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Some identities of degenerate Euler polynomials associated with degenerate Bernstein polynomials

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Abstract

In this paper, we investigate some properties and identities for degenerate Euler polynomials in connection with degenerate Bernstein polynomials by means of fermionic p-adic integrals on $\mathbb{Z}_p$ and generating functions. In addition, we study two variable degenerate Bernstein polynomials and the degenerate Bernstein operators.

MSC: 11B83; 11S80

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1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$, will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. For $\lambda \in \mathbb{C}_p$ with $|\lambda|_p < p^{-\frac{1}{p-1}}$, the degenerate Euler polynomials are defined by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!} \quad \text{(see [1, 2]).} \tag{1}$$

When $x = 0$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers. The degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!} \quad \text{(see [6]),} \tag{2}$$

where

$$(x)_{0,\lambda} = 1, \qquad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad \text{for } n \geq 1. \tag{3}$$

From (1), we note that

$$\mathcal{E}_{n,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l}\mathcal{E}_{l,\lambda}(x)_{n-l,\lambda} \quad (n \geq 0). \tag{4}$$

Recently, Kim–Kim introduced the degenerate Bernstein polynomials given by

$$\frac{(x)_{k,\lambda}}{k!}t^k(1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=k}^{\infty} B_{k,n}(x|\lambda)\frac{t^n}{n!} \quad \text{(see [8–10])}. \tag{5}$$

Thus, by (5), we get

$$B_{k,n}(x|\lambda) = \begin{cases} \binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}, & \text{if } n \geq k, \\ 0, & \text{if } n < k, \end{cases} \tag{6}$$

where n, k are nonnegative integers.

Let f be a continuous function on $\mathbb{Z}_p$. Then the degenerate Bernstein operator of order n is given by

$$\begin{aligned}
\mathbb{B}_{n,\lambda}(f|\lambda) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} \\
&= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}(x|\lambda) \quad \text{(see [8, 9, 12–15, 17–19])}.
\end{aligned} \tag{7}$$

The fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x \quad \text{(see [3, 9])}. \tag{8}$$

By (8), we get

$$\int_{\mathbb{Z}_p} f(x+1)\,d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) = 2f(0) \quad \text{(see [3, 7, 10, 11, 16])}. \tag{9}$$

From (8), we note that

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}}\,d\mu_{-1}(y) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{10}$$

On the other hand,

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}}\,d\mu_{-1}(y) = \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} (x+y)_{n,\lambda}\,d\mu_{-1}(y)\frac{t^n}{n!}. \tag{11}$$

By (10) and (11), we get

$$\int_{\mathbb{Z}_p} (x+y)_{n,\lambda}\,d\mu_{-1}(y) = \mathcal{E}_{n,\lambda}(x) \quad (n \geq 0) \quad \text{(see [8, 9])}. \tag{12}$$

The study of degenerate versions of some special polynomials and numbers began with the work of Carlitz on the degenerate Bernoulli and Euler polynomials and numbers in [1, 2]. As a continuation of this initiative of Carlitz, Kim and his colleagues have been

introducing various degenerate special polynomials and numbers and investigating their properties, some identities related to them and their applications. This research has been carried out by means of generating functions, combinatorial methods, umbral calculus, p-adic analysis and differential equations (see [8, 9] and the references therein). Here, along the same line and by virtue of fermionic p-adic integrals on $\mathbb{Z}_p$ and generating functions, we investigate some properties and identities for degenerate Euler polynomials related to degenerate Bernstein polynomials. In addition, we study two variable degenerate Bernstein polynomials and the degenerate Bernstein operators.

2 Degenerate Euler and Bernstein polynomials

From (1), we note that

$$
\begin{aligned}
2 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_{m,\lambda}(1)_{n-m,\lambda} + \mathcal{E}_{n,\lambda} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\mathcal{E}_{n,\lambda}(1) + \mathcal{E}_{n,\lambda} \right) \frac{t^n}{n!}.
\end{aligned}
\tag{13}
$$

Comparing the coefficients on both sides of (13), we have

$$
\mathcal{E}_{n,\lambda}(1) + \mathcal{E}_{n,\lambda} = 2\delta_{0,n} \quad (n, k \geq 0),
\tag{14}
$$

where $\delta_{n,k}$ is the Kronecker symbol.

