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An improvement of the infinity norm bound for the inverse of $\{P_1, P_2\}$-Nekrasov matrices

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Abstract

A new upper bound for the infinity norm for the inverse of $\{P_1, P_2\}$-Nekrasov matrices is given. It is proved that the upper bound is sharper than those in Cvetković et al. (Open Math. 13:96–105, 2015) and than well-known Varah's bound for strictly diagonally dominant matrices. Numerical examples are given to illustrate the corresponding results.

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Keywords: Infinity norm; $\{P_1, P_2\}$-Nekrasov matrices; H-matrices

1 Introduction

By $\mathbb{C}^{n\times n}(\mathbb{R}^{n\times n})$ we denote the set of all complex (real) matrices of order n. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called an H-matrix if its comparison matrix $\langle A\rangle = [m_{ij}] \in \mathbb{R}^{n\times n}$ defined by

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is a nonsingular M-matrix, i.e., $\langle A\rangle^{-1} \geq 0$ [1, 8, 20].

It is well known that H-matrices are widely used in many subjects such as numerical algebra, the control system, mathematical physics, economics, and dynamical system theory [1, 2, 4, 20]. An important problem among them is to find upper bounds for the infinity norm of the inverse of H-matrices, because it can be used to the convergence analysis of matrix splitting and matrix multi-splitting iterative methods for solving large sparse systems of linear equations [18], as well as linear complementarity problems [10–13, 19]. For example, when solving linear systems in practice, it is important to have an economical method for estimating the condition number $\kappa(A)$ of the matrix of coefficients, which shows how '*ill*' the systems could be. Here, the condition number is defined in the following way:

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|,$$

as the product of a matrix norm and a norm of its inverse. Hence, it can be useful to determine the upper bound for the norm of the inverse matrix without calculating the inverse.

In 1975, Varah provided a simple and elegant upper bound for the infinity norm of the inverse of strictly diagonally dominant (*SDD*) matrices as one of the most important subclass of H-matrices. Here a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is said to be an *SDD* matrix if, for each $i \in N := \{1, 2, \ldots, n\}$,

$$|a_{ii}| > r_i(A),$$

where $r_i(A) = \sum_{j\neq i} |a_{ij}|$.

Theorem 1 ([17]) *If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is SDD, then*

$$\|A^{-1}\|_\infty \leq \frac{1}{\min_{i\in N}(|a_{ii}| - r_i(A))}. \tag{1}$$

Bound (1) is usually called Varah's bound and works only for *SDD* matrices. Moreover, when the class of involved matrices is a wider subclass of H-matrices, such as doubly strictly diagonally dominant (*DSDD*) matrices, S-*SDD* matrices, weakly chained diagonally dominant matrices, Nekrasov matrices, S-Nekrasov matrices, and DZ-type matrices, upper bounds for $\|A^{-1}\|_\infty$ are derived, which sometimes are tighter in the *SDD* case, see [3, 5, 7, 9, 10, 15, 16, 21] and the references therein. Recently, Cvetković et al. [6] presented two upper bounds for $\|A^{-1}\|_\infty$ involved with $\{P_1, P_2\}$-Nekrasov matrices, which are only dependent on the entries of the matrix A.

In this paper, we give a new upper bound for the infinity norm of the inverse of $\{P_1, P_2\}$-Nekrasov matrices. It is shown by the comparison theorems that the new bound improves corresponding bounds of Cvetković et al. (2015) for $\{P_1, P_2\}$-Nekrasov matrices and improves well-known Varah's bound for strictly diagonally dominant matrices. The tested numerical examples show that the new bound is tighter than those derived recently.

