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Weak convergence of explicit extragradient algorithms for solving equilibirum problems

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Abstract

This paper aims to propose two new algorithms that are developed by implementing inertial and subgradient techniques to solve the problem of pseudomonotone equilibrium problems. The weak convergence of these algorithms is well established based on standard assumptions of a cost bi-function. The advantage of these algorithms was that they did not need a line search procedure or any information on Lipschitz-type bifunction constants for step-size evaluation. A practical explanation for this is that they use a sequence of step-sizes that are updated at each iteration based on some previous iterations. For numerical examples, we discuss two well-known equilibrium models that assist our well-established convergence results, and we see that the suggested algorithm has a competitive advantage over time of execution and the number of iterations.

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1 Introduction

Equilibrium problem (shortly, *EP*) can be considered as a general problem in the sense that it comprises many mathematical models such as variational inequality problems (shortly, *VIP*), optimization problems, fixed point problems, complementarity problems, Nash equilibrium of noncooperative games, saddle point, vector minimization problem and the Kirszbraun problem (see e.g., [1–4]). To the best of our knowledge, the term "equilibrium problem" was initiated in 1992 by Mu and Oettli [5] and has been further strengthened by Blum and Oettli [1]. The equilibrium problem (*EP*) is also seen as the Ky Fan inequality, since Fan [6] gives the first existence result regarding the solution of the *EP*. Many results about the existence of the solution of equilibrium problems have been accomplished and generalized by several authors (e.g., see [7, 8]). One of the most useful research directions in the equilibrium problem theory is to develop the iterative methods to find a numerical solution of the equilibrium problems. The research in this direction is continuing to develop new methods, leading weak convergence to strong convergence, providing modification and extension of existing algorithms which are suitable for a specific subclass of equilibrium problems. In recent years, many methods have been developed to solve equilibrium problems in finite and infinite-dimensional spaces (for instance, [9–20]).

In this direction, two approaches are very well known, one of them is the proximal point method (shortly, *PPM*) [21] and the other one is an auxiliary problem principle [22]. The *PPM* was introduced by Martinet [23] for monotone variational inequality problems, and later it was continued by Rockafellar [24] for monotone operators. Moudafi [21] extended the *PPM* to *EP*s involving monotone bifunction. The *PPM* method is implemented to monotone *EP*s, i.e. the bifunction of an equilibrium problem has to be monotone. Thus, each regularized subproblem becomes strongly monotone, and a unique solution exists. This method will not guarantee the existence of the solution if the bifunction is more general monotone, like pseudomonotone. However, the auxiliary problem principle is based on the idea to develop a new problem that is identical and usually simpler to solve compared to the initial problem. This principle was early established by Cohen [25] for optimization problems and later extended for variational inequality problems [26]. Moreover, Mastroeni [22] uses the auxiliary problem principle for strongly monotone equilibrium problems.

In this paper, we focus on the second direction, including projection methods that are well known and practically easy to implement due to their easier numerical computation. As is well known, the earliest well-known projection method for *VIP*s is the gradient projection method. After that, many other projection methods were developed such as the extragradient method [27], the subgradient extragradient method [28], Popov's extragradient method [29], Tseng's extragradient method [30], projection and contraction schemes [31] and other hybrid and projected gradient methods [32–35]. In recent years, the equilibrium problem theory has become an attractive field for many researchers and a lot of numerical methods for solving equilibrium problems have been developed and analyzed by many authors in Hilbert spaces. Thus, Quoc [20] and Flam [36] extended the extragradient method for equilibrium problems. Recently, Hieu [37] extended the Halpern subgradient extragradient method for variational inequality problem to an equilibrium problem and also many other methods were extended and modified for variational inequality problems to equilibrium problems (see [38, 39]).

On the other hand, let us point out inertial-type algorithms, depending on the heavy ball methods of the two-order time dynamical system, Polyak [40] firstly proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial method is a two-step iterative method, and the next iteration is determined by the use of two previous iterates and it can be considered as a procedure of speeding up an iterative sequence (for more details, see [40, 41]). Various inertial-like algorithms previously developed for special classes of the problem (*EP*) can be found (for instance, in [42–44]). For the problem (*EP*), Moudafi [45] has done work in this direction and proposed a new inertial-type method, namely the second-order differential proximal method. This algorithm can be taken as a combination of the relaxed *PPM* [21] and inertial effect [40]. Recently, another type of inertial algorithm has also been introduced by Chbani and Riahi [46], by choosing a suitable inertial term and incorporating a viscosity-like technique in their algorithm.

This paper proposes two modifications of Algorithm 1 (see [47]) for a class of pseudomonotone equilibrium problems motivated from some recent results (see [28, 48, 49]). These resulting algorithms combine the explicit iterative extragradient method with the subgradient method and the inertial term that is used to speed-up the iterative sequence towrads the solution. The major advantage of these methods is that they are independent of line search procedures and also there is no need to have a prior knowledge of Lipschitz-type constants of a bifunction. Instead of that, they use a sequence of step-sizes which is updated at each iteration, based on some previous iterates. We establish the weak convergence of the resulting algorithm under standard assumptions on a cost bifunction.

We organize the rest of this paper in the following manner: In Sect. 2, we give some definitions and preliminary results that will be used throughout the paper. Section 3 comprises our first subgradient algorithm and provides the weak convergence theorem for the proposed algorithm. Section 4 deals with proposing and analyzing the convergence of the inertial subgradient algorithm, involving a pseudomonotone bifunction. Finally, in Sect. 5, we study the numerical experiments to illustrate the computational performance of our suggested algorithms on test problems, which are modeled from a Nash–Cournot oligopolistic equilibrium model and Nash–Cournot equilibrium models of electricity markets.

2 Preliminaries

Let C be a closed and convex subset of a Hilbert space $\mathbb{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ be the set all positive integers. While $\{x_n\}$ is a sequence in $\mathbb{H}$, we denote the strong convergence by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$ as $n \to \infty$. Also, $[t]_+ = \max\{0, t\}$ and $EP(f, C)$ denote the solution set of the equilibrium problem inside C and p is an element of $EP(f, C)$.

Definition 2.1 (Equilibrium problem [1]) Let C be a nonempty closed convex subset of $\mathbb{H}$. Let f be a bifunction from $C \times C$ to the set of real numbers $\mathbb{R}$ such that $f(x, x) = 0$ for all $x \in C$. The equilibrium problem (EP) for the bifunction f on C is to

$$\text{Find } p \in C \quad \text{such that} \quad f(p, y) \geq 0, \quad \forall y \in C.$$

Definition 2.2 ([50]) Let C be a closed convex subset in $\mathbb{H}$ and we denote the metric projection on C by $P_C(x)$, $\forall x \in \mathbb{H}$, i.e.

$$P_C(x) = \arg\min\bigl\{\|y - x\| : y \in C\bigr\}.$$

Lemma 2.1 ([51]) *Let $P_C : \mathbb{H} \to C$ be the metric projection from $\mathbb{H}$ onto C. Then*

(i) *For all $x \in C$, $y \in \mathbb{H}$,*

$$\bigl\|x - P_C(y)\bigr\|^2 + \bigl\|P_C(y) - y\bigr\|^2 \leq \|x - y\|^2.$$

(ii) *$z = P_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \leq 0.$$

Now, we define concepts of monotonicity for a bifunction (see [1, 52] for more details).

Definition 2.3 A bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is said to be

(i) *strongly monotone* on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2, \quad \forall x, y \in C;$$

(ii) *monotone* on C if

$$f(x,y)+f(y,x)\le 0, \quad \forall x,y\in C;$$

(iii) *strongly pseudomonotone* on C if there exists a constant $\gamma>0$ such that

$$f(x,y)\ge 0 \quad\Longrightarrow\quad f(y,x)\le -\gamma\|x-y\|^2, \quad \forall x,y\in C;$$

(iv) *pseudomonotone* on C if

$$f(x,y)\ge 0 \quad\Longrightarrow\quad f(y,x)\le 0, \quad \forall x,y\in C;$$

(v) *a Lipschitz-type condition* on C if there exist two positive constants c_1, c_2 such that

$$f(x,z)\le f(x,y)+f(y,z)+c_1\|x-y\|^2+c_2\|y-z\|^2, \quad \forall x,y,z\in C.$$

Remark 2.1 From Definition 2.3, the following implications hold:

$$\text{(i)}\Longrightarrow\text{(ii)}\Longrightarrow\text{(iv)} \quad \text{and} \quad \text{(i)}\Longrightarrow\text{(iii)}\Longrightarrow\text{(iv)}.$$

Remark 2.2 The converse of the above implications is not true in general.