By (1), we easily get

$$
\mathcal{E}_{n,\lambda}(1 - x) = (-1)^n \mathcal{E}_{n,-\lambda}(x) \quad (n \geq 0).
\tag{15}
$$

From (1), (4) and (14), we note that

$$
\begin{aligned}
\mathcal{E}_{n,\lambda}(2) &= \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_{l,\lambda}(1)(1)_{n-l,\lambda} \\
&= (1)_{n,\lambda} + \sum_{l=1}^{n} \binom{n}{l} \mathcal{E}_{l,\lambda}(1)(1)_{n-l,\lambda} \\
&= 2(1)_{n,\lambda} - \sum_{l=0}^{n} \binom{n}{l} (1)_{n-l,\lambda} \mathcal{E}_{l,\lambda} \\
&= 2(1)_{n,\lambda} + \mathcal{E}_{n,\lambda},
\end{aligned}
\tag{16}
$$

where n is a positive integer.

Therefore, by (16), we obtain the following theorem.

Theorem 2.1 *For $n \in \mathbb{N}$, we have*

$$
\mathcal{E}_{n,\lambda}(2) = 2(1)_{n,\lambda} + \mathcal{E}_{n,\lambda}.
$$

Note that

$$(1-x)_{n,\lambda} = (-1)^n (x-1)_{n,-\lambda} \quad (n \geq 0).$$ (17)

Therefore, by (12), (15) and (17), we easily get

$$\int_{\mathbb{Z}_p} (1-x)_{n,\lambda} \, d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} (x-1)_{n,-\lambda} \, d\mu_{-1}(x)$$

$$= \int_{\mathbb{Z}_p} (x+2)_{n,\lambda} \, d\mu_{-1}(x).$$ (18)

Therefore, by (18) and Theorem 2.1, we obtain the following theorem.

Theorem 2.2 *For $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} (1-x)_{n,\lambda} \, d\mu_{-1}(x) = \int_{\mathbb{Z}_p} (x+2)_{n,\lambda} \, d\mu_{-1}(x) = 2(1)_{n,\lambda} + \int_{\mathbb{Z}_p} (x)_{n,\lambda} \, d\mu_{-1}(x).$$

Corollary 2.3 *For $n \in \mathbb{N}$, we have*

$$(-1)^n \mathcal{E}_{n,-\lambda}(-1) = 2(1)_{n,\lambda} + \mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(2).$$

By (4), we get

$$\begin{aligned}
\mathcal{E}_{n,\lambda}(1-x) &= \sum_{l=0}^{n} \binom{n}{l}(1-x)_{n-l,\lambda} \mathcal{E}_{l,\lambda} \\
&= \sum_{l=0}^{n} \binom{n}{l}(x)_{l,\lambda}(1-x)_{n-l,\lambda} \frac{\mathcal{E}_{l,\lambda}}{(x)_{l,\lambda}} \\
&= \sum_{l=0}^{n} B_{l,n}(x|\lambda)\mathcal{E}_{l,\lambda} \frac{1}{(x)_{l,\lambda}}.
\end{aligned}$$ (19)

Let

$$\frac{1}{(x)_{l,\lambda}} = \frac{1}{x(x-\lambda)(x-2\lambda)\cdots(x-(l-1)\lambda)} = \sum_{k=0}^{l-1} \frac{A_k}{x-k\lambda} \quad (l \in \mathbb{N}).$$ (20)

Then we have

$$A_k = \lambda^{1-l} \prod_{\substack{i=0, \\ i \neq k}}^{l-1} \left(\frac{1}{k-i}\right) = \lambda^{1-l} \frac{(-1)^{k-l-1}}{k!(l-1-k)!} = \frac{\lambda^{1-l}}{(l-1)!} \binom{l-1}{k}(-1)^{k-l-1}.$$ (21)

