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General modified viscosity implicit rules for generalized asymptotically nonexpansive mappings in complete CAT(0) spaces

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Abstract

It is well known that the concept (generalized) asymptotically nonexpansive is closely related to the theory of fixed points in Banach spaces, and the implicit midpoint rule is one of the powerful numerical methods for solving differential equations. The purpose of this paper is to introduce a class of new general modified viscosity implicit rules of generalized asymptotically nonexpansive mappings in complete CAT(0) spaces, and to prove some strong convergence theorems of the procedure generated by the new general modified viscosity implicit rules under some suitable conditions. The results presented in this paper improve and extend varieties of results in the recent literature.

Keywords: General modified viscosity implicit rule; Fixed point; Strong convergence; Generalized asymptotically nonexpansive mapping; CAT(0) space

1 Introduction

In 1972, Goebel and Kirk [1] introduced the concept of asymptotically nonexpansive mapping, which is closely related to the theory of fixed points in Banach spaces. Whereafter, Zhou et al. [2] discussed convergence of modified Ishikawa and Mann iterative sequences for approximating the fixed points of a class of generalized asymptotically nonexpansive mappings. Recently, the iterative approximation problems of fixed points for nonexpansive mapping, asymptotically nonexpansive mapping, and asymptotically nonexpansive type mapping in Hilbert space or Banach spaces have been studied by many authors. See, for example, [3–7] and the references therein. It is well known that *Mann's* and *Ishikawa's* iterations have the only weak convergence theorem even in a Hilbert space. As a counterexample, Bauschke et al. [8] also showed that the algorithm only converges weakly but not strongly. In order to obtain strong convergence theorems, Moudafi [9] introduced the viscosity approximation method for nonexpansive mappings in Hilbert spaces according to the ideas of Attouch [10]. It is an extension of Halpern iteration method, the refinements of which in Hilbert spaces and extensions to Banach spaces were obtained by Xu [11].

Furthermore, Shi and Chen [12] first studied the convergence theorems of the following *Moudafi* viscosity iteration method for a nonexpansive mapping $T : C \to C$ with $F(T) =$

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$\{x \in C | x = T(x)\} \neq \emptyset$ in a CAT(0) space X: For any $x_0 \in C$,

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.1}$$

where $C \subseteq X$ is a nonempty closed convex subset, $f : C \to C$ is a contraction mapping, $\{\alpha_n\} \subseteq [0, 1]$ is a sequence satisfying certain conditions. In connection with work in the Hilbert ball studied by Reich and Shemen [13] and Kopecká and Reich [14], very recently, we considered strong convergence of viscosity iterative approximation methods for set-valued nonexpansive mappings in [15, 16].

On the other hand, for initial value problem of the ordinary differential equation

$$x'(t) = f\big(x(t)\big), \qquad x(0) = x_0, \tag{1.2}$$

the implicit midpoint rule, which is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations, generates a sequence $\{x_n\}$ by the recursion procedure

$$x_{n+1} = x_n + hf\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.3}$$

where $h > 0$ is a step size. It is known that if $f : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{x_n\}$ generated by (1.3) converges to the exact solution of (1.2) as $h \to 0$ uniformly over $t \in [0, u]$ for any fixed $u > 0$. Based on the above fact, Alghamdi et al. [17] proposed the following implicit midpoint rule for nonexpansive mappings in Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.4}$$

where $\alpha_n \in [0, 1]$ and $T : H \to H$ is a nonexpansive mapping. The authors proved the weak convergence of (1.4) under some additional conditions on $\{\alpha_n\}$. In 2015, using the viscosity approximation method associated with implicit midpoint rule, Xu et al. [18] presented the following viscosity implicit midpoint rule for nonexpansive mappings in a Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.5}$$

under some suitable conditions and proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a point $q \in F(T)$, which is also the unique solution of the variational inequality

$$\big\langle (I - f)q, x - q \big\rangle \geq 0, \quad \forall x \in F(T),$$

where I is an identity mapping. Since then, the iterative method (1.5) was generalized to Banach spaces, CAT(0) space, and geodesic spaces, which were studied by Luo et al. [19], Zao et al. [20], and Preechasilp [21], respectively. In connection with variational inequalities, asymptotically nonexpansive mappings, equilibrium problems, algorithm of (modified) viscosity implicit rules, and so on have been studied by many authors. See, for example, [21–29] and the references therein.

Recently, Li and Liu [30] extended viscosity implicit midpoint rule to the following asymptotically nonexpansive mapping in a CAT(0) space X: For any $x_0 \in C \subseteq X$,

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.6}$$

where $T : C \to C$ is an asymptotically nonexpansive mapping and $f : C \to C$ is a contractive mapping and $\alpha_n \in [0,1]$. Moreover, under some suitable conditions, the authors proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a fixed point $q \in F(T)$ such that $q = P_{F(T)}f(q)$, which is also the unique solution of the variational inequality

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0, \quad x \in F(T). \tag{1.7}$$

Similar research was accomplished by Zao et al. [31] in Hilbert spaces. Further, Ke and Ma [32] and Yan and Hu [33] established strong convergence theorems for the generalized viscosity implicit rules of (asymptotically) nonexpansive mappings in Hilbert spaces to a generalized viscosity rule of nonexpansive mappings in Hilbert spaces. Here, the rule is an extension of the implicit midpoint rule presented in [30, 31, 34].

Motivated and inspired by the above work, in this paper, we consider the following new general modified viscosity implicit rules of generalized asymptotically nonexpansive mappings in a complete CAT(0) space (X,d):

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n\big(s_n x_n \oplus (1-s_n)x_{n+1}\big), \quad \forall n \geq 0, \tag{1.8}$$

where $x_0 \in C \subseteq X$ is an arbitrary fixed element, $\{\alpha_n\}, \{s_n\} \subseteq [0,1]$, $T : C \to C$ is a generalized asymptotically nonexpansive mapping, and $f : C \to C$ is a contractive mapping. We shall prove that the sequence $\{x_n\}$ defined by (1.8) converges strongly to $q \in F(T)$ such that $q = P_{F(T)}f(q)$ is the unique solution of the variational inequality

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0, \quad x \in F(T).$$

2 Preliminaries

In this section, we will give some definitions and lemmas for proving our main results.

Throughout this paper, let (X,d) be a metric space, $C \subseteq X$ be a nonempty subset.