Remark 2.3 If $F:C\to\mathbb{H}$ is a Lipschitz continuous operator, then the bifunction $f(x,y)=\langle F(x),y-x\rangle$ satisfies Lipschitz-type condition with $c_1=c_2=\frac{L}{2}$ (see [53], Lemma 6(i)).

Further, we recall that the *subdifferential* of a convex function $g:C\to\mathbb{R}$ at $x\in C$ is defined by

$$\partial g(x)=\bigl\{w\in C: g(y)-g(x)\ge\langle w,y-x\rangle,\forall y\in C\bigr\},$$

and the *normal cone* of C at $x\in C$ is defined by

$$N_C(x)=\bigl\{w\in\mathbb{H}:\langle w,y-x\rangle\le 0,\forall y\in C\bigr\}.$$

Lemma 2.2 ([54], p. 97) *Let C be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$ and $g:C\to\mathbb{R}$ be a convex, subdifferentiable, lower semicontinuous function on C. Then z is a solution to the following convex optimization problem* $\min\{g(x):x\in C\}$ *if and only if $0\in\partial g(z)+N_C(z)$, where $\partial g(z)$ and $N_C(z)$ denote the subdifferential of g at z and the normal cone of C at z, respectively.*

Lemma 2.3 ([55], p. 31) *For all $x,y\in\mathbb{H}$ with $\mu\in\mathbb{R}$ the following relation holds*:

$$\bigl\|\mu x+(1-\mu)y\bigr\|^2=\mu\|x\|^2+(1-\mu)\|y\|^2-\mu(1-\mu)\|x-y\|^2.$$

Lemma 2.4 ([56]) *Let ϕ_n, δ_n and β_n be sequences in* $[0,+\infty)$ *such that*

$$\phi_{n+1}\le\phi_n+\beta_n(\phi_n-\phi_{n-1})+\delta_n, \quad \forall n\ge 1, \sum_{n=1}^{+\infty}\delta_n<+\infty,$$

and there exists a real number β with $0 \leq \beta_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Then the following relations hold:

(i) $\sum_{n=1}^{+\infty}[\phi_n - \phi_{n-1}]_+ < \infty$, *where* $[t]_+ := \max\{t, 0\}$.

(ii) *There exists* $\phi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \phi_n = \phi^*$.

Lemma 2.5 ([57]) *Let C be a nonempty set of $\mathbb{H}$ and $\{x_n\}$ be a sequence in $\mathbb{H}$ such that the following two conditions hold*:

(i) *For every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists.*

(ii) *Every sequentially weak cluster point of $\{x_n\}$ is in C.*

Then $\{x_n\}$ converges weakly to a point in C.

Assumption 2.1 We have the following assumptions on the bifunction $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ which are useful to prove the weak convergence of the iterative sequence $\{x_n\}$ generated by our proposed algorithms.

(A_1) $f(x,x) = 0$, $\forall x \in C$ and f is pseudomonotone on C.

(A_2) f satisfies the Lipschitz-type conditions on $\mathbb{H}$ with two constants c_1 and c_2.

(A_3) $\lim_{n\to\infty} \sup f(x_n, y) \leq f(z, y)$ for each $y \in C$ and $\{x_n\} \subset C$ with $x_n \rightharpoonup z$.

(A_4) $f(x,\cdot)$ is convex and subdifferentiable on C for every fixed $x \in C$.

3 Subgradient explicit iterative algorithm for a class of pseudomonotone *EP*

In this section, we suggest our first algorithm for finding a solution to a pseudomonotone problem (*EP*). This algorithm comprises two convex optimization problems with a subgradient technique, used to make the computation easier, the so-called "subgradient explicit iterative algorithm" for a class of pseudomonotone *EP*. The detailed algorithm is given below.

Remark 3.1 From the definition of λ_n, we can see that this sequence is bounded, non-increasing, and converges to some positive number $\lambda > 0$ (for more details see [47]).

Remark 3.2 It is definite that H_n is a half-space and $C \subset H_n$ (see [37]). If we restrict our constraint set to C in the above convex minimization problem then we have the same algorithm (see Algorithm 1 [47]).

Lemma 3.1 *From Algorithm 1, we have the following useful inequality*:

$$\lambda_n f(y_n, y) - \lambda_n f(y_n, x_{n+1}) \geq \langle x_n - x_{n+1}, y - x_{n+1}\rangle, \quad \forall y \in H_n.$$

Proof It follows from Lemma 2.2 and the definition of x_{n+1} that we have

$$0 \in \partial_2 \left\{ \lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 \right\}(x_{n+1}) + N_{H_n}(x_{n+1}).$$

Thus, for $\upsilon \in \partial f(y_n, x_{n+1})$ there exists $\overline{\upsilon} \in N_{H_n(}x_{n+1})$ such that

$$\lambda_n \upsilon + x_{n+1} - x_n + \overline{\upsilon} = 0,$$

which implies that

$$\langle x_n - x_{n+1}, y - x_{n+1}\rangle = \lambda_n \langle \upsilon, y - x_{n+1}\rangle + \langle \overline{\upsilon}, y - x_{n+1}\rangle, \quad \forall y \in H_n.$$

Algorithm 1 Subgradient explicit iterative algorithm for pseudomontone *EP*

Initialization: Choose $x_0 \in C$, $\lambda_0 > 0$ and $\mu \in (0,1)$.

Iterative steps: Assume x_n and λ_n are known for $n \geq 0$.

Step 1: Compute

$$y_n = \arg\min\left\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\right\}.$$

If $y_n = x_n$ then stop and x_n is the solution of problem (*EP*). Otherwise,

Step 2: construct a half-space first

$$H_n = \left\{w \in \mathbb{H} : \langle x_n - \lambda_n \upsilon_n - y_n, w - y_n\rangle \leq 0\right\},$$

where $\upsilon_n \in \partial f(x_n, y_n)$ and then compute

$$x_{n+1} = \arg\min\left\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in H_n\right\}.$$

Step 3: Compute

$$\lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2[f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1})]_+}\right\}.$$

Set $n := n + 1$ and go back **Step 1**.

Since $\overline{\upsilon} \in N_{H_n}(x_{n+1})$ we have $\langle \overline{\upsilon}, y - x_{n+1}\rangle \leq 0$ for all $y \in H_n$. This implies that

$$\langle x_n - x_{n+1}, y - x_{n+1}\rangle \leq \lambda_n \langle \upsilon, y - x_{n+1}\rangle, \quad \forall y \in H_n. \tag{1}$$

From $\upsilon \in \partial f(y_n, x_{n+1})$ and the definition of the subdifferential, we have

$$f(y_n, y) - f(y_n, x_{n+1}) \geq \langle \upsilon, y - x_{n+1}\rangle, \quad \forall y \in \mathbb{H}. \tag{2}$$

Combining (1) and (2) we obtain

$$\lambda_n f(y_n, y) - \lambda_n f(y_n, x_{n+1}) \geq \langle x_n - x_{n+1}, y - x_{n+1}\rangle, \quad \forall y \in H_n. \tag{3}$$

□

Lemma 3.2 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be generated from the Algorithm* 1, *then the following relation holds*:

$$\lambda_n\left\{f(x_n, x_{n+1}) - f(x_n, y_n)\right\} \geq \langle x_n - y_n, x_{n+1} - y_n\rangle.$$

Proof It follows from the definition of x_{n+1} in Algorithm 1 and by the definition of the hyperplane H_n that $\langle x_n - \lambda_n \upsilon_n - y_n, x_{n+1} - y_n\rangle \leq 0$. Thus, we get

$$\lambda_n \langle \upsilon_n, x_{n+1} - y_n\rangle \geq \langle x_n - y_n, x_{n+1} - y_n\rangle. \tag{4}$$

Further, $\upsilon_n \in \partial f(x_n, y_n)$ and due to definition of the subdifferential, we have

$$f(x_n, y) - f(x_n, y_n) \geq \langle \upsilon_n, y - y_n \rangle, \quad \forall y \in \mathbb{H}.$$

Substitute $y = x_{n+1}$ in the above expression

$$f(x_n, x_{n+1}) - f(x_n, y_n) \geq \langle \upsilon_n, x_{n+1} - y_n \rangle, \quad \forall y \in \mathbb{H}. \tag{5}$$

Combining (4) and (5) we obtain

$$\lambda_n \{ f(x_n, x_{n+1}) - f(x_n, y_n) \} \geq \langle x_n - y_n, x_{n+1} - y_n \rangle. \qquad \square$$

Next, we prove an important inequality that is useful for understanding the pattern and converging analysis of the sequence generated by Algorithm 1.