By (20) and (21), we get

$$A_k = \frac{(-\lambda)^{1-l}}{(l-1)!} \binom{l-1}{k}(-1)^k.$$ (22)

From (20) and (22), we have

$$\frac{1}{(x)_{l,\lambda}} = \sum_{k=0}^{l-1} \frac{(-1)^k}{(l-1)!} \binom{l-1}{k} \frac{(-\lambda)^{1-l}}{x-k\lambda} \quad (l \in \mathbb{N}). \tag{23}$$

By (19) and (20), we get

$$\mathcal{E}_{n,\lambda}(1-x) = \sum_{l=0}^{n} B_{l,n}(x|\lambda)\mathcal{E}_{l,\lambda} \frac{1}{(x)_{l,\lambda}}$$

$$= (1-x)_{n,\lambda} + \sum_{l=1}^{n} B_{l,n}(x|\lambda)\mathcal{E}_{l,\lambda} \frac{1}{(x)_{l,\lambda}}$$

$$= (1-x)_{n,\lambda} + \sum_{l=1}^{n} B_{l,n}(x|\lambda)\mathcal{E}_{l,\lambda} \frac{(-\lambda)^{1-l}}{(l-1)!} \sum_{k=0}^{l-1}(-1)^k \binom{l-1}{k} \frac{1}{x-k\lambda}. \tag{24}$$

Therefore, by (24), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}(1-x) = (1-x)_{n,\lambda} + \sum_{l=1}^{n} B_{l,n}(x|\lambda)\mathcal{E}_{l,\lambda} \frac{(-\lambda)^{1-l}}{(l-1)!} \sum_{k=0}^{l-1}(-1)^k \binom{l-1}{k} \frac{1}{x-k\lambda}.$$

Corollary 2.5 *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda}(2) = (2)_{n,\lambda} - \sum_{l=1}^{n} B_{l,n}(-1|\lambda)\mathcal{E}_{l,\lambda} \frac{(-\lambda)^{1-l}}{(l-1)!} \sum_{k=0}^{l-1}(-1)^k \binom{l-1}{k} \frac{1}{1+k\lambda}.$$

For $k \in \mathbb{N}$, the higher order degenerate Euler polynomials are given by the generating function

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}\right)^k (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} \quad (\text{see } [4, 5]). \tag{25}$$

From (5) and (25), we note that

$$\sum_{n=0}^{\infty} \frac{1}{\binom{n+k}{n}} B_{k,n+k}(x|\lambda) \frac{t^n}{n!} = (x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}}$$

$$= \frac{(x)_{k,\lambda}}{2^k} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}\right)^k (1+\lambda t)^{\frac{1-x+l}{\lambda}}$$

$$= \frac{(x)_{k,\lambda}}{2^k} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{k} \binom{k}{l} \mathcal{E}_{n,\lambda}^{(k)}(1-x+l)\right) \frac{t^n}{n!}. \tag{26}$$

Therefore, by comparing the coefficients on both sides of (26), we obtain the following theorem.

Theorem 2.6 *For $n, k \in \mathbb{N}$, we have*

$$\frac{2^k}{\binom{n+k}{n}} B_{k,n+k}(x|\lambda) = (x)_{k,\lambda} \sum_{l=0}^{k} \binom{k}{l} \mathcal{E}_{n,\lambda}^{(k)}(1-x+l).$$

Let f be a continuous function on $\mathbb{Z}_p$. For $x_1, x_2 \in \mathbb{Z}_p$, we consider the degenerate Bernstein operator of order n given by

$$\mathbb{B}_{n,\lambda}(f|x_1, x_2) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} (x_1)_{k,\lambda} (1-x_2)_{n-k,\lambda}$$

$$= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x_1, x_2|\lambda), \tag{27}$$

where

$$B_{k,n}(x_1, x_2|\lambda) = \binom{n}{k} (x_1)_{k,\lambda} (1-x_2)_{n-k,\lambda}, \tag{28}$$

where n, k are nonnegative integers.