Definition 2.1 A nonlinear mapping $T : C \to C$ is said to be

(i) *contraction* if there exists a constant $\kappa \in [0,1)$ such that

$$d\big(T(x), T(y)\big) \leq \kappa d(x,y), \quad \forall x,y \in C;$$

when $\kappa = 1$, then T is called *nonexpansive*;

(ii) *asymptotically nonexpansive* if there exists a real number sequence $\{k_n\} \subseteq [1,+\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d\big(T^n x, T^n y\big) \leq k_n d(x,y), \quad \forall x,y \in C, n \geq 1;$$

(iii) *asymptotically nonexpansive mapping in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n\to\infty}\sup_{x,y\in C}\bigl\{d\bigl(T^n x, T^n y\bigr)-d(x,y)\bigr\}\leq 0;$$

(iv) *generalized asymptotically nonexpansive* if there exist two real number sequences $\{k_n\}\subseteq[1,+\infty)$ and $\{\xi_n\}\subseteq[0,+\infty)$ with $\lim_{n\to\infty}k_n=1$ and $\lim_{n\to\infty}\xi_n=0$ such that

$$d\bigl(T^n x, T^n y\bigr)\leq k_n d(x,y)+\xi_n,\quad \forall x,y\in C, n\geq 1;$$

(v) $(\{\mu_n\},\{\xi_n\},\zeta)$*-total asymptotically nonexpansive* if there exist two nonnegative real number sequences $\{\mu_n\}$ and $\{\xi_n\}$ with $\lim_{n\to\infty}\mu_n=0$, $\lim_{n\to\infty}\xi_n=0$ and a strictly increasing continuous function $\zeta:[0,+\infty)\to[0,+\infty)$ with $\zeta(0)=0$ such that

$$d\bigl(T^n x, T^n y\bigr)\leq d(x,y)+\mu_n\xi\bigl(d(x,y)\bigr)+\xi_n,\quad \forall x,y\in C, n\geq 1;$$

(vi) uniformly L-Lipschitzian if there exists a constant $L>0$ such that

$$d\bigl(T^n x, T^n y\bigr)\leq Ld(x,y),\quad \forall x,y\in C, n\geq 1.$$

Remark 2.1 (i) For an asymptotically nonexpansive mapping in the intermediate sense, if we let $\xi_n=\max\{0,\sup_{x,y\in C}\{d(T^n x,T^n y)-d(x,y)\}\}$, then $\lim_{n\to\infty}\xi_n=0$. It follows that relational expression of (iii) is reduced to $d(T^n x,T^n y)\leq d(x,y)+\xi_n$ for any $x,y\in C$ and $n\geq 1$.

(ii) By the above definitions, one can know that nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically nonexpansive mapping in the intermediate sense is generalized asymptotically nonexpansive, a generalized asymptotically nonexpansive mapping is totally asymptotically nonexpansive. But the converse does not hold. For other relevant details of asymptotically nonexpansive mapping in the intermediate sense, totally asymptotically nonexpansive mappings, and so on, see, for example, [1, 7, 35–37] and the references therein.

A geodesic path joining $x\in X$ to $y\in X$ (or, more briefly, a geodesic from x to y) is a map ϕ from a closed interval $[0,l]\subseteq\mathbb{R}$ to X such that $\phi(0)=x$, $\phi(l)=y$, and $d(\phi(s),\phi(t))=|s-t|$ for all $s,t\in[0,l]$. In particular, ϕ is an isometry and $d(x,y)=l$. The image of ϕ is called a geodesic segment (or metric) joining x and y when unique is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points in X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y\in X$. A subset C of X is said to be convex if every pair of points $x,y\in C$ can be joined by a geodesic in X and the image of every such geodesic is contained in C.

A geodesic triangle $\triangle(p,q,r)$ in a geodesic space (X,d) consists of three points p, q, r in X (the vertices of $\triangle$) and a choice of three geodesic segments $[p,q]$, $[q,r]$, $[r,p]$ (the edge of $\triangle$) joining them. A comparison triangle for geodesic triangle $\triangle(p,q,r)$ in X is a triangle $\overline{\triangle}(\bar{p},\bar{q},\bar{r})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{p},\bar{q})=d(p,q),\qquad d_{\mathbb{R}^2}(\bar{q},\bar{r})=d(q,r),\quad\text{and}\quad d_{\mathbb{R}^2}(\bar{r},\bar{p})=d(r,p).$$

A point $\bar{u} \in [\bar{p}, \bar{q}]$ is called a comparison point for $u \in [p, q]$ if $d(p, u) = d_{\mathbb{R}^2}(\bar{p}, \bar{u})$. Similarly, the comparison points on $[\bar{q}, \bar{r}]$ and $[\bar{r}, \bar{p}]$ can be defined in the same way.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let $\triangle$ be a geodesic triangle in (X, d), and let $\overline{\triangle}$ be a comparison triangle for $\triangle$, then $\triangle$ is said to satisfy the CAT(0) inequality if, for any $u, v \in \triangle$ and for their comparison points $\bar{u}, \bar{v} \in \overline{\triangle}$, one has

$$d(u, v) \le d_{\mathbb{R}^2}(\bar{u}, \bar{v}).$$

Complete CAT(0) *space* is often called Hadamard space (see [38]). For other equivalent definitions and basic properties of CAT(0) spaces, we refer the reader to standard texts, such as [39]. It is well known that every CAT(0) space is uniquely geodesic and any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces [39], $\mathbb{R}$-trees [40], Euclidean buildings [41], the complex Hilbert ball with a hyperbolic metric [42], and many others.

Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). It follows from Proposition 2.4 of [39] that, for every $x \in X$, there exists a unique point $x_0 \in C$ such that

$$d(x, x_0) = \inf\bigl\{d(x, y) : y \in C\bigr\}.$$

In this case, x_0 is called the *unique nearest point* of x in C. The metric projection of X onto C is the mapping $P_C : X \to C$ defined by

$$P_C(x) := \text{the unique nearest point of } x \text{ in } C.$$

Let (X, d) be a CAT(0) space. For each $x, y \in X$ and $t \in [0, 1]$, it follows from Lemma 2.1 in [43] that there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = (1 - t)d(x, y) \quad \text{and} \quad d(y, z) = td(x, y). \tag{2.1}$$

We shall denote by $tx \oplus (1 - t)y$ the unique point z satisfying (2.1). Now, we collect some elementary facts about CAT(0) spaces which will be used in the proof of our main theorem.

