzaman, M., Nurgali, A.: Some approximation results on Bernstein–Schurer operators defined by  $(p, q)$ -integers. *J. Inequal. Appl.* **2015**, Article ID 249 (2015)
30. Acar, T., Aral, A., Mohiuddine, S.A.: On Kantorovich modifications of  $(p, q)$ -Baskakov operators. *J. Inequal. Appl.* **2016**, Article ID 98 (2016)
31. Acar, T., Aral, A., Mohiuddine, S.A.: Approximation by bivariate  $(p, q)$ -Bernstein–Kantorovich operators. *Iran. J. Sci. Technol., Trans. A, Sci.* (2016). <https://doi.org/10.1007/s40995-016-0045-4>
32. Aral, A., Gupta, V.:  $(p, q)$ -Variant of Szász–beta operators. *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.* **111**, 719–733 (2017). <https://doi.org/10.1007/s13398-016-0325-z>
33. Cai, Q.-B., Zhou, G.: On  $(p, q)$ -analogue of Kantorovich type Bernstein–Stancu–Schurer operators. *Appl. Math. Comput.* **276**, 12–20 (2016)
34. Ilarslan, H.G., Acar, T.: Approximation by bivariate  $(p, q)$ -Baskakov–Kantorovich operators. *Georgian Math. J.* (2016). <https://doi.org/10.1515/gmj-2016-0057>
35. Mursaleen, M., Alotaibi, A., Ansari, K.J.: On a Kantorovich variant of  $(p, q)$ -Szász–Mirakjan operators. *J. Funct. Spaces* **2016**, Article ID 1035253 (2016)
36. Mursaleen, M., Ansari, K.J., Khan, A.: Some approximation results for Bernstein–Kantorovich operators based on  $(p, q)$ -calculus. *Sci. Bull. “Politeh.” Univ. Buchar., Ser. A, Appl. Math. Phys.* **78**(4), 129–142 (2016)
37. Mursaleen, M., Khan, F., Khan, A.: Approximation by  $(p, q)$ -Lorentz polynomials on a compact disk. *Complex Anal. Oper. Theory* **10**(8), 1725–1740 (2016)
38. Acar, T., Aral, A., Mohiuddine, S.A.: On Kantorovich modification of  $(p, q)$ -Bernstein operators. *Iran. J. Sci. Technol., Trans. A, Sci.* (2017). <https://doi.org/10.1007/s40995-017-0154-8>
39. Mursaleen, M., Sarsenbi, A.M., Khan, T.: On  $(p, q)$ -analogue of two parametric Stancu–beta operators. *J. Inequal. Appl.* **2016**, Article ID 190 (2016)
40. Acar, T.:  $(p, q)$ -Generalization of Szász–Mirakjan operators. *Math. Methods Appl. Sci.* **39**(10), 2685–2695 (2016)
41. Sharma, H., Gupta, C.: On  $(p, q)$ -generalization of Szász–Mirakjan Kantorovich operators. *Boll. Unione Mat. Ital.* **8**, 213–222 (2015)
42. Mursaleen, M., Al-Abied, A.A.H., Alotaibi, A.: On  $(p, q)$ -Szász–Mirakjan operators and their approximation properties. *J. Inequal. Appl.* **2017**, Article ID 196 (2017)
43. Acar, T., Mursaleen, M., Mohiuddines, S.A.: Stancu type  $(p, q)$ -Szász–Mirakjan–Baskakov operators. *Commun. Fac. Sci. Univ. Ank. Sér. A1* **67**(1), Article ID 116128 (2018)
44. Acar, T., Agrawal, P.N., Kumar, A.S.: On a modification of  $(p, q)$ -Szász–Mirakjan operators. *Complex Anal. Oper. Theory* **12**, 155–167 (2018)
45. Njionou Sadjang, P.: On the fundamental theorem of  $(p, q)$ -calculus and some  $(p, q)$ -Taylor formulas. *Results Math.* **73**, Article ID 39 (2013). <https://doi.org/10.1007/s00025-018-0783-z>
46. Becker, M.: Global approximation theorems for Szász–Mirakjan and Baskakov operators in polynomial weight spaces. *Indiana Univ. Math. J.* **27**(1), 127–142 (1978)
47. Totik, V.: Uniform approximation by Szász–Mirakjan type operators. *Acta Math. Hung.* **41**, 291–307 (1983)
48. de la Cal, J., Cárcamo, J.: On uniform approximation by some classical Bernstein-type operators. *J. Math. Anal. Appl.* **279**, 625–638 (2003)

Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

---

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)

---