2 Main results

First, some notation and definitions are listed. Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, denote

$$d(A) = \bigl(|a_{11}|, \ldots, |a_{nn}|\bigr)^T; \tag{2}$$

$$z_1(A) = 1, \qquad z_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1, \quad i = 2, 3, \ldots, n; \tag{3}$$

and

$$h_1(A) = \sum_{j\neq 1} |a_{1j}|, \qquad h_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j(A) + \sum_{j=i+1}^{n} |a_{ij}|, \quad i = 2, 3, \ldots, n. \tag{4}$$

Next, we recall the concept of $\{P_1, P_2\}$-Nekrasov matrices.

Definition 1 ([14]) A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called a Nekrasov matrix if, for each $i \in N$,

$$|a_{ii}| > h_i(A).$$

Motivated by Definition 1, Cvetković et al. in [6] presented the following new subclass of H-matrices, called $\{P_1, P_2\}$-Nekrasov matrices, which contains Nekrasov matrices.

Definition 2 ([6]) Given two permutation matrices P_1 and P_2, a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, is called a $\{P_1, P_2\}$-Nekrasov matrix if

$$d(A) > \min\bigl\{h^{P_1}(A), h^{P_2}(A)\bigr\},$$

where

$$h^{P_k}(A) = P_k h\bigl(P_k^T A P_k\bigr), \quad k = 1, 2,$$

with

$$h\bigl(P_k^T A P_k\bigr) = \bigl[h_1\bigl(P_k^T A P_k\bigr), \ldots, h_n\bigl(P_k^T A P_k\bigr)\bigr]^T,$$

in which $h_i(P_k^T A P_k)$, $i \in N$, being defined as (4).

Remark here that if $P_1 = P_2 = I$, where I is an identity matrix, then $h^{P_1}(A) = h^{P_2}(A) = h(A)$, which implies that a Nekrasov matrix is a $\{P_1, P_2\}$-Nekrasov matrix for $P_1 = P_2 = I$. In addition, note that for any permutation matrix P, the matrix $P^T AP$ has the same set of diagonal entries as does A and, moreover, the same set of row sums as does A. Hence, if A is an *SDD* matrix, then $|a_{ii}| > r_i(A) > h_i^{P_k}(A)$ holds for all $i \in N$, which means that an *SDD* matrix is a $\{P_1, P_2\}$-Nekrasov matrix for any $\{P_1, P_2\}$.

Next, we recall two upper bounds for the infinity norm of the inverse of $\{P_1, P_2\}$-Nekrasov matrices which are given by Cvetković et al. in [6].

Theorem 2 ([6]) *Given a set of permutation matrices $\{P_1, P_2\}$, let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be a $\{P_1, P_2\}$-Nekrasov matrix. Then*

$$\|A^{-1}\|_\infty \leq \frac{\max_{i\in N}(\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|})}{\min_{i\in N}(1 - \min\{\frac{h_i^{P_1}(A)}{|a_{ii}|}, \frac{h_i^{P_2}(A)}{|a_{ii}|}\})}, \tag{5}$$

and

$$\|A^{-1}\|_\infty \leq \frac{\max_{i\in N}(z_i^{P_{k_i}}(A))}{\min_{i\in N}(|a_{ii}| - \min\{h_i^{P_1}(A), h_i^{P_2}(A)\})}, \tag{6}$$

where $z^{P_{k_i}}(A) = P_{k_i} z(P_{k_i}^T A P_{k_i}) = [z_i^{P_{k_i}}(A), \ldots, z_n^{P_{k_i}}(A)]^T$ with $z(P_{k_i}^T A P_{k_i})$ being defined as (3), $h_i^{P_1}(A)$ and $h_i^{P_2}(A)$ are given by Definition 2, and for each index i, the corresponding index $k_i \in \{1, 2\}$ is chosen in such a way that

$$\min\bigl\{h_i^{P_1}(A), h_i^{P_2}(A)\bigr\} = h_i^{P_{k_i}}(A).$$

In what follows, we give a new upper bound for the infinity norm of the inverse of $\{P_1, P_2\}$-Nekrasov matrices. Before that, some lemmas and notation which will be used later are listed.

Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, by $A = D - L - U$ we denote the standard splitting of A into its diagonal (D), strictly lower ($-L$), and strictly upper ($-U$) triangular parts, and $|A| = [|a_{ij}|]$.

Given a $\{P_1, P_2\}$-Nekrasov matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, we recall two special matrices $C \in \mathbb{C}^{n\times n}$ and $\widetilde{C} \in \mathbb{C}^{n\times n}$ as follows:

$$C := \begin{bmatrix} C(1) \\ C(2) \\ \vdots \\ C(n) \end{bmatrix} \quad \text{and} \quad \widetilde{C} := \begin{bmatrix} \widetilde{C}(1) \\ \widetilde{C}(2) \\ \vdots \\ \widetilde{C}(n) \end{bmatrix}, \tag{7}$$

where

$$C(i) = e_i^T P_{k_i} \bigl(|D_{k_i}| - |L_{k_i}|\bigr)^{-1} |U_{k_i}| P_{k_i}^T$$

and

$$\widetilde{C}(i) = e_i^T P_{k_i} \bigl(|D_{k_i}| - |L_{k_i}|\bigr)^{-1} P_{k_i}^T,$$

with $e_i = (0, \ldots, 1, \ldots, 0)^T$ and for each index i, the corresponding index $k_i \in \{1, 2\}$ is chosen in the same way given in Theorem 2.

Lemma 1 ([6]) *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, be a $\{P_1, P_2\}$-Nekrasov matrix, then the matrix $I - C$ is an SDD matrix, where I is the identify matrix and C is defined as in (7).*

Lemma 2 ([1]) *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be a nonsingular H-matrix. Then*

$$\bigl|A^{-1}\bigr| \leq \langle A\rangle^{-1}.$$

Lemma 3 ([6]) *Given any $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, with $a_{ii} \neq 0$ for all $i \in N$, and given a permutation matrix $P \in \mathbb{R}^{n\times n}$, then*

$$h_i^P(A) = |a_{ii}| \bigl[P\bigl(|\widetilde{D}| - |\widetilde{L}|\bigr)^{-1} |\widetilde{U}| e\bigr]_i,$$

where $e = (1, 1, \ldots, 1)^T$ and $P^T AP = \widetilde{D} - \widetilde{L} - \widetilde{U}$ is the standard splitting of the matrix $P^T AP$.

The following lemma will be used in the proof of Theorem 3.

Lemma 4 *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ with $a_{ii} \neq 0$ for all $i \in N$, and $P \in \mathbb{R}^{n\times n}$ is a permutation matrix. Then*

$$z^P(A) = |D| \bigl[P\bigl(|\widetilde{D}| - |\widetilde{L}|\bigr)^{-1}\bigr] e,$$

where $z^P(A) = Pz(P^T AP)$ with $z(P^T AP)$ being defined as (3), $P^T AP = \widetilde{D} - \widetilde{L} - \widetilde{U}$ is the standard splitting of the matrix $P^T AP$, and $e = (1, 1, \ldots, 1)^T$.

Proof Let $x := (\widetilde{D} - \widetilde{L})^{-1}e = (x_1, x_2, \dots, x_n)^T$. Then

$$e = (\widetilde{D} - \widetilde{L})x,$$

i.e.,

$$\widetilde{D}x = \widetilde{L}x + e. \tag{8}$$

By (8), we have

$$|\widetilde{a}_{11}|x_1 = 1, \qquad |\widetilde{a}_{ii}|x_i = 1 + \sum_{j=1}^{i-1} |a_{ij}|x_j, \quad i = 2, \dots, n,$$

which implies that

$$|\widetilde{D}|\big(|\widetilde{D}| - |\widetilde{L}|\big)^{-1}e = z\big(P^T AP\big).$$

Therefore,

$$P|\widetilde{D}|\big(|\widetilde{D}| - |\widetilde{L}|\big)^{-1}e = Pz\big(P^T AP\big).$$

Note that $|\widetilde{D}| = P^T|D|P$ and $P^T P = I$, we can see that

$$|D|\big[P\big(|\widetilde{D}| - |\widetilde{L}|\big)^{-1}\big]e = z^P(A).$$

This completes the proof. □

Now, we give the main result of this paper by Lemmas 1, 2, 3, and 4.