Lemma 2.1 ([43]) *Assume that (X, d) is a* CAT(0) *space. Then, for any $x, y, z \in X$ and $t \in [0, 1]$,*
(i) $d(tx \oplus (1 - t)y, z) \le td(x, z) + (1 - t)d(y, z)$,
(ii) $d^2(tx \oplus (1 - t)y, z) \le td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y)$.

Lemma 2.2 ([11]) *Let $\{s_n\}$ be a non-negative real number sequence satisfying*

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \ge 1,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\beta_n\} \subset \mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.
Then $\{s_n\}$ *converges to zero as* $n \to \infty$.

We note that the concept of Δ-convergence due to Lim [44] was shown by Kirk and Panyanak [45] in CAT(0) spaces to be very similar to the weak convergence in Banach space setting. Next, we give the concept of Δ-convergence and collect some basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d). For $x \in X$, we define a function

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(x, \{x_n\}\big) = r\big(\{x_n\}\big)\big\}.$$

It is well known from [46, Proposition 7] that in CAT(0) spaces, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subseteq X$ is called Δ-convergence to $q \in X$ if $A(\{x_{n_k}\}) = \{q\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = q$ and call q the Δ-limit of $\{x_n\}$. For more details, see, for example, [47] and the references therein.

The uniqueness of an asymptotic center implies that a complete CAT(0) space (X, d) satisfies *Opial's* property [48] if, whenever $\{x_n\} \subseteq X$ Δ-converges to $q \in X$ and for given $y \in X$ with $y \neq q$,

$$\limsup_{n\to\infty} d(x_n, q) < \limsup_{n\to\infty} d(x_n, y).$$

It is well known that every Hilbert space satisfies *Opial's* property (see [48]).

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that "$I - T$ is demiclosed at zero" if the conditions $\{x_n\} \subseteq C$ Δ-converges to $q \in X$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$ imply $q \in F(T)$, where C is a nonempty convex closed subset of X.

Lemma 2.3 ([45]) *Each bounded sequence in a complete* CAT(0) *space always has a* Δ-*convergent subsequence.*

Lemma 2.4 ([43]) *If* C *is a closed convex subset of a complete* CAT(0) *space* (X, d) *and* $\{x_n\}$ *is a bounded sequence in* C, *then the asymptotic center* $A(\{x_n\})$ *of* $\{x_n\}$ *is in* C.

Lemma 2.5 ([49]) *Suppose that* C *is a closed convex subset of a complete* CAT(0) *space* (X, d) *and* $T : C \to C$ *is a totally asymptotically nonexpansive (specially,* T *is a generalized asymptotically nonexpansive) and uniformly Lipschitzian mapping, and* $\{x_n\}$ *is a bounded sequence in* C *such that* $\{x_n\}$ Δ-*converges to* q *and* $\lim_{n\to\infty}(x_n, Tx_n) = 0$. *Then* $q \in F(T)$.

The concept of quasi-linearization was introduced by Berg and Nikolaev [50]. Let us denote a pair (a,b) in $X \times X$ by $\overrightarrow{ab}$ and call it a vector. The quasi-linearization is a map $\langle \cdot,\cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\bigl(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\bigr), \quad \forall a,b,c,d \in X.$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a,b,c,d,x \in X$. A geodesic metric space (X,d) is said to satisfy the Cauchy–Schwarz inequality if

$$\bigl|\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle\bigr| \le d(a,b)d(c,d), \quad \forall a,b,c,d \in X.$$

From [50, Corollary 3], it follows that a geodesic space X is a CAT(0) space if and only if X satisfies the Cauchy–Schwarz inequality. Some other properties of quasi-linearization are included as follows.

Lemma 2.6 ([51]) *Let C be a nonempty closed convex subset of a complete* CAT(0) *space (X,d), $u \in X$, and $x \in C$. Then*

$$x = P_C u \quad \textit{if and only if} \quad \langle \overrightarrow{xu}, \overrightarrow{yx} \rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.7 ([52]) *Let (X,d) be a* CAT(0) *space. Then*

$$d^2(x,u) \le d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle, \quad \forall u,x,y \in X.$$

Lemma 2.8 ([53]) *Assume that (X,d) is a complete* CAT(0) *space, $\{x_n\}$ is a sequence in X, and $q \in X$. Then $\{x_n\}$ Δ-converges to q if and only if $\limsup_{n\to\infty} \langle \overrightarrow{x_n q}, \overrightarrow{yq} \rangle \le 0$ for any $y \in X$.*

Lemma 2.9 ([30]) *Let (X,d) be a complete* CAT(0) *space, and $z_1 = \sigma x \oplus (1-\sigma)u$ and $z_2 = \sigma y \oplus (1-\sigma)u$ for each $u,x,y \in X$ and all $\sigma \in [0,1]$. Then the following inequality holds*:

$$\langle \overrightarrow{z_1 z_2}, \overrightarrow{xz_2} \rangle \le \sigma \langle \overrightarrow{xy}, \overrightarrow{xz_2} \rangle.$$

3 Main results

In this section, by using the pre-requisite and elementary facts presented in Sect. 2, we will prove strong convergence of the sequences generated by the new general modified viscosity implicit rules under some suitable conditions.

Theorem 3.1 *Let C be a nonempty closed convex subset of a complete* CAT(0) *space (X,d), $T : C \to C$ be a uniformly Lipschitzian and generalized asymptotically nonexpansive mapping with sequences $\{k_n\} \subset [1,+\infty)$ and $\{\xi_n\} \subset [0,+\infty)$, and $f : C \to C$ be a contraction with coefficient $\kappa \in [0,1)$. If $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$ and $F(T) \neq \emptyset$, and the following conditions hold*:

(C_1) $\{\alpha_n\} \subseteq (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$,
(C_2) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$,
(C_3) $\lim_{n\to\infty} \frac{k_n^2-1}{\alpha_n} = 0$,
(C_4) $s_n \in (0,1]$ *for all* $n \geq 0$ *and* $\lim_{n\to\infty} s_n = s \in (0,1]$,
(C_5) $\sum_{n=1}^{+\infty} \xi_n < +\infty$,

then the sequence $\{x_n\}$ *generated by* (1.8) *converges strongly as* $n \to \infty$ *to* $q = P_{F(T)} f(q)$, *which solves the following variational inequality*:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0, \quad x \in F(T).$$

Proof We divide the proof into five steps as follows.

Step (I) We first prove that $\{x_n\}$ is a bounded sequence.