Here, $B_{k,n}(x_1, x_2|\lambda)$ are called two variable degenerate Bernstein polynomials of degree n. From (28), we note that

$$\sum_{n=k}^{\infty} B_{k,n}(x_1, x_2|\lambda) \frac{t^n}{n!} = \sum_{n=k}^{\infty} \binom{n}{k} (x_1)_{k,\lambda} (1-x_2)_{n-k,\lambda} \frac{t^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{(x_1)_{k,\lambda} (1-x_2)_{n-k,\lambda}}{k!(n-k)!} t^n$$

$$= \frac{(x_1)_{k,\lambda}}{k!} t^k \sum_{n=0}^{\infty} \frac{(1-x_2)_{n,\lambda}}{n!} t^n$$

$$= \frac{(x_1)_{k,\lambda}}{k!} t^k (1+\lambda t)^{\frac{1-x_2}{\lambda}}$$

$$= \frac{(x_1)_{k,\lambda}}{k!} t^k e_\lambda^{1-x_2}(t). \tag{29}$$

Thus, by (29), we get

$$\frac{(x_1)_{k,\lambda}}{k!} t^k (1+\lambda t)^{\frac{1-x_2}{\lambda}} = \sum_{n=k}^{\infty} B_{k,n}(x_1, x_2|\lambda) \frac{t^n}{n!}, \tag{30}$$

where k is a nonnegative integer. By (28), we easily get

$$B_{k,n}(x_1, x_2|\lambda) = \binom{n}{k} \left(1 - (1-x_1)\right)_{n-(n-k),\lambda} (1-x_2)_{n-k,\lambda}$$

$$= B_{n-k,n}(1-x_2, 1-x_1|\lambda). \tag{31}$$

Now, we observe that

$$
\begin{aligned}
&\big(1 - x_2 - (n - k - 1)\lambda\big)B_{k,n-1}(x_1, x_2|\lambda) + \big(x_1 - (k - 1)\lambda\big)B_{k-1,n-1}(x_1, x_2|\lambda) \\
&\quad = \big(1 - x_2 - (n - k - 1)\lambda\big)\binom{n-1}{k}(x_1)_{k,\lambda}(1 - x_2)_{n-1-k,\lambda} \\
&\qquad + \big(x_1 - (k - 1)\lambda\big)\binom{n-1}{k-1}(x_1)_{k-1,\lambda}(1 - x_2)_{n-k,\lambda} \\
&\quad = \binom{n}{k}(x_1)_{k,\lambda}(1 - x_2)_{n-k,\lambda} = B_{k,n}(x_1, x_2|\lambda) \quad (n, k \in \mathbb{N}).
\end{aligned}
\tag{32}
$$

Therefore, by (32), we obtain the following theorem.

Theorem 2.7 *For* $n, k \in \mathbb{N}$, *we have*

$$
\begin{aligned}
&\big(1 - x_2 - (n - k - 1)\lambda\big)B_{k,n-1}(x_1, x_2|\lambda) + \big(x_1 - (k - 1)\lambda\big)B_{k-1,n-1}(x_1, x_2|\lambda) \\
&\quad = B_{k,n}(x_1, x_2|\lambda).
\end{aligned}
$$

If $f = 1$, then we have, from (27),

$$
\begin{aligned}
\mathbb{B}_{n,\lambda}(1|x_1, x_2) &= \sum_{k=0}^{n} B_{k,n}(x_1, x_2|\lambda) = \sum_{k=0}^{n} \binom{n}{k}(x_1)_{k,\lambda}(1 - x_2)_{n-k,\lambda} \\
&= (1 + x_1 - x_2)_{n,\lambda}.
\end{aligned}
\tag{33}
$$