Lemma 3.3 *Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction satisfying the conditions (A_1)–(A_4) (Assumption 2.1). Assume that the solution set $EP(f, C)$ is nonempty. Then for all $p \in EP(f, C)$ we have*

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_n - y_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - y_n\|^2.$$

Proof By Lemma 3.1 and replacing $y = p$ we obtain

$$\lambda_n f(y_n, p) - \lambda_n f(y_n, x_{n+1}) \geq \langle x_n - x_{n+1}, p - x_{n+1} \rangle. \tag{6}$$

Since $f(p, y_n) \geq 0$ and from assumption (A_1) we have $f(y_n, p) \leq 0$, which implies that

$$\langle x_n - x_{n+1}, x_{n+1} - p \rangle \geq \lambda_n f(y_n, x_{n+1}). \tag{7}$$

From the definition of λ_{n+1} we get

$$f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) \leq \frac{\mu(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\lambda_{n+1}}. \tag{8}$$

From Eqs. (7) and (8) we get the following:

$$\begin{aligned}\langle x_n - x_{n+1}, x_{n+1} - p \rangle \geq\ & \lambda_n \{ f(x_n, x_{n+1}) - f(x_n, y_n) \} \\ & - \frac{\mu\lambda_n}{2\lambda_{n+1}}\|x_n - y_n\|^2 - \frac{\mu\lambda_n}{2\lambda_{n+1}}\|x_{n+1} - y_n\|^2. \end{aligned} \tag{9}$$

Next, by Lemma 3.2 and Eq. (9) we obtain

$$\begin{aligned}\langle x_n - x_{n+1}, x_{n+1} - p \rangle \geq\ & \langle x_n - y_n, x_{n+1} - y_n \rangle \\ & - \frac{\mu\lambda_n}{2\lambda_{n+1}}\|x_n - y_n\|^2 - \frac{\mu\lambda_n}{2\lambda_{n+1}}\|x_{n+1} - y_n\|^2. \end{aligned} \tag{10}$$

We have the following facts:

$$\begin{aligned} &\|a \pm b\|^2 = \|a\|^2 + \|b\|^2 \pm 2\langle a, b\rangle, \\ &-2\langle x_n - x_{n+1}, x_{n+1} - p\rangle = -\|x_n - p\|^2 + \|x_{n+1} - x_n\|^2 + \|x_{n+1} - p\|^2, \\ &2\langle y_n - x_n, y_n - x_{n+1}\rangle = \|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 - \|x_n - x_{n+1}\|^2. \end{aligned}$$

Through the above expressions and Eq. (10) we have

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_n - y_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - y_n\|^2. \qquad \square$$

Let us formulate the first main convergence result of this work.

Theorem 3.1 *Under the hypotheses* (A_1)–(A_4) (*Assumption* 2.1) *the sequences* $\{x_n\}$, $\{y_n\}$ *generated from Algorithm* 1 *converge weakly to an element* p *of* $EP(f, C)$. *Moreover,* $\lim_{n\to\infty} P_{EP(f,C)}(x_n) = p$.

Proof By the definition of λ_{n+1} the sequence $\frac{\lambda_n}{\lambda_{n+1}} \to 1$ and $\mu \in (0, 1)$, which implies that

$$\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \to 1 - \mu > 0.$$

Next, we can easily choose $\epsilon \in (0, 1 - \mu)$ such that $(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}) > \epsilon$, $\forall n \ge n_0$. Due to this fact and Lemma 3.3, we obtain

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2, \quad \forall n \ge n_0. \tag{11}$$

Furthermore, we fix an arbitrary number $m \ge n_0$ and consider Lemma 3.3, for all numbers $n_0, n_0 + 1, \ldots, m$. Summing, we obtain

$$\begin{aligned} \|x_{m+1} - p\|^2 &\le \|x_{n_0} - p\|^2 - \sum_{k=n_0}^{m}\left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|x_k - y_k\|^2 \\ &\quad - \sum_{k=n_0}^{m}\left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|x_{k+1} - y_k\|^2 \\ &\le \|x_{n_0} - p\|^2. \end{aligned} \tag{12}$$

Taking $k \to \infty$ in Eq. (12), we can deduce the following results:

$$\sum_n \|x_n - y_n\|^2 < +\infty \quad \Longrightarrow \quad \lim_{n\to\infty} \|x_n - y_n\| = 0 \tag{13}$$

and

$$\sum_n \|x_{n+1} - y_n\|^2 < +\infty \quad \Longrightarrow \quad \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{14}$$

Further, Eqs. (11) and (12) imply that

$$\lim_{n\to\infty} \|x_n - p\| = b, \quad \text{for some finite } b > 0. \tag{15}$$

Moreover, from Eqs. (13), (14) and the Cauchy inequality, we get

$$\lim_{n\to\infty} \|x_{k+1} - x_k\| \Longrightarrow 0. \tag{16}$$

Next, we show that a very sequential weak cluster point of the sequence $\{x_n\}$ is in $EP(f,C)$. Assume that z is a weak cluster point of $\{x_n\}$, i.e. there exists a subsequence, denoted by $\{x_{n_k}\}$ of $\{x_n\}$, weakly converging to z. Then $\{y_{n_k}\}$ also weakly converges to z and $z \in C$. Let us show that $z \in EP(f,C)$. By Lemma 3.1, the definition of λ_{n+1} and Lemma 3.2, we have

$$\begin{aligned}
\lambda_{n_k} f(y_{n_k}, y) &\geq \lambda_{n_k} f(y_{n_k}, x_{n_k+1}) + \langle x_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle \\
&\geq \lambda_{n_k} f(x_{n_k}, x_{n_{k+1}}) - \lambda_{n_k} f(x_{n_k}, y_{n_k}) - \frac{\mu\lambda_{n_k}}{2\lambda_{n_k+1}} \|x_{n_k} - y_{n_k}\|^2 \\
&\quad - \frac{\mu\lambda_{n_k}}{2\lambda_{n_k+1}} \|y_{n_k} - x_{n_k+1}\|^2 + \langle x_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle \\
&\geq \langle x_{n_k} - y_{n_k}, x_{n_k+1} - y_{n_k}\rangle - \frac{\mu\lambda_{n_k}}{2\lambda_{n_k+1}} \|x_{n_k} - y_{n_k}\|^2 \\
&\quad - \frac{\mu\lambda_{n_k}}{2\lambda_{n_k+1}} \|y_{n_k} - x_{n_k+1}\|^2 + \langle x_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle,
\end{aligned} \tag{17}$$

where y is any element in H_n. It follows from (13), (14), (16) and the boundedness of $\{x_n\}$ that the right-hand side of the last inequality tends to zero. Using $\lambda_{n_k} > 0$, condition (A_3) and $y_{n_k} \rightharpoonup z$ we have

$$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(z, y), \quad \forall y \in C.$$

Since $C \subset H_n$ and $z \in C$ we have $f(z,y) \geq 0$, $\forall y \in C$. This shows that $z \in EP(f,C)$. Thus Lemma 2.5, ensures that $\{x_n\}$ and $\{y_n\}$ converge weakly to p as $n \to \infty$.