In fact, by (1.8) and Lemma 2.1, for any $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &\leq \alpha_n d\bigl(f(x_n), p\bigr) + (1-\alpha_n) d\bigl(T^n\bigl(s_n x_n \oplus (1-s_n)x_{n+1}\bigr), p\bigr) \\ &\leq \alpha_n\bigl(d\bigl(f(x_n), f(p)\bigr) + d\bigl(f(p), p\bigr)\bigr) \\ &\quad + (1-\alpha_n)\bigl[k_n d\bigl(s_n x_n \oplus (1-s_n)x_{n+1}, p\bigr) + \xi_n\bigr] \\ &\leq \alpha_n\bigl(\kappa d(x_n, p) + d\bigl(f(p), p\bigr)\bigr) \\ &\quad + (1-\alpha_n)\bigl[k_n s_n d(x_n, p) + k_n(1-s_n) d(x_{n+1}, p) + \xi_n\bigr], \end{aligned}$$

that is,

$$\begin{aligned} &\bigl(1 - k_n(1-s_n)(1-\alpha_n)\bigr) d(x_{n+1}, p) \\ &\quad \leq \bigl(k_n s_n(1-\alpha_n) + \kappa\alpha_n\bigr) d(x_n, p) + \alpha_n d\bigl(f(p), p\bigr) + (1-\alpha_n)\xi_n. \end{aligned} \tag{3.1}$$

By conditions (C_1), (C_3), and (C_4), for any given positive number ϵ $(0 < \epsilon < 1-\kappa)$, there exists a sufficiently large positive integer N such that, for any $n > N$,

$$k_n - 1 \leq \frac{1}{2}\bigl(k_n^2 - 1\bigr) \leq \epsilon\alpha_n \tag{3.2}$$

and

$$\frac{1-\alpha_n}{1 - k_n(1-s_n)(1-\alpha_n)} \leq \frac{2}{s}, \tag{3.3}$$

where $\lim_{n\to\infty} \frac{1-\alpha_n}{1-k_n(1-s_n)(1-\alpha_n)} = \frac{1}{s}$.

By (3.1)–(3.3), after simplifying, for any $n > N$, we have

$$\begin{aligned} d(x_{n+1}, p) &\leq \frac{k_n s_n(1-\alpha_n) + \kappa\alpha_n}{1 - k_n(1-s_n)(1-\alpha_n)} d(x_n, p) \\ &\quad + \frac{\alpha_n d(f(p), p)}{1 - k_n(1-s_n)(1-\alpha_n)} + \frac{(1-\alpha_n)\xi_n}{1 - k_n(1-s_n)(1-\alpha_n)} \\ &\leq \biggl[1 + \frac{k_n - 1 - k_n\alpha_n + \kappa\alpha_n}{1 - k_n(1-s_n)(1-\alpha_n)}\biggr] d(x_n, p) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha_n d(f(p),p)}{1-k_n(1-s_n)(1-\alpha_n)}+\frac{(1-\alpha_n)\xi_n}{1-k_n(1-s_n)(1-\alpha_n)}\\
&\le\left[1-\frac{(k_n-\epsilon-\kappa)\alpha_n}{1-k_n(1-s_n)(1-\alpha_n)}\right]d(x_n,p)\\
&+\frac{\alpha_n d(f(p),p)}{1-k_n(1-s_n)(1-\alpha_n)}+\frac{(1-\alpha_n)\xi_n}{1-k_n(1-s_n)(1-\alpha_n)}\\
&\le\left[1-\frac{(1-\epsilon-\kappa)\alpha_n}{1-k_n(1-s_n)(1-\alpha_n)}\right]d(x_n,p)\\
&+\frac{(1-\epsilon-\kappa)\alpha_n}{1-k_n(1-s_n)(1-\alpha_n)}\frac{d(f(p),p)}{1-\epsilon-\kappa}+\frac{2\xi_n}{s}\\
&\le\max\left\{d(x_n,p),\frac{d(f(p),p)}{1-\kappa-\epsilon}\right\}+\frac{2\xi_n}{s}.
\end{aligned}
$$

By induction, we have

$$
d(x_n,p)\le\max\left\{d(x_{N+1},p),\frac{d(f(p),p)}{1-\kappa-\epsilon}\right\}+\frac{2}{s}\sum_{k=N+1}^{n-1}\xi_k. \tag{3.4}
$$

Since $\sum_{n=1}^{+\infty}\xi_n<+\infty$, there exists a positive constant M_0 such that

$$
\sum_{k=N+1}^{n-1}\xi_k\le M_0. \tag{3.5}
$$

By (3.4) and (3.5), we know that $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$, $\{T^n(x_n)\}$, $\{T^n(x_{n+1})\}$, and $\{T^n(s_nx_n\oplus(1-s_n)x_{n+1})\}$.

Step (II) We show that $\lim_{n\to\infty}d(x_n,x_{n+1})=0$.

Indeed, it follows from (1.8) that

$$
\begin{aligned}
&d(x_n,x_{n+1})\\
&\quad\le d\big(x_{n+1},T^nx_n\big)+d\big(T^nx_n,x_n\big)\\
&\quad\le d\big(\alpha_nf(x_n)\oplus(1-\alpha_n)T^n\big(s_nx_n\oplus(1-s_n)x_{n+1}\big),T^nx_n\big)\\
&\qquad+d\big(T^nx_n,x_n\big)\\
&\quad\le\alpha_nd\big(f(x_n),T^nx_n\big)+(1-\alpha_n)d\big(T^n\big(s_nx_n\oplus(1-s_n)x_{n+1}\big),T^nx_n\big)\\
&\qquad+d\big(T^nx_n,x_n\big)\\
&\quad\le\alpha_nd\big(f(x_n),T^nx_n\big)+d\big(T^nx_n,x_n\big)\\
&\qquad+(1-\alpha_n)\big(k_nd\big(s_nx_n\oplus(1-s_n)x_{n+1},x_n\big)+\xi_n\big)\\
&\quad\le\alpha_nd\big(f(x_n),T^nx_n\big)+d\big(T^nx_n,x_n\big)\\
&\qquad+k_n(1-s_n)(1-\alpha_n)d(x_n,x_{n+1})+(1-\alpha_n)\xi_n.
\end{aligned}
$$

