Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some new discrete Hilbert's inequalities involving Fenchel–Legendre transform

A. Hamiaz[1] and W. Abuelela[2,3]*

*Correspondence:
w_abuelela@yahoo.com
[2]Faculty of Engineering Technology and Science, Higher Colleges of Technology, Abu Dhabi, United Arab Emirates
[3]Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt
Full list of author information is available at the end of the article

Abstract
Some new Hilbert-type inequalities involving Fenchel–Legendre transform are introduced. These inequalities give more general forms of some previously proved inequalities.

Keywords: Hilbert inequality; Legendre transform

1 Introduction

The form of the established classical discrete Hilbert-type inequality is given as follows [1]:

If $a_n, b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}. \tag{1}$$

The integral analogue of inequality (1) is given by

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy \leq \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)\,dx\right)^{1/p}\left(\int_0^{\infty} g^{p^*}(x)\,dx\right)^{1/p^*}, \tag{2}$$

unless $f \equiv 0$ or $g \equiv 0$, where $p > 1$, $p^* = p/(p-1)$. The constant $\pi \operatorname{cosec}(\pi/p)$ in (1) and (2) is optimal, see [1].

Inequalities (1) and (2) have many generalizations, see for instance [2–4] and the references therein, these refinements and ameliorations of the original inequality lead to an important development and improvement of many advanced mathematical branches, see for example [5–7].

In [8] the author gave inequalities that can be considered as an extension to inequality (1), containing a series of positive terms as follows.

Theorem 1 *Let $q \geq 1$, $p \geq 1$, and let (a_n) and (b_m) be two positive sequences of real numbers defined for $n = 1, 2, \ldots, k$ and $m = 1, 2, \ldots, r$, where $k, r \in \mathbb{N}$, and define $A_n = \sum_{s=1}^{n} a_s$,*

$B_m = \sum_{t=1}^{m} b_t$. *Then*

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{n+m} \le C(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}, \quad (3)$$

unless (a_n) *or* (b_m) *is null, where* $C(p,q,k,r) = \frac{1}{2}pq\sqrt{kr}$.

In [7], the author gave an improvement of the inequality given in Theorem 1 as follows.

Theorem 2 *Let* $q \ge 1$, $p \ge 1$, *and let* (a_n) *and* (b_m) *be two positive sequences of real numbers defined for* $n = 1,2,\ldots,k$ *and* $m = 1,2,\ldots,r$, *where* $k,r \in \mathbb{N}$, *and define* $A_n = \sum_{s=1}^{n} a_s$, $B_m = \sum_{t=1}^{m} b_t$. *Then, for* $\alpha > 0$,

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{(n^\alpha + m^\alpha)^{1/\alpha}} \le C(p,q,k,r;\alpha)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}, \quad (4)$$

unless (a_n) *or* (b_m) *is null, where* $C(p,q,k,r;\alpha) = (\frac{1}{2})^{1/\alpha}pq\sqrt{kr}$.

In this paper, through Fenchel–Legendre transform and by utilizing Jensen's and Schwarz's inequalities, we give some improvements of the inequalities given in Theorems 1 and 2. In addition, some new Hilbert-type inequalities are obtained alongside some applications.

2 Preliminaries

In this section we introduce the Fenchel–Legendre transform, which will have an important role in later sections. For more details, we refer, for instance, to [9–11].

Definition 1 Let $h : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $h \not\equiv +\infty$, i.e., $\operatorname{dom}(h) = \{x \in \mathbb{R}^n | h(x) < +\infty\} \neq \emptyset$. Then the Fenchel–Legendre transform is defined as follows:

$$\begin{aligned} &h^* : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\} \\ &\quad y \longrightarrow h^*(y) = \sup\bigl\{\langle y,x\rangle - h(x), x \in \operatorname{dom}(h)\bigr\}, \end{aligned} \quad (5)$$

where $\langle \cdot,\cdot \rangle$ denotes the scalar product on $\mathbb{R}^n$. The mapping $h \longrightarrow h^*$ will often be called the conjugate operation.

In addition, the domain of h^*, *i.e.*, $\operatorname{dom}(h^*)$ is the set of slopes of all the affine functions minorizing the function h over $\mathbb{R}^n$.

With more hypotheses on h we can give, in the next corollary, an equivalent formula for (5) called Legendre transform.

Corollary 1 *Let $h:\mathbb{R}^n \longrightarrow \mathbb{R}$ be strictly convex, differentiable, and 1-coercive function. Then*

$$h^*(y) = \langle y, (\nabla h)^{-1}(y)\rangle - h\bigl((\nabla h)^{-1}(y)\bigr)$$

for all $y \in \operatorname{dom}(h^*)$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $\mathbb{R}^n$.