Next, we show that $\lim_{n\to\infty} P_{EP(f,C)}(x_n) = p$. Define $t_n := P_{EP(f,C)}(x_n)$ for all $n \in \mathbb{N}$. Since $p \in EP(f,C)$, we have

$$\|t_n\| \leq \|t_n - x_n\| + \|x_n\| \leq \|p - x_n\| + \|x_n\|. \tag{18}$$

Thus, $\{t_n\}$ is bounded. In fact, by Lemma 3.3 for $n \geq n_0$, we deduce that

$$\|x_{n+1} - t_{n+1}\|^2 \leq \|x_{n+1} - t_n\|^2 \leq \|x_n - t_n\|^2, \quad \forall n \geq n_0. \tag{19}$$

Equations (18) and (19) imply the existence of the $\lim_{n\to\infty} \|x_n - t_n\|$. By using Lemma 3.3, for all $m > n \geq n_0$, we have

$$\|t_n - x_m\|^2 \leq \|t_n - x_{m-1}\|^2 \leq \cdots \leq \|t_n - x_n\|^2. \tag{20}$$

Next, we show that $\{t_n\}$ is a Cauchy sequence. Let us take $t_m, t_n \in EP(f,C)$, for $m > n \geq n_0$, and Lemma 2.1(i) with (20) gives

$$\|t_n - t_m\|^2 \leq \|t_n - x_m\|^2 - \|t_m - x_m\|^2 \leq \|t_n - x_n\|^2 - \|t_m - x_m\|^2. \tag{21}$$

The existence of $\lim_{n\to\infty} \|t_n - x_n\|$ implies that $\lim_{m,n\to\infty} \|t_n - t_m\| = 0$, for all $m > n$. Consequently, $\{t_n\}$ is a Cauchy sequence. Since $EP(f,C)$ is closed, we find that $\{t_n\}$ converges strongly to $p^* \in EP(f,C)$. Now, we prove that $p^* = p$. It follows from Lemma 2.1(ii) and $p, p^* \in EP(f,C)$ that

$$\langle x_n - t_n, p - t_n \rangle \leq 0. \tag{22}$$

Since $t_n \to p^*$ and $x_n \rightharpoonup p$, we have

$$\langle p - p^*, p - p^* \rangle \leq 0,$$

which implies that $p = p^* = \lim_{n\to\infty} P_{EP(f,C)}(x_n)$. Further, $\|x_n - y_n\| \to 0$, implies $\lim_{n\to\infty} P_{EP(f,C)}(y_n) = p$. □

Remark 3.3 In the case when the bifunction f is strongly pseudomonotone and satisfies the Lipschitz-type condition, the linear rate of convergence can be achieved for Algorithm 1 (for more details see [47]).

4 Modified subgradient explicit iterative algorithm for a class of pseudomonotone *EP*

In this section, we propose an iterative scheme that involves two strong convex optimization problems with an inertial term that is used to speed up the iterative sequence, so we refer to it as a "modified explicit iterative algorithm" for a class of pseudomonotone equilibrium problems. This algorithm is a modification of Algorithm 1 that performs better than the earlier algorithm due to the inertial term. The detailed Algorithm 2 is given belowthes.

Lemma 4.1 *From Algorithm 2 we have the following useful inequality*:

$$\lambda_n f(y_n, y) - \lambda_n f(y_n, x_{n+1}) \geq \langle w_n - x_{n+1}, y - x_{n+1} \rangle, \quad \forall y \in H_n.$$

Proof The proof is very similar to Lemma 3.1. □

Lemma 4.2 *Let $\{x_n\}$ and $\{y_n\}$ generated from the Algorithm 2, then the following relation holds*:

$$\lambda_n \{ f(w_n, x_{n+1}) - f(w_n, y_n) \} \geq \langle w_n - y_n, x_{n+1} - y_n \rangle.$$

Proof The proof is similar to Lemma 3.2. □

Algorithm 2 Modified subgradient explicit iterative algorithm for pseudomontone *EP*

Initialization: Choose $x_{-1}, x_0 \in \mathbb{H}$, $\lambda_0 > 0$ and $\alpha_n \in [0, \sqrt{5}-2)$. Set

$$w_0 = x_0 + \alpha_0(x_0 - x_{-1}).$$

Iterative steps: Assume that x_n, x_{n-1} and $\lambda_n > 0$ are known for $n \geq 0$.

Step 1: Compute

$$y_n = \arg\min\left\{\lambda_n f(w_n, y) + \frac{1}{2}\|w_n - y\|^2 : y \in C\right\},$$

where $w_n = x_n + \alpha_n(x_n - x_{n-1})$. If $y_n = w_n$ then stop and w_n is the solution of problem (EP). Otherwise,

Step 2: first construct a half-space

$$H_n = \left\{w \in \mathbb{H} : \langle w_n - \lambda_n \upsilon_n - y_n, w - y_n\rangle \leq 0\right\},$$

where $\upsilon_n \in \partial f(w_n, y_n)$ and then compute

$$x_{n+1} = \arg\min\left\{\lambda_n f(y_n, y) + \frac{1}{2}\|w_n - y\|^2 : y \in H_n\right\}.$$

Step 3: Assume $\mu(\alpha) \in (0,1)$ and compute

$$\lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu(\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2[f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1})]_+}\right\}.$$

Set $n := n + 1$ and go back **Step 1**.

Lemma 4.3 *Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction satisfying the conditions (A_1)–(A_4) as in Assumption 2.1. Assume that the solution set $EP(f, C)$ is nonempty. Then for all $p \in EP(f, C)$ we have*

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - y_n\|^2.$$

Proof By Lemma 4.1 and replacing $y = p$, we obtain

$$\lambda_n f(y_n, p) - \lambda_n f(y_n, x_{n+1}) \geq \langle w_n - x_{n+1}, p - x_{n+1}\rangle. \quad (23)$$

Since $f(p, y_n) \geq 0$ and from (A_1) we have $f(y_n, p) \leq 0$, which implies that

$$\langle w_n - x_{n+1}, x_{n+1} - p\rangle \geq \lambda_n f(y_n, x_{n+1}). \quad (24)$$

From the definition of λ_{n+1} we get

$$f(w_n, x_{n+1}) - f(w_n, y_n) - f(y_n, x_{n+1}) \leq \frac{\mu(\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\lambda_{n+1}}. \quad (25)$$

Combining (24) and (25) we get

$$\langle w_n - x_{n+1}, x_{n+1} - p\rangle \geq \lambda_n\{f(w_n, x_{n+1}) - f(w_n, y_n)\} - \frac{\mu\lambda_n}{2\lambda_{n+1}}\|w_n - y_n\|^2 - \frac{\mu\lambda_n}{2\lambda_{n+1}}\|x_{n+1} - y_n\|^2. \tag{26}$$

Next, by Lemma 4.2 and Eq. (26) we have

$$\langle w_n - x_{n+1}, x_{n+1} - p\rangle \geq \langle w_n - y_n, x_{n+1} - y_n\rangle - \frac{\mu\lambda_n}{2\lambda_{n+1}}\|w_n - y_n\|^2 - \frac{\mu\lambda_n}{2\lambda_{n+1}}\|x_{n+1} - y_n\|^2. \tag{27}$$

Furthermore, we have the following facts:

$$2\langle w_n - x_{n+1}, x_{n+1} - p\rangle = \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2 - \|x_{n+1} - p\|^2,$$
$$2\langle w_n - y_n, x_{n+1} - y_n\rangle = \|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 - \|w_n - x_{n+1}\|^2.$$

Using the above facts and Eq. (27) after multiplying by 2, we get the desired result. □

Now, let us formulate the second main convergence result for Algorithm 2.

Theorem 4.1 *The sequences $\{w_n\}$, $\{y_n\}$ and $\{x_n\}$ generated by Algorithm 2 converge weakly to the solution p of the problem (EP), where*

$$0 < \mu < \frac{\frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2}{\frac{1}{2} - \alpha + \frac{1}{2}\alpha^2} \quad \text{and} \quad 0 \leq \alpha_n \leq \alpha < \sqrt{5} - 2.$$

Proof From Lemma 4.3, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - y_n\|^2 \\ &\leq \|w_n - p\|^2 - \frac{1}{2}\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - w_n\|^2.\end{aligned} \tag{28}$$

By the definition of w_n in Algorithm 2, we get

$$\begin{aligned}\|w_n - p\|^2 &= \left\|x_n + \alpha_n(x_n - x_{n-1}) - p\right\|^2 \\ &= \left\|(1 + \alpha_n)(x_n - p) - \alpha_n(x_{n-1} - p)\right\|^2 \\ &= (1 + \alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1 + \alpha_n)\|x_n - x_{n-1}\|^2.\end{aligned} \tag{29}$$

Further, by the definition w_n and using the Cauchy inequality, we have

$$\begin{aligned}\|x_{n+1} - w_n\|^2 &= \left\|x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})\right\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \qquad (30) \\ &\geq \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\|x_{n+1} - x_n\|\|x_n - x_{n-1}\|\end{aligned}$$

$$
\begin{aligned}
&\geq \|x_{n+1} - x_n\|^2 + \alpha_n^2 \|x_n - x_{n-1}\|^2 - \alpha_n \|x_{n+1} - x_n\|^2 - \alpha_n \|x_n - x_{n-1}\|^2 \\
&\geq (1 - \alpha_n)\|x_{n+1} - x_n\|^2 + \left(\alpha_n^2 - \alpha_n\right)\|x_n - x_{n-1}\|^2. \qquad (31)
\end{aligned}
$$