Since $\{f(x_n)\}$ and $\{T^nx_n\}$ are bounded, there exists $M_1>0$ such that $M_1\ge\sup_{n\ge1}d(f(x_n),T^nx_n)$. Thus,

$$
d(x_n,x_{n+1})\le\alpha_nM_1+d\big(T^nx_n,x_n\big)+k_n(1-s_n)(1-\alpha_n)d(x_n,x_{n+1})+\xi_n,
$$

and so, for all $n > N$,

$$d(x_n, x_{n+1}) \leq \frac{\alpha_n M_1}{1 - k_n(1 - s_n)(1 - \alpha_n)} + \frac{d(T^n x_n, x_n)}{1 - k_n(1 - s_n)(1 - \alpha_n)} + \frac{\xi_n}{1 - k_n(1 - s_n)(1 - \alpha_n)}. \tag{3.6}$$

From $\sum_{n=1}^{+\infty} \xi_n < +\infty$, one can clearly see

$$\lim_{n\to\infty} \xi_n = 0. \tag{3.7}$$

By virtue of conditions (C_1), (C_3), and (C_4), $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$ and (3.6)–(3.7), we now know

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Step (III) $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ will be displayed.

From (1.8), we have

$$\begin{aligned} d(x_{n+1}, T^n x_{n+1}) &= d\big(\alpha_n f(x_n) \oplus (1 - \alpha_n) T^n\big(s_n x_n \oplus (1 - s_n) x_{n+1}\big), T^n x_{n+1}\big) \\ &\leq \alpha_n d\big(f(x_n), T^n x_{n+1}\big) \\ &\quad + (1 - \alpha_n) d\big(T^n\big(s_n x_n \oplus (1 - s_n) x_{n+1}\big), T^n x_{n+1}\big) \\ &\leq \alpha_n d\big(f(x_n), T^n x_{n+1}\big) \\ &\quad + (1 - \alpha_n)\big(k_n d\big(s_n x_n \oplus (1 - s_n) x_{n+1}, x_{n+1}\big) + \xi_n\big) \\ &\leq \alpha_n d\big(f(x_n), T^n x_{n+1}\big) \\ &\quad + k_n s_n (1 - \alpha_n) d(x_n, x_{n+1}) + (1 - \alpha_n)\xi_n \\ &\leq \alpha_n M_1 + k_n s_n (1 - \alpha_n) d(x_n, x_{n+1}) + \xi_n. \end{aligned}$$

It follows from conditions (C_1), (C_3), (C_4) and (3.7), (3.8) that

$$\lim_{n\to\infty} d\big(x_{n+1}, T^n x_{n+1}\big) = 0. \tag{3.9}$$

Further, by the continuity of the generalized asymptotically nonexpansive mapping T, we obtain

$$\lim_{n\to\infty} d\big(Tx_{n+1}, T^{n+1} x_{n+1}\big) = 0. \tag{3.10}$$

Moreover,

$$d(Tx_{n+1}, x_{n+1}) \leq d\big(Tx_{n+1}, T^{n+1} x_{n+1}\big) + d\big(T^{n+1} x_{n+1}, x_{n+1}\big). \tag{3.11}$$

It follows from (3.9)–(3.11) that $\lim_{n\to\infty} d(Tx_{n+1}, x_{n+1}) = 0$, which implies that

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{3.12}$$

Step (IV) We prove that

$$w_{\Delta}\{x_n\} := \bigcup_{\{u_n\}\subset\{x_n\}} \{A(\{u_n\})\} \subset F(T), \tag{3.13}$$

where $A(\{u_n\})$ is the asymptotic center of $\{u_n\}$. Let $u \in w_{\Delta}\{x_n\}$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. From Lemma 2.3, it follows that there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = u$. In view of (3.12), we have

$$\lim_{n\to\infty} d(Tv_n, v_n) = 0.$$

By Lemma 2.5 and demi-closedness of $I - T$ at zero, also by Lemma 2.4, we know that $u \in C$ and $Tu = u$. Hence, $u \in F(T)$, i.e., $w_{\Delta}\{x_n\} \subset F(T)$.

Step (V) We show $x_n \to q$ as $n \to \infty$, where $q \in F(T)$ is the unique fixed point of contraction $P_{F(T)}f$, that is, $q = P_{F(T)}f(q)$, which solves the following variational inequality:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0, \quad x \in F(T).$$

Firstly, mapping $P_{F(T)}f$ is a contraction, thus, obviously, there exists a unique $q \in F(T)$ such that $q = P_{F(T)}f(q)$. It follows from Lemma 2.6 that, for any $x \in F(T)$,

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0.$$

Next, we reveal

$$\limsup_{n\to\infty} \langle \overrightarrow{qf(q)}, \overrightarrow{qx_n} \rangle \leq 0.$$

As a matter of fact, since $\{x_n\}$ is bounded, it follows from Lemma 2.3 that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$, which Δ-converges to a point p. By Lemma 2.8 and (3.13), we get

$$\limsup_{n\to\infty} \langle \overrightarrow{qf(q)}, \overrightarrow{qx_n} \rangle = \lim_{k\to\infty} \langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n_k}} \rangle = \langle \overrightarrow{qf(q)}, \overrightarrow{qp} \rangle \leq 0. \tag{3.14}$$

Finally, for each $n \geq 0$, let $z_n = \alpha_n q \oplus (1-\alpha_n)T^n(s_n x_n \oplus (1-s_n)x_{n+1})$. By Lemma 2.9, we know

$$\langle \overrightarrow{z_n x_{n+1}}, \overrightarrow{qx_{n+1}} \rangle \leq \alpha_n \langle \overrightarrow{qf(x_n)}, \overrightarrow{qx_{n+1}} \rangle.$$