Lemma 1 (Fenchel–Young inequality [11]) *Let h be a function and h^* be its Fenchel–Legendre transform, then*

$$\langle x, y\rangle \le h(x) + h^*(y) \tag{6}$$

for all $x \in \operatorname{dom}(h)$ and $y \in \operatorname{dom}(h^*)$.

Corollary 2 (Jensen's inequality [12, 13]) *Let $\Phi : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on a convex set U, with $x_i \in U$, $i = 1, 2, \dots, n$, and $P_n = \sum_{i=1}^n p_i > 0$ for $p_i \ge 0$, then*

$$\Phi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^n p_i \Phi(x_i). \tag{7}$$

Definition 2 A function Φ is called a submultiplicative function on $[0,\infty)$ if

$$\Phi(xy) \le \Phi(x)\Phi(y) \quad \text{for all } x, y \ge 0.$$

3 Main results

We begin this section by proving the following simple and useful lemma.

Lemma 2 *For x and $y \in \mathbb{R}$. Assume that $x + y \ge 1$, then*

$$\forall \alpha \ge \beta \ge \frac{1}{2}: \quad \bigl(|x|^{\frac{1}{2\beta}} + |y|^{\frac{1}{2\beta}}\bigr)^{\alpha} \ge (x+y)^{\frac{1}{2}}. \tag{8}$$

Proof First, we use $x + y \ge 1$ and $\frac{\alpha}{\beta} \ge 1$ to write $(x+y)^{\frac{1}{2}} \le (x+y)^{\frac{\alpha}{2\beta}}$. Then the well-known inequality $\forall n \ge 1$, $(|x| + |y|)^{\frac{1}{n}} \le |x|^{\frac{1}{n}} + |y|^{\frac{1}{n}}$ gives the result for $\alpha \ge \beta \ge \frac{1}{2}$. □

Theorem 3 *Let $q \ge 1, p \ge 1, \alpha \ge \beta \ge \frac{1}{2}$ and $(a_n)_{1\le n\le k}$, $(b_m)_{1\le m\le r}$ be two positive sequences of real numbers where $k, r \in \mathbb{N}$. Define $A_n = \sum_{s=1}^n a_s$, $B_m = \sum_{t=1}^m b_t$. Then the following inequalities hold:*

$$\begin{aligned}\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^{2p}B_m^{2q}}{h(n)+h^*(m)} \le C_1(p,q)&\left[\sum_{n=1}^{k}(k-n+1)\bigl(A_n^{p-1}a_n\bigr)^2\right]\\ &\times\left[\sum_{m=1}^{r}(r-m+1)\bigl(B_n^{q-1}b_m\bigr)^2\right]\end{aligned} \tag{9}$$

and

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{(|h(n)|^{\frac{1}{2\beta}}+|h^*(m)|^{\frac{1}{2\beta}})^{\alpha}} \le \sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{\sqrt{h(n)+h^*(m)}}$$
$$\le C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}, \tag{10}$$

unless (a_n) *or* (b_m) *is null, where*

$$C_1(p,q)=p^2q^2, \qquad C_2(p,q,k,r)=pq\sqrt{kr}.$$

Proof By exploiting the following inequality [14, 15]

$$\left(\sum_{i=1}^{n} z_i\right)^{\gamma} \le \gamma\sum_{i=1}^{n} z_i\left(\sum_{j=1}^{i} z_j\right)^{\gamma-1},$$

where $z_i \ge 0$ and $\gamma \ge 1$ is a constant, we have

$$A_n^p \le p\left(\sum_{s=1}^{n} a_s\right)A_s^{p-1}, \quad n=1,2,\dots,k, \tag{11}$$

$$B_m^q \le q\left(\sum_{t=1}^{m} b_t\right)B_t^{q-1}, \quad m=1,2,\dots,k. \tag{12}$$

Using (11), (12), and the Schwarz inequality, we observe that

$$A_n^p B_m^q \le pq\sum_{s=1}^{n} a_s A_s^{p-1}\sum_{t=1}^{m} b_t B_t^{q-1}$$
$$\le pq(nm)^{\frac{1}{2}}\left[\sum_{s=1}^{n}\left(a_s A_s^{p-1}\right)^2\right]^{\frac{1}{2}}\left[\sum_{t=1}^{m}\left(b_t B_t^{q-1}\right)^2\right]^{\frac{1}{2}}, \tag{13}$$

squaring both sides of inequality (13) gives

$$A_n^{2p} B_m^{2q} \le p^2q^2mn\left[\sum_{s=1}^{n}\left(a_s A_s^{p-1}\right)^2\right]\left[\sum_{t=1}^{m}\left(b_t B_t^{q-1}\right)^2\right]. \tag{14}$$

