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q-Non uniform difference calculus and classical integral inequalities

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Abstract

We first establish q-non uniform difference versions of the integral inequalities of Hölder, Cauchy–Schwarz, and Minkowski of classical mathematical analysis and then integral inequalities of Grönwall and Bernoulli based on the Lagrange method of linear q-non uniform difference equations of first order. Finally, we prove the Lyapunov inequality for the solutions of the q-non uniform Sturm–Liouville equation.

Keywords: q-Non uniform difference calculus; Hölder; Cauchy–Schwarz; Minkowski; Grönwall; Bernoulli; Lyapunov inequalities; Lagrange method; Sturm–Liouville equation

1 Introduction

Considering the most general divided difference derivative [1, 2],

$$\mathcal{D}f\big(t(s)\big) = \frac{f(t(s+\frac{1}{2})) - f(t(s-\frac{1}{2}))}{t(s+\frac{1}{2}) - t(s-\frac{1}{2})}, \tag{1}$$

admitting the property that if $f(t) = P_n(t(s))$ is a polynomial of degree n in $t(s)$, then $\mathcal{D}f(t(s)) = \tilde{P}_{n-1}(t(s))$ is a polynomial in $t(s)$ of degree $n-1$, one is led to the following most important canonical forms for $t(s)$ in order of increasing complexity:

$$t(s) = t(0); \tag{2}$$

$$t(s) = s; \tag{3}$$

$$t(s) = q^s; \tag{4}$$

$$t(s) = \frac{q^s + q^{-s}}{2}; \quad q \in \mathbb{C}, s \in \mathbb{Z}. \tag{5}$$

When the function $t(s)$ is given by (2)–(4), the divided difference derivative (1) leads to the ordinary differential derivative $Df(t) = \frac{d}{dt}f(t)$, finite difference derivative

$$\Delta f(s) = f(s+1) - f(s) = \big(e^{\frac{d}{ds}} - 1\big)f(s), \tag{6}$$

and q-difference derivative (or Jackson derivative [3])

$$\mathcal{D}_q f(t) = \frac{f(qt) - f(t)}{qt - t} = \frac{q^{\frac{d}{dt}} - 1}{qt - t} f(t), \tag{7}$$

respectively (see also [4–6]). When $t(s) = x(q^s)$ is given by (5), the corresponding derivative is usually referred to as the *Askey–Wilson* first order divided difference operator [7] that one can write as follows:

$$\mathcal{D}f\big(x(z)\big) = \frac{f(x(q^{\frac{1}{2}}z)) - f(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)}, \tag{8}$$

where $x(z) = \frac{z+z^{-1}}{2}$ is the well-known *Joukowski* transformation and $z = q^s$.

The calculus related to the differential derivative, the continuous or differential calculus, is clearly the classical one. The one related to derivatives (6),(7), (8) (difference, q-difference, and q-non uniform difference, respectively) is referred to as the *discrete calculus*. Its interest is twofold: On the one hand, it generalizes the continuous calculus; on the other hand, it uses a discrete variable.

This work is concerned with the q-non uniform difference calculus. We particularly aim to establish q-non uniform difference versions of the well-known in differential calculus integral inequalities of Hölder, Cauchy–Schwarz, Minkowski, Grönwall, Bernoulli, and Lyapunov. Another captivating work on the raised inequalities can be found in [8] for a calculus based on a derivative also generalizing (6) and (7), but one will remark that even if that work greatly inspired us, there is not any hierarchic relationship between the calculus considered there (see [9] for a general theory) and the one considered here (see also [10–12]) based on (8). We will note also that (8) is at our best knowledge the most general known divided difference derivative having the property of sending a polynomial of degree n in a polynomial of degree $n-1$.