Thus, it follows from Lemmas 2.7 and 2.1, the definition of quasi-linearization, and the Cauchy–Schwarz inequality that

$$\begin{aligned} &d^2(x_{n+1}, q) \\ &\quad \leq d^2(z_n, q) + 2\langle \overrightarrow{z_n x_{n+1}}, \overrightarrow{qx_{n+1}} \rangle \\ &\quad \leq (1-\alpha_n)^2 d^2\big(T^n\big(s_n x_n \oplus (1-s_n)x_{n+1}\big), q\big) + 2\langle \overrightarrow{z_n x_{n+1}}, \overrightarrow{qx_{n+1}} \rangle \end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)^2\big(k_n d\big(s_n x_n \oplus (1-s_n)x_{n+1}, q\big) + \xi_n\big)^2 + 2\alpha_n\langle \overrightarrow{qf(x_n)}, \overrightarrow{qx_{n+1}}\rangle \\
&\le (1-\alpha_n)^2\big(k_n^2 d^2\big(s_n x_n \oplus (1-s_n)x_{n+1}, q\big) + M\xi_n\big) \\
&\quad + 2\alpha_n d\big(f(q), f(x_n)\big) d(x_{n+1}, q) + 2\alpha_n\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle \\
&\le (1-\alpha_n)^2 k_n^2 d^2\big(s_n x_n \oplus (1-s_n)x_{n+1}, q\big) \\
&\quad + M(1-\alpha_n)^2\xi_n + 2\kappa\alpha_n d(x_n, q) d(x_{n+1}, q) + 2\alpha_n\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle \\
&\le (1-\alpha_n)^2 k_n^2\big(s_n d^2(x_n, q) + (1-s_n) d^2(x_{n+1}, q) - s_n(1-s_n) d^2(x_{n+1}, x_n)\big) \\
&\quad + M\xi_n + \kappa\alpha_n\big(d^2(x_n, q) + d^2(x_{n+1}, q)\big) + 2\alpha_n\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle \\
&\le \big[k_n^2 s_n(1-2\alpha_n) + \kappa\alpha_n\big] d^2(x_n, q) + \big[k_n^2(1-s_n)(1-2\alpha_n) + \kappa\alpha_n\big] d^2(x_{n+1}, q) \\
&\quad + M\xi_n + k_n^2\alpha_n^2\big(s_n d^2(x_n, q) + (1-s_n) d^2(x_{n+1}, q)\big) + 2\alpha_n\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, there exists $M > 0$ such that

$$
\sup_{n\ge 1}\big\{k_n^2 d^2(x_n, q), 2k_n d\big(s_n x_n \oplus (1-s_n)x_{n+1}, q\big) + \xi_n\big\} \le M,
$$

and so

$$
\begin{aligned}
d^2(x_{n+1}, q) \le{}& \big[k_n^2 s_n(1-2\alpha_n) + \kappa\alpha_n\big] d^2(x_n, q) \\
&+ \big[k_n^2(1-s_n)(1-2\alpha_n) + \kappa\alpha_n\big] d^2(x_{n+1}, q) \\
&+ M\xi_n + \alpha_n^2 M + 2\alpha_n\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle.
\end{aligned}
\tag{3.15}
$$

From conditions (C_1), (C_3), and (C_4), it follows that $\lim_{n\to\infty}(1-k_n^2(1-s_n)(1-2\alpha_n)-\kappa\alpha_n) = s > 0$, for sufficiently large $n > N$, we have

$$
1 - k_n^2(1-s_n)(1-2\alpha_n) - \kappa\alpha_n > 0,
$$

and so

$$
\begin{aligned}
&\frac{k_n^2 s_n(1-2\alpha_n) + \kappa\alpha_n}{1 - k_n^2(1-s_n)(1-2\alpha_n) - \kappa\alpha_n} \\
&\quad = \left[1 + \frac{(k_n^2 - 1) - 2k_n^2\alpha_n + 2\kappa\alpha_n}{1 - k_n^2(1-s_n)(1-2\alpha_n) - \kappa\alpha_n}\right] \\
&\quad \le \left[1 - \frac{2(1-\epsilon-\kappa)\alpha_n}{1 - k_n^2(1-s_n)(1-2\alpha_n) - \kappa\alpha_n}\right].
\end{aligned}
\tag{3.16}
$$

It follows from (3.15) and (3.16) that

$$
\begin{aligned}
d^2(x_{n+1}, q) \le{}& \frac{[k_n^2 s_n(1-2\alpha_n) + \kappa\alpha_n] d^2(x_n, q)}{1 - k_n^2(1-s_n)(1-2\alpha_n) - \kappa\alpha_n} \\
&+ \frac{M\xi_n + M\alpha_n^2 + 2\alpha_n\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle}{1 - k_n^2(1-s_n)(1-2\alpha_n) - \kappa\alpha_n}
\end{aligned}
$$

$$\begin{aligned}
&\le \left[1 - \frac{2(1-\epsilon-\kappa)\alpha_n}{1-k_n^2(1-s_n)(1-2\alpha_n)-\kappa\alpha_n}\right] d^2(x_n,q) \\
&\quad + \frac{\alpha_n(M\frac{\xi_n}{\alpha_n} + M\alpha_n + 2\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle)}{1-k_n^2(1-s_n)(1-2\alpha_n)-\kappa\alpha_n} \\
&= (1-\gamma_n)d^2(x_n,q) + \delta_n,
\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
\gamma_n &= \frac{2(1-\epsilon-\kappa)\alpha_n}{1-k_n^2(1-s_n)(1-2\alpha_n)-\kappa\alpha_n}, \\
\delta_n &= \frac{\alpha_n(M\frac{\xi_n}{\alpha_n} + M\alpha_n + 2\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle)}{1-k_n^2(1-s_n)(1-2\alpha_n)-\kappa\alpha_n}.
\end{aligned}$$

By conditions (C_1), (C_2), and (C_5),

$$\lim_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0. \tag{3.18}$$

It follows from conditions (C_1)–(C_4), (3.14), and (3.18) that $\gamma_n \subset (0,1)$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$,

$$\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty} \frac{\frac{\xi_n}{\alpha_n}M + \alpha_n M + 2\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}}\rangle}{2(1-\epsilon-\kappa)} \le 0.$$

By (3.17) and Lemma 2.2, we know that $x_n \to q = P_{F(T)}f(q)$, which solves the following variational inequality:

$$\langle \overrightarrow{f(q)q}, \overrightarrow{qx}\rangle \ge 0, \quad \forall x \in F(T).$$

This completes the proof of Theorem 3.1. □

If $k_n \equiv 1$, for any $n \ge 1$, that is, T is an asymptotically nonexpansive mapping in the intermediate sense, then from Theorem 3.1 we have the following result.

Corollary 3.1 *Assume that f, C, and (X,d) are the same as in Theorem 3.1. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense with sequence $\{\xi_n\} \subset [0,+\infty)$. If $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$, $F(T) \neq \emptyset$, and conditions (C_1), (C_2), (C_4), and (C_5) in Theorem 3.1 hold, then, for an arbitrary initial point $x_0 \in C$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n\bigl(s_n x_n \oplus (1-s_n)x_{n+1}\bigr), \quad \forall n \ge 0,$$

converges strongly to $q = P_{F(T)}f(q)$, which is the unique solution of the following variational inequality:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq}\rangle \ge 0, \quad x \in F(T).$$

Remark 3.1 Corollary 3.1 still is a new consequence.