Using (6) (for nonnegative real numbers x and y) in (13) and (14) produces

$$A_n^p B_m^q \le pq\left(h(n)+h^*(m)\right)^{\frac{1}{2}}\left[\sum_{s=1}^{n}\left(a_s A_s^{p-1}\right)^2\right]^{\frac{1}{2}}\left[\sum_{t=1}^{m}\left(b_t B_t^{q-1}\right)^2\right]^{\frac{1}{2}}, \tag{15}$$

$$A_n^{2p} B_m^{2q} \le p^2q^2\left(h(n)+h^*(m)\right)\left[\sum_{s=1}^{n}\left(a_s A_s^{p-1}\right)^2\right]\left[\sum_{t=1}^{m}\left(b_t B_t^{q-1}\right)^2\right]. \tag{16}$$

Let us divide both sides of (15) by $(h(n) + h^*(m))^{\frac{1}{2}}$, take the sum over n from 1 to k afterwards and the sum over m from 1 to r subsequently. Besides, we use the Schwarz inequality, and then we interchange the order of the summations (see[14, 15]). We obtain

$$\begin{aligned}
&\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{A_n^pB_m^q}{(h(n)+h^*(m))^{\frac{1}{2}}}\\
&\quad\le pq\left[\sum_{n=1}^{k}\left[\sum_{s=1}^{n}\left(a_sA_s^{p-1}\right)^2\right]^{\frac{1}{2}}\right]\left[\sum_{m=1}^{r}\left[\sum_{t=1}^{m}\left(b_tB_t^{q-1}\right)^2\right]^{\frac{1}{2}}\right]\\
&\quad\le pq\sqrt{kr}\left[\sum_{n=1}^{k}\sum_{s=1}^{n}\left(a_sA_s^{p-1}\right)^2\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\sum_{t=1}^{m}\left(b_tB_t^{q-1}\right)^2\right]^{\frac{1}{2}}\\
&\quad\le pq\sqrt{kr}\left[\sum_{s=1}^{k}\left(a_sA_s^{p-1}\right)^2\left(\sum_{n=s}^{k}1\right)\right]^{\frac{1}{2}}\left[\sum_{t=1}^{r}\left(b_tB_t^{q-1}\right)^2\left(\sum_{m=t}^{r}1\right)\right]^{\frac{1}{2}}\\
&\quad\le pq\sqrt{kr}\left[\sum_{s=1}^{k}\left(a_sA_s^{p-1}\right)^2(k-s+1)\right]^{\frac{1}{2}}\left[\sum_{t=1}^{r}\left(b_tB_t^{q-1}\right)^2(r-t+1)\right]^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{A_n^pB_m^q}{(h(n)+h^*(m))^{\frac{1}{2}}}\\
&\quad\le pq\sqrt{kr}\left[\sum_{n=1}^{k}\left(a_nA_n^{p-1}\right)^2(k-n+1)\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\left(b_mB_m^{q-1}\right)^2(r-m+1)\right]^{\frac{1}{2}}.
\end{aligned}\tag{17}$$

Now apply Lemma 2 on L.H.S. of (17) to obtain (10). To prove (9), divide both sides of (16) by $h(n) + h^*(m)$, take the sum over n from 1 to k afterwards, then the sum over m from 1 to r, and then interchange the order of the summations to obtain

$$\begin{aligned}
&\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{A_n^{2p}B_m^{2q}}{h(n)+h^*(m)}\\
&\quad\le p^2q^2\left[\sum_{n=1}^{k}\sum_{s=1}^{n}\left(a_sA_s^{p-1}\right)^2\right]\left[\sum_{m=1}^{r}\sum_{t=1}^{m}\left(b_tB_t^{q-1}\right)^2\right]\\
&\quad\le p^2q^2\left[\sum_{s=1}^{n}\left(a_sA_s^{p-1}\right)^2\left(\sum_{n=s}^{k}1\right)\right]\left[\sum_{t=1}^{r}\left(b_tB_t^{q-1}\right)^2\left(\sum_{m=t}^{r}1\right)\right]\\
&\quad\le p^2q^2\left[\sum_{s=1}^{n}\left(a_sA_s^{p-1}\right)^2(k-s+1)\right]\left[\sum_{t=1}^{r}\left(b_tB_t^{q-1}\right)^2(r-t+1)\right].
\end{aligned}\tag{18}$$