Theorem 3 *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, be a $\{P_1, P_2\}$-Nekrasov matrix. Then*

$$\|A^{-1}\|_\infty \leq \max_{i\in N} \frac{z_i^{P_{k_i}}(A)}{|a_{ii}| - \min\{h_i^{P_1}(A), h_i^{P_2}(A)\}}, \tag{9}$$

where $z_i^{P_{k_i}}(A)$ and $h_i^{P_{k_i}}(A)$, $i \in N$, $k_i \in \{1, 2\}$ are given by Theorem 2.

Proof Since $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is a $\{P_1, P_2\}$-Nekrasov matrix, then from Lemma 1 we have

$$B := I - C$$

is an *SDD* matrix, where C is given by (7). By the proof of Theorem 3.1 (see[6]), we have that, for a fixed $k \in \{1, 2\}$,

$$I - P_k\big(|D_k| - |L_k|\big)^{-1}|U_k|P_k^T = P_k\big(|D_k| - |L_k|\big)^{-1}P_k^T\langle A\rangle. \tag{10}$$

By (7) and (10), we have

$$B := I - C = \widetilde{C}\langle A\rangle,$$

which implies that

$$\langle A\rangle^{-1} = B^{-1}\widetilde{C} = B^{-1}\triangle \cdot \triangle^{-1}\widetilde{C},$$

where

$$\triangle = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n), \quad \delta_i > 0, i = 1, 2, \ldots, n.$$

Since a $\{P_1, P_2\}$-Nekrasov matrix is an H-matrix, we have from Lemma 2 that

$$\|A^{-1}\|_\infty \le \|\langle A\rangle^{-1}\|_\infty \le \|B^{-1}\triangle\|_\infty \cdot \|\triangle^{-1}\widetilde{C}\|_\infty. \tag{11}$$

First, we estimate $\|\triangle^{-1}\widetilde{C}\|_\infty$. Because $|D_{k_i}| - |L_{k_i}|$ for $k_i \in \{1, 2\}$ is an M-matrix, so we can take a positive diagonal matrix $\triangle = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ with

$$\delta_i = \left[P_{k_i}\left(|D_{k_i}| - |L_{k_i}|\right)^{-1} e\right]_i \quad \text{for all } i \in N. \tag{12}$$

Note that

$$\widetilde{C} := \begin{bmatrix} \widetilde{C}(1) \\ \widetilde{C}(2) \\ \vdots \\ \widetilde{C}(n) \end{bmatrix} \in \mathbb{C}^{n\times n},$$

where

$$\widetilde{C}(i) = e_i^T P_{k_i}\left(|D_{k_i}| - |L_{k_i}|\right)^{-1} P_{k_i}^T, \quad k_i \in \{1, 2\}.$$

It follows from (12) that

$$\begin{aligned} \|\triangle^{-1}\widetilde{C}\|_\infty &= \max_{i\in N}\left\{\left[\triangle^{-1}\widetilde{C}e\right]_i\right\} = \max_{i\in N}\left\{\frac{[P_{k_i}(|D_{k_i}| - |L_{k_i}|)^{-1}P_{k_i}^T e]_i}{\delta_i}\right\} \\ &= \max_{i\in N}\left\{\frac{[P_{k_i}(|D_{k_i}| - |L_{k_i}|)^{-1} e]_i}{\delta_i}\right\} \\ &= 1. \end{aligned} \tag{13}$$