If $T: C \to C$ is an asymptotically nonexpansive mapping, then Theorem 3.1 should be rewritten as the following two corollaries.

Corollary 3.2 *Let f, C, and (X,d) be the same as in Theorem 3.1, $T: C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,+\infty)$. If $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$, $F(T) \neq \emptyset$, and conditions (C_1)–(C_4) in Theorem 3.1 hold, then for an arbitrary initial point $x_0 \in C$, the following sequence $\{x_n\}$*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T^n \bigl(s_n x_n \oplus (1-s_n) x_{n+1}\bigr), \quad \forall n \geq 0,$$

converges strongly to $q = P_{F(T)} f(q)$, which is the unique solution of the variational inequality

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0, \quad x \in F(T).$$

Proof Take $\xi_n \equiv 0$ for any $n \geq 1$ in Theorem 3.1 and note that condition (C_5) in Theorem 3.1 is satisfied automatically. Hence the conclusion of Corollary 3.2 can be obtained from Theorem 3.1 immediately. □

Remark 3.2 Corollary 3.2 improves and extends the main results of [34] in regard to parameter α_n.

Corollary 3.3 *Suppose that T, f, C, and (X,d) are the same as in Corollary 3.2. If $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$, $F(T) \neq \emptyset$, and conditions (C_1)–(C_3) in Theorem 3.1 hold, then for any given initial point $x_0 \in C$, the sequence $\{x_n\}$ generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T^n \left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad \forall n \geq 0,$$

converges strongly as $n \to \infty$ to $q = P_{F(T)} f(q)$, which solves the following variational inequality:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0, \quad x \in F(T).$$

Proof Take $s_n = \frac{1}{2}$ for any $n \geq 1$ in Corollary 3.2. Then condition (C_4) in Corollary 3.2 holds. From Corollary 3.2, this completes the proof. □

Remark 3.3 Corollary 3.3 is the main result of Li and Liu [30]. It also extends and improves the main results of [31] from a Hilbert space to a CAT(0) space.

Corollary 3.4 *Let f, C, and (X,d) be the same as in Theorem 3.1, $T: C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If conditions (C_1), (C_2), and (C_4) in Theorem 3.1 hold, then for any initial point $x_0 \in C$, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T \bigl(s_n x_n \oplus (1-s_n) x_{n+1}\bigr), \quad \forall n \geq 0,$$

converges strongly to $q = P_{F(T)} f(q)$, which is the unique solution of the following variational inequality:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0, \quad x \in F(T).$$

Proof For each $n \geq 1$, let $\xi_n \equiv 0$ and $k_n \equiv 1$ in Theorem 3.1. Then conditions (C_3) and (C_5) in Theorem 3.1 are satisfied. The condition of $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$ is not needed, because it can be proved very similarly to [31]. Thus, the conclusion of Corollary 3.4 can be obtained from Theorem 3.1 immediately. □

Remark 3.4 Corollary 3.4 improves and extends the corresponding results of Theorem 2.1 in [32] from a Hilbert space to a CAT(0) space, monotonic increase of sequence $\{s_n\}$ and condition $\sum_{n=1}^{+\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ are not also needed.

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The authors declare that they have no competing interests.

Authors' contributions
T-JX carried out the proof of the corollaries and gave some examples to show the main results. H-YL conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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References

1. Goebel, K., Kirk, W.A.: A fixed point theorem for asymptotically non-expansive mappings. Proc. Am. Math. Soc. **35**, 171–174 (1972)
2. Zhou, H.Y., Cho, Y.J., Grabiec, M.: Iterative processes for generalized asymptotically nonexpansive mappings in Banach spaces. Panam. Math. J. **13**(4), 99–107 (2003)
3. Sahin, A., Basarir, M.: On the strong and Δ convergence of SP-iteration on CAT(0) spaces. J. Inequal. Appl. **2013**, 311 (2013)
4. Kumam, P., Saluja, G.S., Nashine, H.K.: Convergence of modified *S*-iteration process for two asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces. J. Inequal. Appl. **2014**, 368 (2014)
5. Cholamjiak, W., Chutibutr, N., Weerakham, S.: Weak and strong convergence theorems for the modified Ishikawa iteration for two hybrid multivalued mappings in Hilbert spaces. Commun. Korean Math. Soc. **33**(3), 767–786 (2018)
6. Sharma, A.: Approximating fixed points of nearly asymptotically nonexpansive mappings in CAT(0) spaces. Arab J. Math. Sci. **24**(2), 166–181 (2018)
7. Bruck, R.E., Kuczumow, T., Reich, S.: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**, 169–179 (1993)
8. Bauschke, H.H., Matoušková, E., Reich, S.: Projection and proximal point methods: convergence results and counterexamples. Nonlinear Anal. **56**, 715–738 (2004)
9. Moudafi, A.: Viscosity approximation methods for fixed points problems. J. Math. Anal. Appl. **241**, 46–55 (2000)
10. Attouch, H.: Viscosity solutions of minimization problems. SIAM J. Optim. **6**, 769–806 (1996)
11. Xu, H.K.: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279–291 (2004)
12. Shi, L.Y., Chen, R.D.: Strong convergence of viscosity approximation methods for nonexpansive mappings in CAT(0) spaces. J. Appl. Math. **2012**, Article ID 421050 (2012)
13. Reich, S., Shemen, L.: A note on Halpern's algorithm in the Hilbert ball. J. Nonlinear Convex Anal. **14**, 853–862 (2013)
14. Kopecká, E., Reich, S.: Approximating fixed points in the Hilbert ball. J. Nonlinear Convex Anal. **15**, 819–829 (2014)
15. Xiong, T.J., Lan, H.Y.: Strong convergence of new two-step viscosity iterative approximation methods for set-valued nonexpansive mappings in CAT(0) spaces. J. Funct. Spaces **2018**, Article ID 1280241 (2018)