Next, combining (28), (29) and (31), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 + \alpha_n)\|x_n - p\|^2 - \alpha_n \|x_{n-1} - p\|^2 + \alpha_n (1 + \alpha_n)\|x_n - x_{n-1}\|^2 \\
&\quad - \varrho_n (1 - \alpha_n)\|x_{n+1} - x_n\|^2 - \varrho_n \left(\alpha_n^2 - \alpha_n\right)\|x_n - x_{n-1}\|^2 \qquad (32) \\
&= (1 + \alpha_n)\|x_n - p\|^2 - \alpha_n \|x_{n-1} - p\|^2 - \varrho_n (1 - \alpha_n)\|x_{n+1} - x_n\|^2 \\
&\quad + \left[\alpha_n (1 + \alpha_n) - \varrho_n \left(\alpha_n^2 - \alpha_n\right)\right]\|x_n - x_{n-1}\|^2 \\
&= (1 + \alpha_n)\|x_n - p\|^2 - \alpha_n \|x_{n-1} - p\|^2 - Q_n \|x_{n+1} - x_n\|^2 \\
&\quad + R_n \|x_n - x_{n-1}\|^2, \qquad (33)
\end{aligned}
$$

where

$$
\begin{aligned}
&\varrho_n := \frac{1}{2}\left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right), \\
&Q_n := \varrho_n (1 - \alpha_n),
\end{aligned}
$$

and

$$
R_n := \alpha_n (1 + \alpha_n) - \varrho_n \left(\alpha_n^2 - \alpha_n\right).
$$

Next, we take

$$
\Lambda_n = \|x_n - p\|^2 - \alpha_n \|x_{n-1} - p\|^2 + R_n \|x_n - x_{n-1}\|^2,
$$

and compute

$$
\begin{aligned}
\Lambda_{n+1} - \Lambda_n &= \|x_{n+1} - p\|^2 - \alpha_{n+1} \|x_n - p\|^2 + R_{n+1} \|x_{n+1} - x_n\|^2 \\
&\quad - \|x_n - p\|^2 + \alpha_n \|x_{n-1} - p\|^2 - R_n \|x_n - x_{n-1}\|^2 \\
&\leq \|x_{n+1} - p\|^2 - (1 + \alpha_n)\|x_n - p\|^2 + \alpha_n \|x_{n-1} - p\|^2 \\
&\quad + R_{n+1} \|x_{n+1} - x_n\|^2 - R_n \|x_n - x_{n-1}\|^2. \qquad (34)
\end{aligned}
$$

Using Eq. (33) in (34), we obtain

$$
\begin{aligned}
\Lambda_{n+1} - \Lambda_n &\leq -Q_n \|x_{n+1} - x_n\|^2 + R_{n+1} \|x_{n+1} - x_n\|^2 \\
&= -(Q_n - R_{n+1})\|x_{n+1} - x_n\|^2. \qquad (35)
\end{aligned}
$$

Furthermore, we have to compute

$$\begin{aligned} Q_n - R_{n+1} &= \varrho_n(1-\alpha_n) - \alpha_{n+1}(1+\alpha_{n+1}) + \varrho_{n+1}\bigl(\alpha_{n+1}^2 - \alpha_{n+1}\bigr) \\ &\geq \varrho_n(1-\alpha_{n+1}) - \alpha_{n+1}(1+\alpha_{n+1}) + \varrho_n\bigl(\alpha_{n+1}^2 - \alpha_{n+1}\bigr) \\ &= \varrho_n(1-\alpha_{n+1})^2 - \alpha_{n+1}(1+\alpha_{n+1}) \\ &\geq \varrho_n(1-\alpha)^2 - \alpha(1+\alpha) \\ &= \bigl(\varrho_n - \alpha - \alpha^2\bigr) + \varrho_n\alpha^2 - 2\varrho_n\alpha \\ &= \left(\frac{1}{2} - \alpha - \alpha^2 + \frac{\alpha^2}{2} - \alpha\right) - \mu\left(\frac{\lambda_n}{2\lambda_{n+1}} + \frac{\lambda_n}{2\lambda_{n+1}}\alpha^2 - \frac{\lambda_n}{\lambda_{n+1}}\alpha\right) \\ &= \left(\frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2\right) - \mu\left(\frac{\lambda_n}{2\lambda_{n+1}} - \frac{\lambda_n}{\lambda_{n+1}}\alpha + \frac{\lambda_n}{2\lambda_{n+1}}\alpha^2\right). \end{aligned} \tag{36}$$

We have

$$0 < \mu < \frac{\frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2}{\frac{1}{2} - \alpha + \frac{1}{2}\alpha^2} \quad \text{and} \quad 0 \leq \alpha < \sqrt{5} - 2.$$

This implies that, for every $0 \leq \alpha < \sqrt{5} - 2$, there exist $n_0 \geq 1$ and a fixed number

$$\epsilon \in \left(0, \frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2 - \mu\left(\frac{1}{2} - \alpha + \frac{1}{2}\alpha^2\right)\right),$$

such that

$$Q_n - R_{n+1} \geq \epsilon, \quad \forall n \geq n_0. \tag{37}$$

Equations (35) and (37) imply that, for all $n \geq n_0$, we have

$$\Lambda_{n+1} - \Lambda_n \leq -(Q_n - R_{n+1})\|x_{n+1} - x_n\|^2 \leq 0. \tag{38}$$

Hence the sequence $\{\Lambda_n\}$ is nonincreasing for $n \geq n_0$. Further, from the definition of Λ_{n+1} we have

$$\begin{aligned} \Lambda_{n+1} &= \|x_{n+1} - p\|^2 - \alpha_{n+1}\|x_n - p\|^2 + R_{n+1}\|x_{n+1} - x_n\|^2 \\ &\geq -\alpha_{n+1}\|x_n - p\|^2. \end{aligned} \tag{39}$$

Also, from Λ_n we have

$$\begin{aligned} \Lambda_n &= \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + R_n\|x_n - x_{n-1}\|^2 \\ &\geq \|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2. \end{aligned} \tag{40}$$

Equation (40) implies that, for $n \geq n_0$, we have

$$\begin{aligned}\|x_n - p\|^2 &\leq \Lambda_n + \alpha_n \|x_{n-1} - p\|^2 \\ &\leq \Lambda_{n_0} + \alpha \|x_{n-1} - p\|^2 \\ &\leq \cdots \leq \Lambda_{n_0}\left(\alpha^{n-n_0} + \cdots + 1\right) + \alpha^{n-n_0}\|x_{n_0} - p\|^2 \\ &\leq \frac{\Lambda_{n_0}}{1-\alpha} + \alpha^{n-n_0}\|x_{n_0} - p\|^2. \end{aligned} \tag{41}$$

Combining (39) and (41) we obtain

$$\begin{aligned}-\Lambda_{n+1} &\leq \alpha_{n+1}\|x_n - p\|^2 \\ &\leq \alpha \|x_n - p\|^2 \\ &\leq \alpha \frac{\Lambda_{n_0}}{1-\alpha} + \alpha^{n-n_0+1}\|x_{n_0} - p\|^2. \end{aligned} \tag{42}$$

It follows from (38) and (42) that

$$\begin{aligned}\epsilon \sum_{n=n_0}^{k} \|x_{n+1} - x_n\|^2 &\leq \Lambda_{n_0} - \Delta_{k+1} \\ &\leq \Lambda_{n_0} + \alpha \frac{\Delta_{n_0}}{1-\alpha} + \alpha^{k-n_0+1}\|x_{n_0} - p\|^2 \\ &\leq \frac{\Lambda_{n_0}}{1-\alpha} + \|x_{n_0} - p\|^2. \end{aligned} \tag{43}$$

Letting $k \to \infty$ in (43) we have $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty$. This implies that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{44}$$