Next, we estimate $\|B^{-1}\triangle\|_\infty$. Observe that $B := I - C$, where C is given by (7). Then, for each $i \in N$, we have

$$\begin{aligned} [Be]_i &= \left[(I - C)e\right]_i = 1 - \left[P_{k_i}\left(|D_{k_i}| - |L_{k_i}|\right)^{-1}|U_{k_i}|P_{k_i}^T e\right]_i \\ &= 1 - \left[P_{k_i}\left(|D_{k_i}| - |L_{k_i}|\right)^{-1}|U_{k_i}|e\right]_i \\ &= 1 - \frac{h_i^{P_{k_i}}(A)}{|a_{ii}|} \quad \text{(by Lemma 3)} \\ &= 1 - \min\left\{\frac{h_i^{P_1}(A)}{|a_{ii}|}, \frac{h_i^{P_2}(A)}{|a_{ii}|}\right\}. \end{aligned}$$

By Lemma 1, we have B is an SDD matrix. Since $\triangle^{-1}$ is a positive diagonal matrix, it holds that $\triangle^{-1}B$ is also an SDD matrix. Hence, applying Varah's bound (1), we have

$$\|B^{-1}\triangle\|_\infty \leq \frac{1}{\min_{i\in N}(\triangle^{-1}Be)_i} = \frac{1}{\min_{i\in N}[\frac{1}{\delta_i}\cdot(Be)_i]} = \max_{i\in N}\frac{\delta_i}{1-\min\{\frac{h_i^{P_1}(A)}{|a_{ii}|},\frac{h_i^{P_2}(A)}{|a_{ii}|}\}}$$
$$= \max_{i\in N}\frac{\delta_i|a_{ii}|}{|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\}}.$$

By Lemma 4, we have

$$z_i^{P_{k_i}}(A) = |a_{ii}|\big[P_{k_i}\big(|D_{k_i}|-|L_{k_i}|\big)^{-1}e\big]_i, \quad i\in N,$$

which together with (12) implies that

$$\delta_i|a_{ii}| = z_i^{P_{k_i}}(A), \quad i\in N.$$

Hence,

$$\|B^{-1}\triangle\|_\infty \leq \max_{i\in N}\frac{\delta_i|a_{ii}|}{|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\}} = \max_{i\in N}\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\}}, \tag{14}$$

where $k_i \in \{1,2\}$ is chosen in such a way that

$$\min\big\{h_i^{P_1}(A),h_i^{P_2}(A)\big\} = h_i^{P_{k_i}}(A) \quad \text{for each } i\in N.$$

Now, the conclusion follows from (11), (13), and (14). □

The following comparison theorem shows that bound (9) of Theorem 3 is better than bounds (5) and (6) of Theorem 2 (bound (8) of Theorem 3.1 and bound (10) of Theorem 3.2 in [6]).

Theorem 4 *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, be a $\{P_1,P_2\}$-Nekrasov matrix. Then*

$$\max_{i\in N}\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\}} \leq \frac{\max_{i\in N}(\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|})}{\min_{i\in N}(1-\min\{\frac{h_i^{P_1}(A)}{|a_{ii}|},\frac{h_i^{P_2}(A)}{|a_{ii}|}\})} \tag{15}$$

and

$$\max_{i\in N}\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\}} \leq \frac{\max_{i\in N}(z_i^{P_{k_i}}(A))}{\min_{i\in N}(|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\})}. \tag{16}$$

Furthermore, equality in (15) holds if and only if, for certain $l\in N$, the following two conditions hold:

$$\max_{i\in N}\left(\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|}\right) = \frac{z_l^{P_{k_l}}(A)}{|a_{ll}|}$$

and

$$\max_{i\in N}\left(\min\left\{\frac{h_i^{P_1}(A)}{|a_{ii}|},\frac{h_i^{P_2}(A)}{|a_{ii}|}\right\}\right)=\min\left\{\frac{h_l^{P_1}(A)}{|a_{ll}|},\frac{h_l^{P_2}(A)}{|a_{ll}|}\right\}.$$