Therefore,

$$\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{A_n^{2p}B_m^{2q}}{h(n)+h^*(m)}\le p^2q^2\left[\sum_{n=1}^{n}\left(a_nA_n^{p-1}\right)^2(k-n+1)\right]\left[\sum_{m=1}^{r}\left(b_mB_m^{q-1}\right)^2(r-m+1)\right],$$

which is (9). This completes the proof. □

Theorem 4 *Under the hypotheses of Theorem 3, for $\sqrt{n} \in \operatorname{dom}(h)$, $\sqrt{m} \in \operatorname{dom}(h^*)$, the following inequality holds*:

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{h(\sqrt{n})+h^*(\sqrt{m})} \le C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}, \tag{19}$$

unless (a_n) or (b_m) is null, where

$$C_2(p,q,k,r) = pq\sqrt{kr}.$$

Proof By the hypothesis that $\sqrt{n} \in \operatorname{dom}(h)$, $\sqrt{m} \in \operatorname{dom}(h^*)$, inequality (6) gives

$$\sqrt{mn} \le h(\sqrt{n}) + h^*(\sqrt{m}).$$

Complete the proof as we did to obtain inequality (10) in Theorem 3 with appropriate changes. □

Corollary 3 *Let (a_n), (b_m), A_n, and B_m be as defined in Theorem 3. Then the inequalities*

$$\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{A_n^2 B_m^2}{h(n)+h^*(m)} \le \left[\sum_{n=1}^{n}(a_n)^2(k-n+1)\right]\left[\sum_{m=1}^{r}(b_m)^2(r-m+1)\right] \tag{20}$$

and

$$\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{A_n B_m}{(h(n)+h^*(m))^{\frac{1}{2}}} \le \sqrt{kr}\left[\sum_{n=1}^{k}(a_n)^2(k-n+1)\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}(b_m)^2(r-m+1)\right]^{\frac{1}{2}} \tag{21}$$

hold.

Proof Put $p = q = 1$ in (9) and (10). This completes the proof. □

The following theorem treats the further generalization of the inequality obtained in Corollary 3. Furthermore, suppose that Φ and Ψ are nonnegative, convex, and submultiplicative functions on $[0,\infty)$.

Theorem 5 *Let (a_n), (b_m), A_n, and B_m be as defined in Theorem 3, and $(p_n)_{1\le n\le k}$, $(q_m)_{1\le m\le r}$ be positive sequences. Define $P_n = \sum_{s=1}^{n} p_s$, $Q_m = \sum_{t=1}^{m} q_t$. Then the following inequality* *holds*:

$$\begin{aligned}\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{\Phi(A_n)\Psi(B_m)}{(|h(n)|^{\frac{1}{2\beta}}+|h^*(m)|^{\frac{1}{2\beta}})^{\alpha}} &\le \sum_{m=1}^{r}\sum_{n=1}^{k}\frac{\Phi(A_n)\Psi(B_m)}{(h(n)+h^*(m))^{\frac{1}{2}}}\\ &\le M_1(k,r)\left[\sum_{n=1}^{k}\left[p_n\Phi\left(\frac{a_n}{p_n}\right)\right]^2(k-n+1)\right]^{\frac{1}{2}}\\ &\quad\times\left[\sum_{m=1}^{r}\left[q_m\Psi\left(\frac{b_m}{q_m}\right)\right]^2(r-m+1)\right]^{\frac{1}{2}},\end{aligned} \tag{22}$$

where

$$M_1(k,t)=\left[\sum_{n=1}^{k}\left[\Phi\left(\frac{a_n}{P_n}\right)\right]^2\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\left[\Psi\left(\frac{b_m}{Q_m}\right)\right]^2\right]^{\frac{1}{2}}.$$

Proof Using the fact that Φ is a submultiplicative function, we have

$$\begin{aligned}\Phi(A_n)&=\Phi\left(\frac{P_n\sum_{s=1}^{n}p_sa_s/p_s}{\sum_{s=1}^{n}p_s}\right)\\ &\le\Phi(P_n)\Phi\left(\frac{\sum_{s=1}^{n}p_sa_s/p_s}{\sum_{s=1}^{n}p_s}\right),\end{aligned} \tag{23}$$

then by Jensen's and Schwarz's inequalities we have that

$$\begin{aligned}\Phi(A_n)&\le\frac{\Phi(P_n)}{P_n}\sum_{s=1}^{n}p_s\Phi\left(\frac{a_s}{p_s}\right)\\ &\le\frac{\Phi(P_n)}{P_n}n^{\frac{1}{2}}\left[\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2\right]^{\frac{1}{2}};\end{aligned} \tag{24}$$

similarly, we can get

$$\Psi(B_m)\le\frac{\Psi(Q_m)}{Q_n}m^{\frac{1}{2}}\left[\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2\right]^{\frac{1}{2}}. \tag{25}$$