From Eqs. (30) and (44) we have

$$\|x_{n+1} - w_n\| \to 0 \quad \text{as } n \to \infty. \tag{45}$$

Moreover, by Lemma 2.4, Eq. (32) and $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty$,

$$\lim_{n\to\infty} \|x_n - p\|^2 = b. \tag{46}$$

Thus, from Eqs. (29), (44) and (46),

$$\lim_{n\to\infty} \|w_n - p\|^2 = b, \tag{47}$$

also

$$0 \leq \|x_n - w_n\| = \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0 \quad \text{as } n \to \infty. \tag{48}$$

To show $\lim_{n\to\infty}\|y_n - p\|^2 = b$, we use Lemma 4.3 for $n \geq n_0$, which gives

$$\begin{aligned}
&\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 \\
&\quad\leq \|w_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad\leq \bigl(\|w_n - p\| + \|x_{n+1} - p\|\bigr)\bigl(\|w_n - p\| - \|x_{n+1} - p\|\bigr) \\
&\quad\leq \bigl(\|w_n - p\| + \|x_{n+1} - p\|\bigr)\|x_{n+1} - w_n\| \to 0 \quad \text{as } n \to \infty
\end{aligned} \tag{49}$$

and

$$0 \leq \|x_n - y_n\| = \|x_n - w_n\| + \|w_n - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{50}$$

Further, (44), (46) and (50) imply that

$$\|x_{n+1} - y_n\| \to 0 \quad \text{as } n \to \infty, \quad \text{and} \quad \lim_{n\to\infty}\|y_n - p\|^2 = b. \tag{51}$$

This implies that the sequences $\{x_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded, and for every $p \in EP(f,C)$, the $\lim_{n\to\infty}\|x_n - p\|^2$ exists. Now, further we show that for very sequential weak cluster point of the sequence $\{x_n\}$ is in $EP(f,C)$. Assume that z is a weak cluster point of $\{x_n\}$, i.e., there exists a subsequence, denoted by $\{x_{n_k}\}$, of $\{x_n\}$ weakly converging to z. Then $\{y_{n_k}\}$ also weakly converges to z and $z \in C$. Let us show that $z \in EP(f,C)$. By Lemma 4.1, the definition of λ_{n+1} and Lemma 4.2, we have

$$\begin{aligned}
\lambda_{n_k} f(y_{n_k}, y) &\geq \lambda_{n_k} f(y_{n_k}, x_{n_k+1}) + \langle w_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle \\
&\geq \lambda_{n_k} f(w_{n_k}, x_{n_{k+1}}) - \lambda_{n_k} f(w_{n_k}, y_{n_k}) - \frac{\mu\lambda_{n_k}}{2\lambda_{n_k+1}}\|w_{n_k} - y_{n_k}\|^2 \\
&\quad - \frac{\mu\lambda_{n_k}}{2\lambda_{n_k+1}}\|y_{n_k} - x_{n_k+1}\|^2 + \langle w_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle \\
&\geq \langle w_{n_k} - y_{n_k}, x_{n_k+1} - y_{n_k}\rangle - \frac{\mu\lambda_{n_k}}{2\lambda_{n_k+1}}\|w_{n_k} - y_{n_k}\|^2 \\
&\quad - \frac{\mu\lambda_{n_k}}{2\lambda_{n_k+1}}\|y_{n_k} - x_{n_k+1}\|^2 + \langle w_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle,
\end{aligned} \tag{52}$$

where y is any element in H_n. It follows from (45), (49), (51) and the boundedness of $\{x_n\}$ that the right-hand side of the last inequality tends to zero. Using $\lambda_{n_k} > 0$, condition (A_3) and $y_{n_k} \rightharpoonup z$, we have

$$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(z,y), \quad \forall y \in \mathbb{H}_n.$$

Since $C \subset H_n$ and $z \in C$, we have $f(z,y) \geq 0$, $\forall y \in C$. This shows that $z \in EP(f,C)$. Thus Lemma 2.5, ensures that $\{w_n\}$, $\{x_n\}$ and $\{y_n\}$ converge weakly to p as $n \to \infty$. □

Remark 4.1 The knowledge of the Lipschitz-type constants is not mandatory to build up the sequence $\{x_n\}$ in Algorithm 2 and to get the convergence result in Theorem 4.1.

5 Computational experiment

In this section, some numerical results will be presented in order to test Algorithms 1 and 2 with the recent Heiu algorithm in [47]. The MATLAB codes run on a PC (with Intel(R) Core(TM)i3-4010U CPU @ 1.70 GHz 1.70 GHz, RAM 4.00 GB) under MATLAB version 9.5 (R2018b).

5.1 Nash–Cournot oligopolistic equilibrium model

We consider an extension of a Nash–Cournot oligopolistic equilibrium model [2]. Assume that there are m companies that are producing the same commodity. Let x denote the vector whose entry x_j stands for the quantity of the commodity produced by company j. We suppose that the price $p_j(s)$ is a decreasing affine function of s with $s = \sum_{j=1}^{m} x_j$ i.e. $p_j(s) = \alpha_j - \beta_j s$, where $\alpha_j > 0$, $\beta_j > 0$. Then the profit made by company j is given by $f_j(x) = p_j(s)x_j - c_j(x_j)$, where $c_j(x_j)$ is the tax and fee for generating x_j. Suppose that $C_j = [x_j^{\min}, x_j^{\max}]$ is the strategy set of company j. Then the strategy set of the model is $C := C_1 \times C_2 \times \cdots \times C_m$. Actually, each company wants to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies is a parameter input. A frequently used approach to dealing with this model is based upon the well-known Nash equilibrium concept. We recall that a point $x^* \in C = C_1 \times C_2 \times \cdots \times C_m$ is an equilibrium point of the model if

$$f_j\bigl(x^*\bigr) \geq f_j\bigl(x^*[x_j]\bigr), \quad \forall x_j \in C_j, \forall j = 1, 2, \ldots, m.$$

where the vector $x^*[x_j]$ stands for the vector attain from x^* by replacing x_j^* with x_j. By taking $f(x,y) := \psi(x,y) - \psi(x,x)$ with $\psi(x,y) := -\sum_{j=1}^{m} f_j(x[y_j])$, the problem of finding a Nash equilibrium point of the model can be formulated as

$$\text{Find } x^* \in C : f\bigl(x^*, y\bigr) \geq 0, \quad \forall y \in C.$$

Now, assume that the tax-fee function $c_j(x_j)$ is increasing and affine for every j. This assumption means that both of the tax and fee for producing a unit are increasing as the quantity of the production gets larger. As in [20, 53], the bifunction f can be formulated in the form of $f(x,y) = \langle Px + Qy + q, y - x\rangle$, where $q \in \mathbb{R}^m$ and P, Q are two matrices of order m such that Q is symmetric positive semidefinite and $Q - P$ is symmetric negative semidefinite.

For Experiment 5.1 we take $x_{-1} = (10, 0, 10, 1, 10)^T$, $x_0 = (1, 3, 1, 1, 2)^T$, $C = \{x : -2 \leq x_i \leq 5\}$ and y-axes represent for the value of $D_n = \|w_n - y_n\|$ while the x-axes represent for the number of iterations or elapsed time (in seconds).

5.1.1 Algorithm 2 nature in terms of different values of α_n

Figures 1 and 2 illustrate the numerical results for the first 120 iterations of Algorithm 2 (shortly, MSgEIA) with respect to using different values of α_n. For these results, we use parameters $\alpha_n = 0.22, 0.16, 0.11, 0.07, 0.03$, $\lambda_0 = 1$ and $\mu = 0.11$. These two figures are useful for choosing the best possible value of α_n.