Similarly, equality in (16) *holds if and only if, for certain* $l\in N$, *the following two conditions hold*:

$$\max_{i\in N}\bigl(z_i^{P_{k_i}}(A)\bigr)=z_l^{P_{k_l}}(A) \tag{17}$$

and

$$\min_{i\in N}\bigl(|a_{ii}|-\min\bigl\{h_i^{P_1}(A),h_i^{P_2}(A)\bigr\}\bigr)=|a_{ll}|-\min\bigl\{h_l^{P_1}(A),h_l^{P_2}(A)\bigr\}. \tag{18}$$

Proof It is easy to see that inequality in (15) holds, and the inequality in (16) also holds if we use the following relation:

$$\max_{i\in N}\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\}}=\max_{i\in N}\frac{\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|}}{1-\min\{\frac{h_i^{P_1}(A)}{|a_{ii}|},\frac{h_i^{P_2}(A)}{|a_{ii}|}\}}.$$

Next, we prove that the case of equality in (16) holds if and only if (17) and (18) hold. Suppose that

$$\max_{i\in N}\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\}}=\frac{\max_{i\in N}(z_i^{P_{k_i}}(A))}{\min_{i\in N}(|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\})}.$$

Note that

$$\max_{i\in N}\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\}}=\frac{z_l^{P_{k_l}}(A)}{|a_{ll}|-\min\{h_l^{P_1}(A),h_l^{P_2}(A)\}}\quad\text{for some } l\in N.$$

Therefore,

$$\frac{\max_{i\in N}(z_i^{P_{k_i}}(A))}{\min_{i\in N}(|a_{ii}|-\min\{h_i^{P_1}(A),h_i^{P_2}(A)\})}=\frac{z_l^{P_{k_l}}(A)}{|a_{ll}|-\min\{h_l^{P_1}(A),h_l^{P_2}(A)\}}. \tag{19}$$

Since

$$|a_{ll}|-\min\bigl\{h_l^{P_1}(A),h_l^{P_2}(A)\bigr\}\ge\min_{i\in N}\bigl(|a_{ii}|-\min\bigl\{h_i^{P_1}(A),h_i^{P_2}(A)\bigr\}\bigr),$$

it follows from (21) that

$$z_l^{P_{k_l}}(A)\ge\max_{i\in N}\bigl(z_i^{P_{k_i}}(A)\bigr),$$

which implies (17) holds, and thus (18) holds from (17) and (19).

Conversely, if conditions (17) and (18) hold for some $l \in N$, then we have

$$\begin{aligned}\frac{\max_{i\in N}(z_i^{P_{k_i}}(A))}{\min_{i\in N}(|a_{ii}| - \min\{h_i^{P_1}(A), h_i^{P_2}(A)\})} &= \frac{z_l^{P_{k_l}}(A)}{|a_{ll}| - \min\{h_l^{P_1}(A), h_l^{P_2}(A)\}} \\ &\le \max_{i\in N} \frac{z_i^{P_{k_i}}(A)}{|a_{ii}| - \min\{h_i^{P_1}(A), h_i^{P_2}(A)\}} \\ &\le \frac{\max_{i\in N}(z_i^{P_{k_i}}(A))}{\min_{i\in N}(|a_{ii}| - \min\{h_i^{P_1}(A), h_i^{P_2}(A)\})},\end{aligned}$$

this implies that the equality in (16) holds. The equality in (15) can also be proved in a similar way. The proof is completed. □

Since an *SDD* matrix is a $\{P_1, P_2\}$-Nekrasov matrix, by Theorem 3, a new upper bound for $\|A^{-1}\|_\infty$ when A is an *SDD* matrix can be obtained. As expected, the following theorem shows that this new bound works better than Varah's bound of Theorem 1.