From inequalities (24), (25) and the Fenchel–Young inequality (for nonnegative reals x and y), we have

$$\begin{aligned}\Phi(A_n)\Psi(B_m)\le\big(h(n)+h^*(m)\big)^{\frac{1}{2}}\cdot\frac{\Phi(P_n)}{P_n}\left[\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2\right]^{\frac{1}{2}}\\ \cdot\frac{\Psi(Q_m)}{Q_m}\left[\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2\right]^{\frac{1}{2}}.\end{aligned} \tag{26}$$

Let us divide both sides of (26) by $(h(n)+h^*(m))^{\frac{1}{2}}$, take the sum over n from 1 to k afterwards, then take the sum over m from 1 to r. Additionally, use the Schwarz inequality and then interchange the order of the summations to have

$$\begin{aligned}
\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{\Phi(A_n)\Psi(B_m)}{(h(n)+h^*(m))^{\frac{1}{2}}} &\le \sum_{n=1}^{k}\frac{\Phi(P_n)}{P_n}\left[\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2\right]^{\frac{1}{2}} \\
&\quad\cdot\sum_{m=1}^{r}\frac{\Psi(Q_m)}{Q_m}\left[\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2\right]^{\frac{1}{2}} \\
&\le \left[\sum_{n=1}^{k}\left[\frac{\Phi(P_n)}{P_n}\right]^2\right]^{\frac{1}{2}}\left[\sum_{n=1}^{k}\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2\right]^{\frac{1}{2}} \\
&\quad\times\left[\sum_{m=1}^{r}\left[\frac{\Psi(Q_m)}{Q_m}\right]^2\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2\right]^{\frac{1}{2}}. \qquad (27)
\end{aligned}$$

Now define $M_1(k,r)$ as

$$M_1(k,r)=\left[\sum_{n=1}^{k}\left[\frac{\Phi(P_n)}{P_n}\right]^2\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\left[\frac{\Psi(Q_m)}{Q_m}\right]^2\right]^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}
\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{\Phi(A_n)\Psi(B_m)}{(h(n)+h^*(m))^{\frac{1}{2}}} &\le M_1(k,r)\left[\sum_{n=1}^{k}\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2\right]^{\frac{1}{2}} \\
&\quad\times\left[\sum_{m=1}^{r}\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2\right]^{\frac{1}{2}} \\
&\le M_1(k,r)\left[\sum_{s=1}^{k}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2\left(\sum_{n=s}^{k}1\right)\right]^{\frac{1}{2}} \\
&\quad\times\left[\sum_{t=1}^{r}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2\left(\sum_{m=t}^{r}1\right)\right]^{\frac{1}{2}} \\
&= M_1(k,r)\left[\sum_{s=1}^{k}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2(k-s+1)\right]^{\frac{1}{2}} \\
&\quad\times\left[\sum_{t=1}^{r}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2(r-t+1)\right]^{\frac{1}{2}}. \qquad (28)
\end{aligned}$$

Now apply Lemma 2 on the L.H.S. of (28) to obtain (22). This completes the proof. □

Lemma 3 *Under the hypotheses of Theorem 5, the following inequality holds*:

$$\begin{aligned}
\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{\Phi(A_n)^2\Psi(B_m)^2}{h(n)+h^*(m)} &\le M_2(k,r)\left[\sum_{n=1}^{k}\left[p_n\Phi\left(\frac{a_n}{p_n}\right)\right]^4(k-n+1)\right]^{\frac{1}{2}} \\
&\quad\times\left[\sum_{m=1}^{r}\left[q_m\Psi\left(\frac{b_m}{q_m}\right)\right]^4(r-m+1)\right]^{\frac{1}{2}}, \qquad (29)
\end{aligned}$$

where

$$M_2(k,r)=\left[\sum_{n=1}^{k}\left[n\Phi\left(\frac{a_n}{P_n}\right)\right]^4\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\left[m\Psi\left(\frac{b_m}{Q_m}\right)\right]^4\right]^{\frac{1}{2}}.$$

Proof From inequalities (24), (25) and the Fenchel–Young inequality (for nonnegative reals x and y), we have

$$\begin{aligned}\Phi\left(A_n^2\right)\Psi\left(B_m^2\right) &\le \Phi(A_n)^2\Psi(B_m)^2\\ &\le \left(h(n)+h^*(m)\right)\left[\frac{\Phi(P_n)^2}{P_n^2}\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2\right]\\ &\quad\times\left[\frac{\Psi(Q_m)^2}{Q_m^2}\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2\right].\end{aligned}\tag{30}$$