If $f(t) = t$, then we also get from (27) that, for $n \in \mathbb{N}$ and $x_1, x_2 \in \mathbb{Z}_p$,

$$
\begin{aligned}
\mathbb{B}_{n,\lambda}(t|x_1, x_2) &= \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k}(x_1)_{k,\lambda}(1 - x_2)_{n-k,\lambda} \\
&= (x_1)_{1,\lambda}(x_1 + 1 - \lambda - x_2)_{n-1,\lambda}.
\end{aligned}
\tag{34}
$$

Hence,

$$
(x_1)_{1,\lambda} = \frac{1}{(x_1 + 1 - \lambda - x_2)_{n-1,\lambda}}\mathbb{B}_{n,\lambda}(t|x_1, x_2).
\tag{35}
$$

By the same method, we get

$$
\begin{aligned}
&\mathbb{B}_{n,\lambda}\big(t^2|x_1, x_2\big) \\
&\quad = \frac{1}{n}(x_1)_{1,\lambda}(1 + x_1 - \lambda - x_2)_{n-1,\lambda} + \frac{n-1}{n}(x_1)_{2,\lambda}(1 + x_1 - 2\lambda - x_2)_{n-2,\lambda}.
\end{aligned}
\tag{36}
$$

Note that

$$
\lim_{n \to \infty}\Big(\lim_{\lambda \to 0}\mathbb{B}_{n,\lambda}\big(t^2|x, x\big)\Big) = \lim_{n \to \infty}\Big(\frac{x}{n} + \frac{n-1}{n}x^2\Big) = x^2.
$$

Now, we observe that

$$\sum_{k=1}^{n} \frac{\binom{k}{1}}{\binom{n}{1}} B_{k,n}(x_1,x_2|\lambda) = \sum_{k=1}^{n} \binom{n-1}{k-1}(x_1)_{k,\lambda}(1-x_2)_{n-k,\lambda}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k}(x_1)_{k+1,\lambda}(1-x_2)_{n-1-k,\lambda}$$

$$= (x_1)_{1,\lambda}(x_1 + 1 - \lambda - x_2)_{n-1,\lambda}. \tag{37}$$

Thus, by (37), we get

$$(x_1)_{1,\lambda} = \frac{1}{(1 + x_1 - x_2 - \lambda)_{n-1,\lambda}} \sum_{k=1}^{n} \frac{\binom{k}{1}}{\binom{n}{1}} B_{k,n}(x_1,x_2|\lambda).$$

By the same method, we get

$$(x_1)_{2,\lambda} = \frac{1}{(1 + x_1 - x_2 - 2\lambda)_{n-2,\lambda}} \sum_{k=2}^{n} \frac{\binom{k}{2}}{\binom{n}{2}} B_{k,n}(x_1,x_2|\lambda).$$

Continuing this process, we have

$$(x_1)_{i,\lambda} = \frac{1}{(1 + x_1 - x_2 - i\lambda)_{n-i,\lambda}} \sum_{k=i}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x_1,x_2|\lambda) \quad (i \in \mathbb{N}). \tag{38}$$

Theorem 2.8 *For $i \in \mathbb{N}$, we have*

$$(x)_{i,\lambda} = \frac{1}{(1 + x_1 - x_2 - i\lambda)_{n-i,\lambda}} \sum_{k=i}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x_1,x_2|\lambda) \quad (i \in \mathbb{N}).$$

Taking the double fermionic p-adic integral on $\mathbb{Z}_p$, we get the following equation:

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|\lambda)\, d\mu_{-1}(x_1)\, d\mu_{-1}(x_2)$$

$$= \binom{n}{k} \int_{\mathbb{Z}_p} (x_1)_{k,\lambda}\, d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} (1-x_2)_{n-k,\lambda}\, d\mu_{-1}(x_2). \tag{39}$$

Therefore, by (39) and Theorem 2.2, we obtain the following theorem.