Theorem 5 *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be an SDD matrix. Then, for any set of permutation matrices $\{P_1, P_2\}$,*

$$\|A^{-1}\|_\infty \le \max_{i\in N} \frac{z_i^{P_{k_i}}(A)}{|a_{ii}| - \min\{h_i^{P_1}(A), h_i^{P_2}(A)\}}. \tag{20}$$

Furthermore,

$$\max_{i\in N} \frac{z_i^{P_{k_i}}(A)}{|a_{ii}| - \min\{h_i^{P_1}(A), h_i^{P_2}(A)\}} \le \frac{1}{\min_{i\in N}\{|a_{ii}| - r_i(A)\}}. \tag{21}$$

Proof Since an *SDD* matrix is a $\{P_1, P_2\}$-Nekrasov matrix for any permutation matrices P_1 and P_2, so (20) directly follows from Theorem 2. We next prove that (21) holds.

Let $d := d(A)e = |D|e$ and $P_{k_i}^T A P_{k_i} = D_{k_i} - U_{k_i} - L_{k_i}$ be the standard splitting of the matrix $P_{k_i}^T A P_{k_i}$ for $k_i \in \{1, 2\}$. Then, by Lemma 3, Lemma 4, and $\langle P_{k_i}^T A P_{k_i}\rangle = |D_{k_i}| - |U_{k_i}| - |L_{k_i}|$, we have

$$\begin{aligned} d - h^{P_{k_i}}(A) &= |D|e - P_{k_i}|D_{k_i}|\big(|D_{k_i}| - |L_{k_i}|\big)^{-1}|U_{k_i}|e \\ &= P_{k_i}|D_{k_i}|P_{k_i}^T e - P_{k_i}|D_{k_i}|\big(|D_{k_i}| - |L_{k_i}|\big)^{-1}|U_{k_i}|e \\ &= P_{k_i}|D_{k_i}|e - P_{k_i}|D_{k_i}|\big(|D_{k_i}| - |L_{k_i}|\big)^{-1}|U_{k_i}|e \\ &= P_{k_i}|D_{k_i}|\big(I - \big(|D_{k_i}| - |L_{k_i}|\big)^{-1}|U_{k_i}|\big)e \\ &= P_{k_i}|D_{k_i}|\big(|D_{k_i}| - |L_{k_i}|\big)^{-1} \cdot \langle P_{k_i}^T A P_{k_i}\rangle e \\ &\ge \min_{i\in N}\big\{\langle P_{k_i}^T A P_{k_i}\rangle e\big\}_i \cdot P_{k_i}|D_{k_i}|\big(|D_{k_i}| - |L_{k_i}|\big)^{-1} e \\ &= \min_{i\in N}\big\{\langle P_{k_i}^T A P_{k_i}\rangle e\big\}_i \cdot |D|P_{k_i}\big(|D_{k_i}| - |L_{k_i}|\big)^{-1} e \\ &= \min_{i\in N}\big\{\langle P_{k_i}^T A P_{k_i}\rangle e\big\}_i \cdot z^{P_{k_i}}(A), \end{aligned}$$

which implies that

$$\begin{aligned} |a_{ii}| - h_i^{P_{k_i}}(A) &\geq \min_{i\in N}\bigl\{\bigl(P_{k_i}^T A P_{k_i}\bigr)e\bigr\}_i \cdot z_i^{P_{k_i}}(A) \\ &= \min_{i\in N}\bigl\{\bigl|\bigl(P_{k_i}^T A P_{k_i}\bigr)_{ii}\bigr| - r_i\bigl(P_{k_i}^T A P_{k_i}\bigr)\bigr\} \cdot z_i^{P_{k_i}}(A), \quad i \in N. \end{aligned} \tag{22}$$