Now divide both sides of (30) by $h(n)+h^*(m)$, then take the sum over n from 1 to k first and the sum over m from 1 to r, then use the Schwarz inequality to obtain

$$\begin{aligned}\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{\Phi(A_n)^2\Psi(B_m)^2}{h(n)+h^*(m)} &\le \left[\sum_{n=1}^{k}\frac{\Phi(P_n)^2}{P_n^2}\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^2\right]\\ &\quad\times\left[\sum_{m=1}^{r}\frac{\Psi(Q_m)^2}{Q_m^2}\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^2\right]\\ &\le \sum_{n=1}^{k}\frac{\Phi(P_n)^2}{P_n^2}n^{\frac{1}{2}}\left[\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^4\right]^{\frac{1}{2}}\\ &\quad\times\sum_{m=1}^{r}\frac{\Psi(Q_m)^2}{Q_m^2}m^{\frac{1}{2}}\left[\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^4\right]^{\frac{1}{2}}\\ &\le \left[\sum_{n=1}^{k}\frac{\Phi(P_n)^4}{P_n^4}n\right]^{\frac{1}{2}}\left[\sum_{n=1}^{k}\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^4\right]^{\frac{1}{2}}\\ &\quad\times\left[\sum_{m=1}^{r}\frac{\Psi(Q_m)^4}{Q_m^4}m\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^4\right]^{\frac{1}{2}}\\ &\le M_2(k,r)\left[\sum_{n=1}^{k}\sum_{s=1}^{n}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^4\right]^{\frac{1}{2}}\\ &\quad\times\left[\sum_{m=1}^{r}\sum_{t=1}^{m}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^4\right]^{\frac{1}{2}},\end{aligned}\tag{31}$$

where

$$M_2(k,r)=\left[\sum_{n=1}^{k}\left[n\Phi\left(\frac{a_n}{P_n}\right)\right]^4\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\left[m\Psi\left(\frac{b_m}{Q_m}\right)\right]^4\right]^{\frac{1}{2}}.$$

Therefore, if we interchange the order of the summations in (31), we obtain (29). This completes the proof. □

We believe that the inequalities in the next theorem are new to the literature.

Theorem 6 *Under the hypotheses of Theorems 3 and 5, the following inequalities hold*:

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^{2p}B_m^{2q}}{h(n)+h^*(m)} \le C_1(p,q)\times\left[h\left(\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right)+h^*\left(\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right)\right], \tag{32}$$

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^{p}B_m^{q}}{\sqrt{h(n)+h^*(m)}} \le C_2(p,q,k,r)\times\left[h\left(\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right)+h^*\left(\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right)\right]^{\frac{1}{2}}, \tag{33}$$

$$\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{\Phi(A_n)\Psi(B_m)}{(h(n)+h^*(m))^{\frac{1}{2}}} \le M_1(k,r)\times\left[h\left(\sum_{n=1}^{k}\left[p_n\Phi\left(\frac{a_n}{p_n}\right)\right]^2(k-n+1)\right)+h^*\left(\sum_{m=1}^{r}\left[q_m\Psi\left(\frac{b_m}{q_m}\right)\right]^2(r-m+1)\right)\right]^{\frac{1}{2}}, \tag{34}$$

and

$$\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{\Phi(A_n)^2\Psi(B_m)^2}{h(n)+h^*(m)} \le M_2(k,r)\times\left[h\left(\sum_{n=1}^{k}\left[p_n\Phi\left(\frac{a_n}{p_n}\right)\right]^4(k-n+1)\right)+h^*\left(\sum_{m=1}^{r}\left[q_m\Psi\left(\frac{b_m}{q_m}\right)\right]^4(r-m+1)\right)\right]^{\frac{1}{2}}. \tag{35}$$

Proof Using Fenchel–Young inequality (6) in (9), (10), (22), and (29) produces inequalities (32), (33), (34), and (35) respectively. This completes the proof. □

The following theorem deals with slight changes of the inequality given in Theorem 9.

Theorem 7 *Let* $(a_n)_{1\le n\le k}$, $(b_m)_{1\le m\le r}$, $(p_n)_{1\le n\le k}$, *and* $(q_m)_{1\le m\le r}$ *be nonnegative sequences of real numbers where* $k,r\in\mathbb{N}$. *Suppose that* Φ *and* Ψ *are nonnegative, convex, and submultiplicative functions on* $[0,\infty)$. *Let* A_n, B_m *be defined as follows*:

$$A_n=\frac{1}{P_n}\sum_{s=1}^{n}p_sa_s,\qquad B_m=\frac{1}{Q_m}\sum_{t=1}^{m}q_tb_t,$$