5.1.2 Algorithm 2 comparison with existing algorithms

Figures 3 and 4 describe the numerical results for the first 100 iterations of Algorithm 2 [Modified subgradient explicit iterative algorithm (shortly, MSgEIA)] compared with Al-

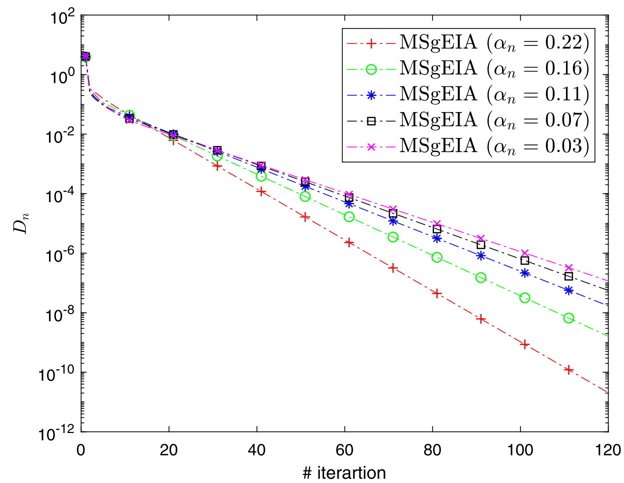

Figure 1 Experiment 5.1: Algorithm 2 behavior in terms of iterations relative to different values of α_n

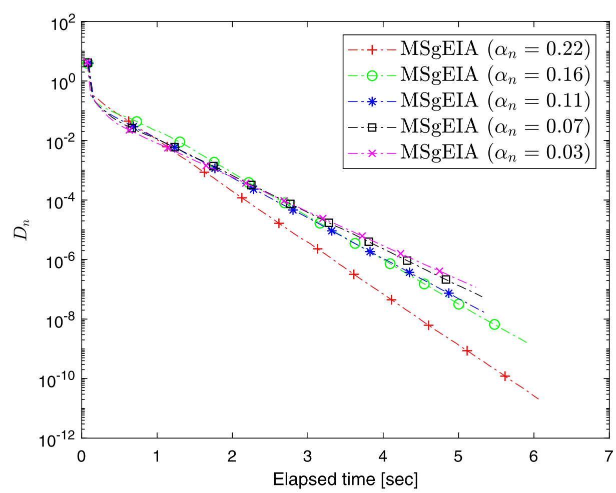

Figure 2 Experiment 5.1: Algorithm 2 behavior in terms of elapsed time relative to different values of α_n

gorithm 1 [Subgradient explicit iterative algorithm (shortly, SgEIA)] and explicit Algorithm 1 [Explicit iterative algorithm (shortly, EIA) [47]] in terms of no. of iterations and elapsed time in seconds.

(i) For Explicit iterative algorithm (EIA) we use the parameters $\mu = 0.11$, $\lambda_0 = 1$ and $D_n = \|x_n - y_n\|$.

(ii) For Subgradient explicit iterative algorithm (SgEIA) we use the parameters $\mu = 0.11$, $\lambda_0 = 1$ and $D_n = \|x_n - y_n\|$.

(iii) For Modified subgradient explicit iterative algorithm (MSgEIA) we use the parameters $\alpha_n = 0.12$, $\mu = 0.11$, $\lambda_0 = 1$ and $D_n = \|w_n - y_n\|$.

5.2 Nash–Cournot equilibrium models of electricity markets

In this experiment, we apply our proposed algorithm to a Nash–Cournot equilibrium model of electricity markets as in [13]. In this model, it is considered that there are three

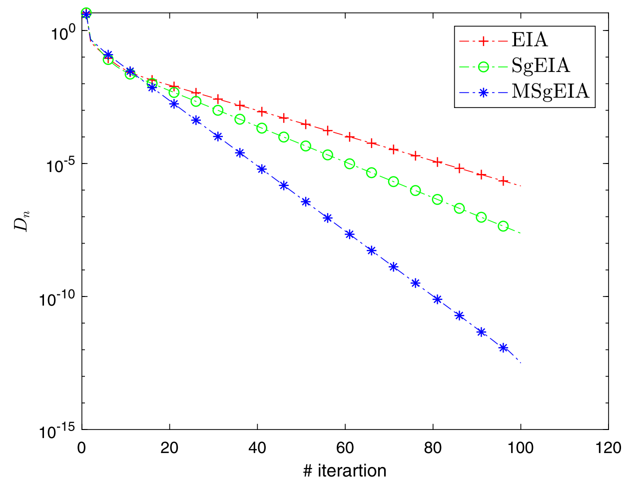

Figure 3 Experiment 5.1: Algorithm 2 comparison in terms of iterations

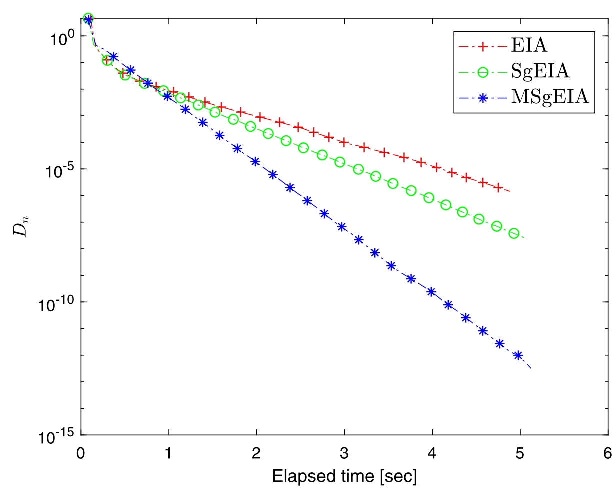

Figure 4 Experiment 5.1: Algorithm 2 comparison in terms of elapsed time

electricity companies i ($i = 1, 2, 3$). Each company i has its own, several generating units with index set I_i. In this experiment, suppose that $I_1 = \{1\}$, $I_2 = \{2, 3\}$ and $I_3 = \{4, 5, 6\}$. Let x_j be the power generation of units j ($j = 1, \dots, 6$) and suppose that the electricity price p can be expressed as by $p = 378.4 - 2\sum_{j=1}^{6} x_j$. The cost of a generating unit j is illustrated by

$$c_j(x_j) := \max\left\{\mathring{c}_j(x_j), \dot{c}_j(x_j)\right\},$$

with

$$\mathring{c}_j(x_j) := \frac{\mathring{\alpha}_j}{2} x_j^2 + \mathring{\beta}_j x_j + \mathring{\gamma}_j$$

Table 1 The parameter values used in this experiment

j	$\overset{\circ}{\alpha}_j$	$\overset{\circ}{\beta}_j$	$\overset{\circ}{\gamma}_j$	$\overset{\bullet}{\alpha}_j$	$\overset{\bullet}{\beta}_j$	$\overset{\bullet}{\gamma}_j$
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

Table 2 The parameter values used in this experiment

	j					
	1	2	3	4	5	6
$x_j^{\min}$	0	0	0	0	0	0
$x_j^{\max}$	80	80	50	55	30	40

and

$$\overset{\bullet}{c}_j(x_j) := \overset{\bullet}{\alpha}_j x_j + \frac{\overset{\bullet}{\beta}_j}{\overset{\bullet}{\beta}_j + 1} \overset{\bullet}{\gamma}_j^{\frac{-1}{\overset{\bullet}{\beta}_j}} (x_j)^{\frac{(\overset{\bullet}{\beta}_j+1)}{\overset{\bullet}{\beta}_j}},$$

where the parameter values are given in $\overset{\circ}{\alpha}_j$, $\overset{\circ}{\beta}_j$, $\overset{\circ}{\gamma}_j$, $\overset{\bullet}{\alpha}_j$, $\overset{\bullet}{\beta}_j$ and $\overset{\bullet}{\gamma}_j$ are given in Table 1. Suppose the profit of company i is given by

$$f_i(x) := p \sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j) = \left(378.4 - 2\sum_{l=1}^{6} x_l\right) \sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j),$$

where $x = (x_1, \ldots, x_6)^T$ subject to the constraint $x \in C := \{x \in \mathbb{R}^6 : x_j^{\min} \leq x_j \leq x_j^{\max}\}$, with $x_j^{\min}$ and $x_j^{\max}$ given in Table 2.

Next, we define the equilibrium function f by

$$f(x,y) := \sum_{i=1}^{3} \bigl(\phi_i(x,x) - \phi_i(x,y)\bigr),$$

where

$$\phi_i(x,y) := \left[378.4 - 2\left(\sum_{j\notin I_i} x_j + \sum_{j\in I_i} y_j\right)\right] \sum_{j\in I_i} y_j - \sum_{j\in I_i} c_j(y_j).$$

The Nash–Cournot equilibrium models of electricity markets can be reformulated as an equilibrium problem (see [58]):

$$\text{Find } x^* \in C \quad \text{such that} \quad f\bigl(x^*, y\bigr) \geq 0, \quad \forall y \in C.$$

For Experiment 5.2, we take $x_{-1} = (10, 0, 10, 1, 10, 1)^T$, $x_0 = (48, 48, 30, 27, 18, 24)^T$, and the y-axes represent for the value of D_n while the x-axes represent the number of iterations or elapsed time (in seconds).