Theorem 2.9 *For $n, k \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|\lambda)\, d\mu_{-1}(x_1)\, d\mu_{-1}(x_2)$$

$$= \begin{cases} \binom{n}{k}\mathcal{E}_{k,\lambda}(2(1)_{n-k,\lambda} + \mathcal{E}_{n-k,\lambda}), & \text{if } n > k, \\ \mathcal{E}_{n,\lambda}, & \text{if } n = k. \end{cases}$$

We see from the symmetric properties of two variable degenerate Bernstein polynomials that, for $n, k \in \mathbb{N}$ with $n > k$,

$$
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2 | \lambda) \, d\mu_{-1}(x_1) \, d\mu_{-1}(x_2)
$$

$$
= \sum_{l=0}^{k} \binom{n}{k} \binom{k}{l} (-1)^{k+l} (1)_{l,\lambda}
$$

$$
\times \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (1 - x_1)_{k-l,-\lambda} (1 - x_2)_{n-k,\lambda} \, d\mu_{-1}(x_1) \, d\mu_{-1}(x_2)
$$

$$
= \binom{n}{k} \int_{\mathbb{Z}_p} (1 - x_2)_{n-k,\lambda} \, d\mu_{-1}(x_2) \left\{ (1)_{k,\lambda} + \sum_{l=0}^{k-1} \binom{k}{l} (-1)^{k+l} (1)_{l,\lambda} \mathcal{E}_{k-l,-\lambda}(2) \right\}
$$

$$
= \binom{n}{k} \mathcal{E}_{n-k,\lambda}(2) \left\{ (1)_{k,\lambda} + \sum_{l=0}^{k-1} \binom{k}{l} (-1)^{k+l} (1)_{l,\lambda} \mathcal{E}_{k-l,-\lambda}(2) \right\}. \tag{40}
$$

Therefore, by Theorem 2.9 and (40), we obtain the following theorem.

Theorem 2.10 *For $k \in \mathbb{N}$, we have*

$$
\mathcal{E}_{k,\lambda} = (1)_{k,\lambda} + \sum_{l=0}^{k-1} \binom{k}{l} (-1)^{k+l} (1)_{l,\lambda} \left(\mathcal{E}_{k-l,-\lambda} + 2(1)_{k-l,-\lambda} \right).
$$

Note that

$$
\sum_{l=0}^{k-1} \binom{k}{l} (-1)^{k+l} (1)_{l,\lambda} (1)_{k-l,-\lambda}
$$

$$
= (-1)^{k} \left(\sum_{l=0}^{k} \binom{k}{l} (-1)_{l,-\lambda} (1)_{k-l,-\lambda} - (-1)_{k,-\lambda} \right)
$$

$$
= (-1)^{k} \left((0)_{k,-\lambda} - (-1)_{k,-\lambda} \right)
$$

$$
= -(1)_{k,\lambda}.
$$

Corollary 2.11 *For $k \in \mathbb{N}$, we have*

$$
\mathcal{E}_{k,\lambda} = -(1)_{k,\lambda} + \sum_{l=0}^{k-1} \binom{k}{l} (-1)^{k+l} (1)_{l,\lambda} \mathcal{E}_{k-l,-\lambda}.
$$

3 Conclusions

In [1, 2], Carlitz initiated the study of degenerate versions of some special polynomials and numbers, namely the degenerate Bernoulli and Euler polynomials and numbers. Here we would like to draw the attention of the reader to the fact that Kim et al. have introduced various degenerate polynomials and numbers and investigating their properties, some identities related to them and their applications by means of generating functions, combinatorial methods, umbral calculus, p-adic analysis and differential equations (see [8, 9] and the references therein). It is amusing that this line of study led them even to

the introduction of degenerate gamma functions and degenerate Laplace transforms (see [7]). These already demonstrate that studying various degenerate versions of known special numbers and polynomials can be very promising and rewarding. Furthermore, we can hope that many applications will be found not only in mathematics but also in sciences and engineering.

In this paper, we investigated some properties and identities for degenerate Euler polynomials in connection with degenerate Bernstein polynomials and operators which were recently introduced as degenerate versions of the classical Bernstein polynomials and operators. This has been done by means of fermionic p-adic integrals on $\mathbb{Z}_p$ and generating functions. In addition, we studied two variable degenerate Bernstein polynomials and the degenerate Bernstein operators.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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