where $P_n = \sum_{s=1}^{n} p_s$ *and* $Q_m = \sum_{t=1}^{m} q_t$. *Then*

$$\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{P_nQ_m\Phi(A_n)\Psi(B_m)}{(h(n)+h^*(m))^{\frac{1}{2}}} \le M_3(k,r)\left[\sum_{n=1}^{k}\left[p_n\Phi(a_n)\right]^2(k-n+1)\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}\left[q_m\Psi(b_m)\right]^2(r-m+1)\right]^{\frac{1}{2}}, \tag{36}$$

where

$$M_3(k,r) = \sqrt{kr}\left[\sum_{n=1}^{k}\left[\frac{1}{P_n}\right]^2\right]^{\frac{1}{2}}\left[\sum_{m=1}^{r}\left[\frac{1}{Q_m}\right]^2\right]^{\frac{1}{2}}.$$

Proof Using Jensen's and Schwarz's inequalities, we observe that

$$\begin{aligned}\Phi(A_n) &= \Phi\left(\frac{\sum_{s=1}^{n} p_s a_s}{P_n}\right)\\ &\le \frac{1}{P_n}\sum_{s=1}^{n} p_s\Phi(a_s)\\ &\le \frac{\sqrt{n}}{P_n}\left[\sum_{s=1}^{n}\left(p_s\Phi(a_s)\right)^2\right]^{\frac{1}{2}};\end{aligned} \tag{37}$$

similarly,

$$\Phi(B_n) \le \frac{\sqrt{m}}{Q_m}\left[\sum_{t=1}^{m}\left(q_t\Phi(b_t)\right)^2\right]^{\frac{1}{2}}. \tag{38}$$

The rest of the proof is similar to the proof of Theorems 3 and 5 with suitable changes. □

Corollary 4 *Under the hypotheses of Theorem 7, the following inequality holds*:

$$\sum_{m=1}^{r}\sum_{n=1}^{k}\frac{nm\Phi(A_n)\Psi(B_m)}{(h(n)+h^*(m))^{\frac{1}{2}}} \le \frac{\pi^2\sqrt{kr}}{6}\left[\sum_{n=1}^{k}\left[\Phi(a_n)\right]^2(k-n+1)\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}\left[\Psi(b_m)\right]^2(r-m+1)\right]^{\frac{1}{2}}. \tag{39}$$

Proof To prove this result, take $p_s = q_t = 1$ for all $s \ge 1$, $t \ge 1$, then $P_n = n$, $Q_m = m$ and use the fact that

$$\sum_{n=1}^{k}\left[\frac{1}{n}\right]^2 \le \frac{\pi^2}{6}, \qquad \sum_{m=1}^{r}\left[\frac{1}{m}\right]^2 \le \frac{\pi^2}{6}.$$

□

4 Some applications

In this section we try to show the beauty behind our results. We achieve this by utilizing inequality (10) and inequality (19) through substituting $h(x)$ and $h^*(y)$ by suitable functions. In what follows recall that $\alpha \ge \beta \ge \frac{1}{2}$.

Example 1 We can derive inequality (3) from inequality (19). To attain this purpose, choose $h(x) = \frac{x^2}{2}$; then $h^*(y) = \frac{y^2}{2}$ for $x, y \in \mathbb{R}$ (see [10]), then inequality (19) gives

$$\begin{aligned}\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{h(\sqrt{n}) + h^*(\sqrt{m})} &= 2\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{n+m}\\ &\le C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}}\\ &\quad\times\left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{n+m} &\le \frac{1}{2}C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}}\\ &\quad\times\left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}},\end{aligned} \tag{40}$$

which is inequality (3) as desired.

Example 2 If we take $h(x) = \frac{x^s}{s}$, $s > 1$, then $h^*(y) = \frac{y^t}{t}$, $t > 1$, where $\frac{1}{s} + \frac{1}{t} = 1$ and $x, y \in \mathbb{R}_+$ (see [10]), then inequality (10) gives

$$\begin{aligned}\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{(|h(n)|^{\frac{1}{2\beta}} + |h^*(m)|^{\frac{1}{2\beta}})^\alpha} &= \left(\frac{1}{st}\right)^{\frac{-\alpha}{2\beta}}\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{((tn^s)^{\frac{1}{2\beta}} + (sm^t)^{\frac{1}{2\beta}})^\alpha}\\ &\le C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}}\\ &\quad\times\left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}.\end{aligned} \tag{41}$$

Clearly,

$$\begin{aligned}\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{((tn^s)^{\frac{1}{2\beta}} + (sm^t)^{\frac{1}{2\beta}})^\alpha} &\le \left(\frac{1}{st}\right)^{\frac{\alpha}{2\beta}}C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}}\\ &\quad\times\left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}.\end{aligned} \tag{42}$$