It is easy to see that, for a given permutation matrix P_{k_i}, the matrix $P_{k_i}^T A P_{k_i}$ has the same set of diagonal entries as does A and the same set of row (and column) sums as does A. Therefore,

$$\min_{i\in N}\bigl\{\bigl|\bigl(P_{k_i}^T A P_{k_i}\bigr)_{ii}\bigr| - r_i\bigl(P_{k_i}^T A P_{k_i}\bigr)\bigr\} = \min_{i\in N}\bigl\{|a_{ii}| - r_i(A)\bigr\},$$

which together with (22) implies that

$$\frac{z_i^{P_{k_i}}(A)}{|a_{ii}| - \min\{h_i^{P_1}(A), h_i^{P_2}(A)\}} \leq \frac{z_i^{P_{k_i}}(A)}{|a_{ii}| - h_i^{P_{k_i}}(A)} \leq \frac{1}{\min_{i\in N}(|a_{ii}| - r_i(A))}, \quad i \in N.$$

This completes the proof. □

3 Numerical examples

In this section, we give the numerical example to show that bound (9) in Theorem 3 improves bounds (5) and (6) of Theorem 2, and Varah's bound of Theorem 1.

Example 1 Consider the following matrices in [6]:

$$A_1 = \begin{pmatrix} 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 12 & 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 114 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 14 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 814 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 8 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 8 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 7 & -2 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 4 & 1 & -2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 2 & 8 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 0 & -1 & 0 & 8 \end{pmatrix},$$

Table 1 The upper bounds for $\|A_i^{-1}\|_\infty$, $i = 1, 2, 3$

Matrix	Exact $\|A_i^{-1}\|_\infty$	Varah's bound	$\{P_1, P_2\}$-Nek I	$\{P_1, P_2\}$-Nek II	Bound (9)
A_1	0.1796	0.5000	0.2132	0.2433	**0.1869**
A_2	0.3455	–	0.7726	0.5992	**0.5989**
A_3	1.0578	–	1.1140	1.1255	**1.0646**

$$A_3 = \begin{pmatrix} -1.5 & -0.1 & 0 & -0.1 & 0 & 0 \\ -0.1 & 2 & -0.1 & -1.9 & 0 & 0 \\ 0 & -0.1 & 23 & -0.1 & -0.1 & -0.1 \\ 0 & 0 & -0.5 & 44 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 44 & -0.4 \\ 0 & 0 & -0.5 & 0 & -1 & 1 \end{pmatrix}.$$

Obviously, A_1 is an *SDD* matrix, and thus it is a $\{P_1, P_2\}$-Nekrasov matrix for any set of permutations $\{P_1, P_2\}$. As reported in [6], A_2 is a Nekrasov matrix and A_3 is neither *SDD* nor Nekrasov matrix, but they are both $\{P_1, P_2\}$-Nekrasov matrices for choosing identical permutation P_1 and counteridentical permutation P_2. Hence, by bound (1) of Theorem 1, bounds (5) and (6) of Theorem 2, and bound (9) of Theorem 3, we can compute the upper bounds for the infinity norm of the inverse of A_i, $i = 1, 2, 3$, which are shown in Table 1 (in Table 1 we call bounds (5) and (6) $\{P_1, P_2\}$-Nek I and $\{P_1, P_2\}$-Nek II).

It can be seen from Table 1 that bound (9) in Theorem 3 is better than Varah's bound for strictly diagonally dominant matrices, and it is also better than (5) and (6) in Theorem 2 (Theorem 3.1 and Theorem 3.2 in [6]) for $\{P_1, P_2\}$-Nekrasov matrices.

4 Conclusions

In this paper, we presented a new upper bound for the infinity norm of the inverse of $\{P_1, P_2\}$-Nekrasov matrices and proved that the new bound improves those bounds obtained in [6] for $\{P_1, P_2\}$-Nekrasov matrices and well-known Varah's bound for strictly diagonally dominant matrices. Numerical examples were included to illustrate the corresponding results.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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