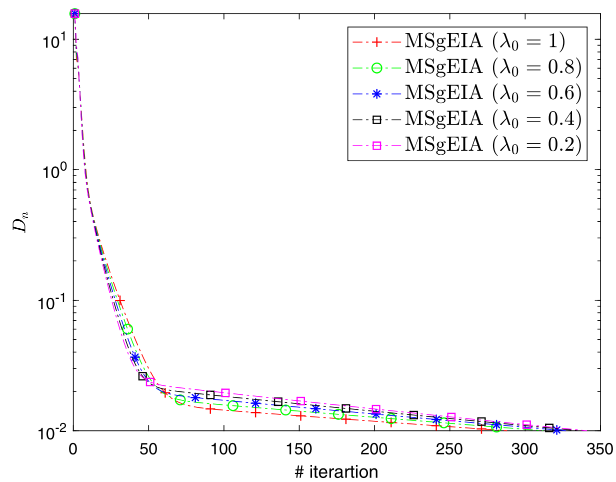

Figure 5 Experiment 5.2: Algorithm 2 behavior in terms of iterations relative to different values of λ_0

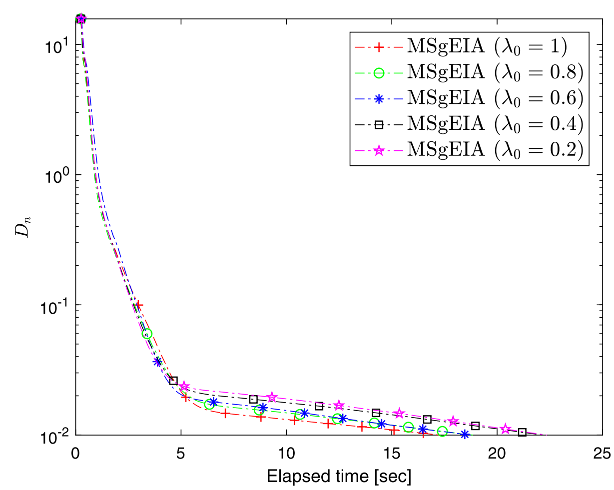

Figure 6 Experiment 5.2: Algorithm 2 behavior in terms of elapsed time relative to different values of λ_0

5.2.1 Algorithm 2 nature in terms of different values of λ_0

Figures 5 and 6 describe the numerical results of Algorithm 2 (MSgEIA) with respect to using different values of λ_0, in terms of no. of iterations and elapsed time in seconds relative to $D_n = \|x_{n+1} - x_n\|$. For these results, we use the parameters $\alpha_n = 0.20$ $\lambda_0 = 1, 0.8, 0.6, 0.4, 0.2$, $\mu = 0.24$ and $\epsilon = 10^{-2}$.

5.2.2 Algorithm 2 comparison with existing algorithms

Figures 7 and 8 describe the numerical results of Algorithm 2 [Modified subgradient explicit iterative algorithm (MSgEIA)] compared with Algorithm 1 [Subgradient explicit iterative algorithm (SgEIA)] and Algorithm 1 [Explicit iterative algorithm (EIA) [47]] in terms of no. of iterations and elapsed time in seconds.

(i) For the Explicit iterative algorithm (EIA) we use the parameters $\mu = 0.2$, $\lambda_0 = 0.6$ and $D_n = \|x_{n+1} - x_n\|$.

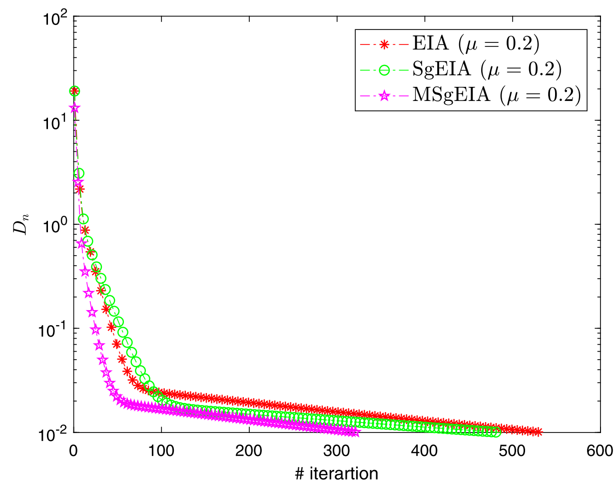

Figure 7 Experiment 5.2: Algorithm 2 compared in terms of iterations

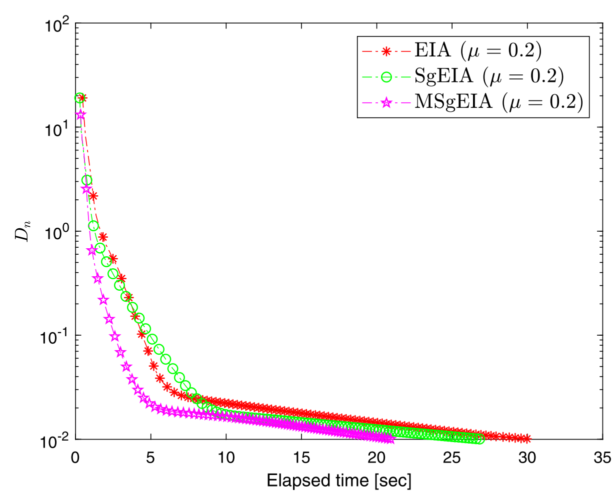

Figure 8 Experiment 5.2: Algorithm 2 compared in terms of elapsed time

(ii) For Subgradient explicit iterative algorithm (SgEIA) we use the parameters $\mu = 0.2$, $\lambda_0 = 0.6$ and $D_n = \|x_{n+1} - x_n\|$.

(iii) For Modified subgradient explicit iterative algorithm (MSgEIA) we use the parameters $\alpha_n = 0.20$, $\mu = 0.2$, $\lambda_0 = 0.6$ and $D_n = \|x_{n+1} - x_n\|$.

5.3 Two-dimensional (2-D) pseudomonotone *EP*

Let us consider the following bifunction:

$$f(x,y) = \langle F(x), y - x \rangle,$$

where

$$F(x) = \begin{pmatrix} (x_1^2 + (x_2 - 1)^2)(1 + x_2) \\ -x_1^3 - x_1(x_2 - 1)^2 \end{pmatrix} \quad \text{with } C = \{x \in \mathbb{R}^2 : -10 \le x_i \le 10\}.$$

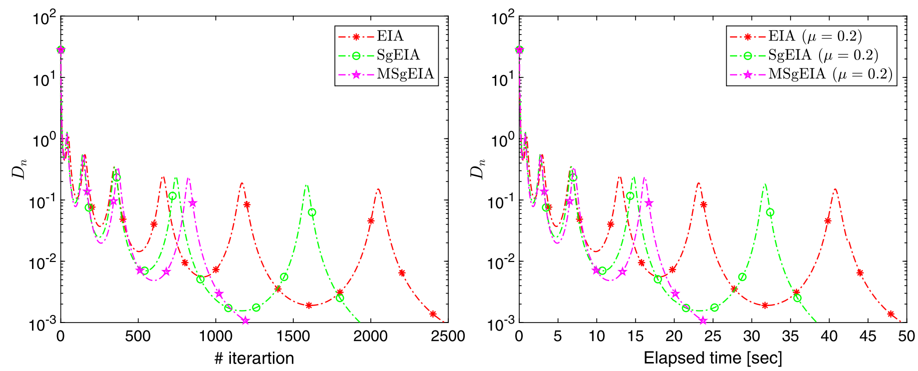

Figure 9 Algorithm 2 compared in terms of no. of iterations and elapsed time

The bifunction is not monotone on C but pseudomonotone (for more details see p. 10, [59, 60]). Figure 9 illustrates the numerical results of comparison of Algorithm 2 with two other algorithms, with $x_{-1} = (5,5)^T$ and $x_0 = (10,10)^T$.

6 Conclusion

In this paper, we propose two algorithms by incorporating the subgradient and inertial technique with an explicit iterative algorithm, which can solve the problem of a pseudomonotone equilibrium. The evaluation of the step-size did not require a line search procedure or information on the Lipchitz-type constants of the bifunction. Rather, one uses a step-size sequence that can be updated on each iteration with the help of previous iterations. We have presented various numerical results to show the computational performance of our algorithm in comparison with other algorithms. These numerical results have also explained that the algorithm with inertial effects seems to perform better than without inertial effects.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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