When $\beta = \frac{1}{2\alpha}$, inequality (42) becomes

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{((tn^s)^\alpha + (sm^t)^\alpha)^\alpha} \le \left(\frac{1}{st}\right)^{\alpha^2} C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}. \tag{43}$$

It is obvious that, if $\alpha = \beta = 1$, inequality (42) yields

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{(tn^s)^{\frac{1}{2}} + (sm^t)^{\frac{1}{2}}} \le \left(\frac{1}{st}\right)^{\frac{1}{2}} C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}. \tag{44}$$

If in addition $s = t = 2$, inequality (44) produces

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{n+m} \le \frac{1}{\sqrt{2}} C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}. \tag{45}$$

Example 3 We put $h(x) = e^x$ and $h^*(y) = y\log(y) - y$, see [10], in inequality (10) to get

$$\sum_{n=1}^{k}\sum_{m=1}^{r}\frac{A_n^p B_m^q}{(|e^n|^{\frac{1}{2\beta}} + |m\log(m) - m|^{\frac{1}{2\beta}})^\alpha} \le C_2(p,q,k,r)\left[\sum_{n=1}^{k}(k-n+1)\left(A_n^{p-1}a_n\right)^2\right]^{\frac{1}{2}} \times \left[\sum_{m=1}^{r}(r-m+1)\left(B_m^{q-1}b_m\right)^2\right]^{\frac{1}{2}}. \tag{46}$$

5 Conclusion

Using Fenchel–Young inequality (6) helped in obtaining some inequalities that cover a wide range of Hilbert-type inequalities through choosing the functions $h(x)$ and $h^*(x)$ suitably.

Although the left-hand sides in inequalities (10) and (22) depend on some parameters (α and β), we obtained upper bounds that are free of those parameters. The effect of these parameters appears on the right-hand side only if the chosen functions have some constant component.

Some results proved in this paper are generalizations of previously proved results. For example, inequality (19) is a generalization of inequality (3).

Integral analogues to all results in this paper can be obtained following the same spirit of the proofs mentioned here with slight changes. For instance, the integral version of Theorem 4 has been proved in [16].

Acknowledgements
Not applicable.

Funding
The authors declare that they have received no funding from any funding body.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors have contributed equally. They read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia. [2]Faculty of Engineering Technology and Science, Higher Colleges of Technology, Abu Dhabi, United Arab Emirates. [3]Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 9 April 2019 Accepted: 10 February 2020 Published online: 18 February 2020

References

1. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
2. Zhong, J., Yang, B.: An extension of a multidimensional Hilbert-type inequality. J. Inequal. Appl. **2017**, 78 (2017)
3. Waleed, A.: Short note on Hilbert's inequality. J. Egypt. Math. Soc. **22**, 174–176 (2014)
4. Chen, Q., Yang, B.: A survey on the study of Hilbert-type inequalities. J. Inequal. Appl. **2015**, 302 (2015)
5. Zhao, C.J., Cheung, W.-S.: On Hilbert type inequality. J. Inequal. Appl. **2012**, 145 (2012)
6. Huang, Q., Yang, B.: A multiple Hilbert-type integral inequality with a non-homogeneous kernel. J. Inequal. Appl. **2013**, 73 (2013)
7. Kim, Y.-H.: An improvement of some inequalities similar to Hilbert's inequality. Int. J. Math. Math. Sci. **28**(4), 211–221 (2001)
8. Pachpatte, B.G.: On some new inequalities similar to Hilbert's inequality. J. Math. Anal. Appl. **226**(1), 166–179 (1998)
9. Hiriart-Urruty, J.B., Lemaréchal, C. (eds.): Fundamentals of Convex Analysis Springer, Berlin (2012)
10. Borwein, J., Lewi, A.S.: Convex Analysis and Nonlinear Optimization: Theory and Examples. Springer, Berlin (2010)
11. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Springer, Berlin (2013)
12. Pachpatte, B.G.: Mathematical Inequalities. North-Holland Mathematical Library, vol. 67. Elsevier, Amsterdam (2005)
13. Mitrinović, D.S.: Analytic Inequalities, vol. 1. Springer, Berlin (1970)
14. Nemeth, J.: Generalizations of the Hardy–Littlewood inequality. Acta Sci. Math. **32**(3–4), 295–299 (1971)
15. Pachpatte, B.G.: A note on some series inequalities. Tamkang J. Math. **27**(1), 77–79 (1996)
16. Hamiaz, A., Abuelela, W., Bahaa, G.M.: Integral inequalities of Hilbert's type involving Fenchel–Legendre transform with applications. J. Taibah Univ. Sci. **13**(1), 390–395 (2019)