16. Xiong, T.J., Lan, H.Y.: New two-step viscosity approximation methods of fixed points for set-valued nonexpansive mappings associated with contraction mappings in CAT(0) spaces. J. Comput. Anal. Appl. **26**(5), 899–909 (2019)
17. Alghamdi, M.A., Alghamdi, M.A., Shahzad, N., Xu, H.K.: The implicit midpoint rule for nonexpansive mappings. Fixed Point Theory Appl. **2014**, 96 (2014)
18. Xu, H.K., Alghamdi, M.A., Shahzad, N.: The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces. Fixed Point Theory Appl. **2015**, 41 (2015)
19. Luo, P., Cai, G., Shehu, Y.: The viscosity iterative algorithms for the implicit midpoint rule of nonexpansive mappings in uniformly smooth Banach spaces. J. Inequal. Appl. **2017**, 154 (2017)
20. Zao, L.C., Chang, S.S., Wang, L., Wang, G.: Viscosity approximation methods for the implicit midpoint rule of nonexpansive mappings in CAT(0) spaces. J. Nonlinear Sci. Appl. **10**, 386–394 (2017)
21. Preechasilp, P.: Viscosity approximation methods for implicit midpoint rule of nonexpansive mappings in geodesic spaces. Bull. Malays. Math. Sci. Soc. **41**(3), 1561–1579 (2018)
22. Cai, G., Shehu, Y., Iyiola, O.S.: Strong convergence results for variational inequalities and fixed point problems using modified viscosity implicit rules. Numer. Algorithms **77**(2), 535–558 (2018)
23. Yan, Q., Cai, G.: Convergence analysis of modified viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**(4), 1125–1140 (2018)
24. Cai, G., Shehu, Y., Iyiola, O.S.: Modified viscosity implicit rules for nonexpansive mappings in Hilbert spaces. J. Fixed Point Theory Appl. **19**(4), 2831–2846 (2017)
25. Marino, G., Rugiano, A.: Strong convergence of a generalized viscosity implicit midpoint rule for nonexpansive mappings and equilibrium problems. J. Nonlinear Convex Anal. **17**(11), 2255–2275 (2016)
26. Takahashi, W., Wen, C.F., Yao, J.C.: The split common fixed point problem for families of generalized demimetric mappings in Banach spaces. Appl. Anal. Optim. **2**(3), 467–486 (2018)
27. Takahashi, W., Wen, C.F., Yao, J.C.: An implicit algorithm for the split common fixed point problem in Hilbert spaces and applications. Appl. Anal. Optim. **1**(3), 423–439 (2017)
28. Qin, X.L., Petrusel, A., Yao, J.C.: CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces. J. Nonlinear Convex Anal. **19**(1), 157–165 (2018)
29. Narashirad, E., Takahashi, W., Yao, J.C.: Strong convergence of projection methods for Bregman asymptotically quasi-nonexpansive mappings and equilibrium problems in Banach spaces. Pac. J. Optim. **10**, 321–342 (2014)
30. Li, Y., Liu, H.B.: Viscosity approximation methods for the implicit midpoint rule of asymptotically nonexpansive mappings in complete CAT(0) spaces. J. Nonlinear Sci. Appl. **10**, 1270–1280 (2017)
31. Zao, L.C., Chang, S.S., Wen, C.F.: Viscosity approximation methods for the implicit midpoint rule of asymptotically nonexpansive mappings in Hilbert spaces. J. Nonlinear Sci. Appl. **9**, 4478–4488 (2016)
32. Ke, Y.F., Ma, C.F.: The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces. Fixed Point Theory Appl. **2015**, 190 (2015)
33. Yan, Q., Hu, S.T.: Strong convergence theorems for the generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces. J. Comput. Anal. Appl. **24**(3), 486–496 (2018)
34. Pakkaranang, N., Kumam, P., Cho, Y.J., Saipara, P., Padcharoen, A., Khaofong, C.: Strong convergence of modified viscosity implicit approximation methods for asymptotically nonexpansive mappings in complete CAT(0) spaces. J. Math. Comput. Sci. **17**, 345–354 (2017)
35. Kaczor, W., Kuczumow, T., Reich, S.: A mean ergodic theorem for mappings which are asymptotically nonexpansive in the intermediate sense. Nonlinear Anal. **47**, 2731–2742 (2001)
36. Alber, Y.I., Chidume, C.E., Zegeye, H.: Approximating fixed points of total asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2006**, Article ID 10673 (2006)
37. Xiong, T.J., Lan, H.Y.: Convergence analysis of new modified iterative approximating processes for two finite families of total asymptotically nonexpansive mappings in hyperbolic spaces. J. Nonlinear Sci. Appl. **10**, 1407–1423 (2017)
38. Khamsi, M.A., Kirk, W.A.: An Introduction to Metric Spaces and Fixed Point Theory. Pure Appl. Math. Wiley, New York (2001)
39. Bridson, M., Haefliger, A.: Metric Spaces of Non-positive Curvature. Springer, Berlin (1999)
40. Kirk, W.A.: Fixed point theorems in CAT(0) spaces and $\mathbb{R}$-trees. Fixed Point Theory Appl. **4**, 309–316 (2004)
41. Brown, K.S. (ed.): Buildings. Springer, New York (1989)
42. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Monographs and Textbooks in Pure and Applied Mathematics, vol. 83. Dekker, New York (1984)
43. Dhompongsa, S., Panyanak, B.: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**, 2572–2579 (2008)
44. Lim, T.C.: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179–182 (1976)
45. Kirk, W.A., Panyanak, B.: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689–3696 (2008)
46. Dhompongsa, S., Kirk, W.A., Sims, B.: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762–772 (2006)
47. Reich, S., Salinas, Z.: Weak convergence of infinite products of operators in Hadamard spaces. Rend. Circ. Mat. Palermo **65**, 55–71 (2016)
48. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)
49. Chang, S.S., Wang, L., Lee, H.W.J., Chan, C.K., Yang, L.: Demiclosed principle and Δ-convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Appl. Math. Comput. **219**(5), 2611–2617 (2012)
50. Berg, I.D., Nikolaev, I.G.: Quasilinearization and curvature of Alexandrov spaces. Geom. Dedic. **133**, 195–218 (2008)
51. Dehghan, H., Rooin, J.: A characterization of metric projection in CAT(0) spaces. In: Proceedings of International Conference on Functional Equation, Geometric Functions and Applications (ICFGA 2012), Iran, 10–12 May 2012, pp. 41–43. Payame Noor University, Tabriz (2012)
52. Wangkeeree, R., Preechasilp, P.: Viscosity approximation methods for nonexpansive mappings in CAT(0) spaces. J. Inequal. Appl. **2013**, 93 (2013)
53. Kakavandi, B.A.: Weak topologies in complete CAT(0) metric spaces. Proc. Am. Math. Soc. **141**, 1029–1039 (2013)