In the following lines, we first introduce basic concepts of q-non uniform difference calculus necessary for the sequel, and then study the mentioned integral inequalities. The functions that are considered in the q-non uniform difference calculus are clearly defined on the set

$$\mathbb{T} = \left\{x\big(q^k\big), k \in \frac{\mathbb{Z}}{2} = \left\{\frac{n}{2}\right\}_{n\in\mathbb{Z}}\right\}. \tag{9}$$

2 q-Non uniform difference calculus

In this section, we discus the essential elements of q-non uniform integral calculus and q-non uniform linear difference equations of first order.

2.1 q-Non uniform integral calculus

2.1.1 Integration

We consider the q-non uniform divided difference derivative defined by (from now on $q \in \mathbb{R}^+$, $s \in \mathbb{Z}^+$, as indicated below)

$$\mathcal{D}f\big(x(z)\big) \stackrel{\text{def}}{=} \frac{f(x(q^{\frac{1}{2}}z)) - f(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)}, \tag{10}$$

$$x(z) = \frac{z+z^{-1}}{2} = x\left(q^s\right) = \frac{q^s+q^{-s}}{2}, \quad q \in \mathbb{R}^+, s \in \mathbb{Z}^+. \tag{11}$$

A function $f(x(z))$ is said to be *q-non uniform differentiable* on $x(z)$ iff the ratio on the r.h.s. (right-hand side) of (10) exists and is finite. Clearly, every continuous function on $\mathbb{T}$ (in the topology of $\mathbb{R}$) is q-non uniform differentiable on that set.

Let us suppose that $\mathcal{D}F(x(z)) = f(x(z))$; $z = q^s$ or equivalently

$$\frac{F(x(q^{s+\frac{1}{2}})) - F(x(q^{s-\frac{1}{2}}))}{x(q^{s+\frac{1}{2}}) - x(q^{s-\frac{1}{2}})} = f\left(x\left(q^s\right)\right). \tag{12}$$

We have

$$F\left(x\left(q^{s-\frac{1}{2}}\right)\right) - F\left(x\left(q^{s+\frac{1}{2}}\right)\right) = \left[x\left(q^{s-\frac{1}{2}}\right) - x\left(q^{s+\frac{1}{2}}\right)\right]f\left(x\left(q^s\right)\right),$$

then we have

$$\begin{aligned}
&F\left(x\left(q^s\right)\right) - F\left(x\left(q^{s+1}\right)\right) = \left[x\left(q^s\right) - x\left(q^{s+1}\right)\right]f\left(x\left(q^{s+\frac{1}{2}}\right)\right)\\
&F\left(x\left(q^{s+1}\right)\right) - F\left(x\left(q^{s+2}\right)\right) = \left[x\left(q^{s+1}\right) - x\left(q^{s+2}\right)\right]f\left(x\left(q^{s+\frac{3}{2}}\right)\right)\\
&\vdots\\
&F\left(x\left(q^{N-1}\right)\right) - F\left(x\left(q^{N}\right)\right) = \left[x\left(q^{N-1}\right) - x\left(q^{N}\right)\right]f\left(x\left(q^{N-\frac{1}{2}}\right)\right).
\end{aligned}$$

By adding member by member, we get

$$F\left(x\left(q^s\right)\right) - F\left(x\left(q^N\right)\right) = \sum_{r=s}^{N-1}\left[x\left(q^r\right) - x\left(q^{r+1}\right)\right]f\left(x\left(q^{r+\frac{1}{2}}\right)\right).$$

Hence

$$\begin{aligned}
\int_{x(q^N)}^{x(z)} f\left(x(z)\right)\,d_q x(z) &\stackrel{\text{def}}{=} \sum_{x(t)=x(z)}^{x(q^{N-1})}\left[x(t) - x(qt)\right]f\left(x\left(tq^{\frac{1}{2}}\right)\right)\\
&= \sum_{t=z}^{q^{N-1}}\left[x(t) - x(qt)\right]f\left(x\left(tq^{\frac{1}{2}}\right)\right).
\end{aligned} \tag{13}$$

This integral sends a polynomial (in $x(z)$) of degree n in a polynomial of degree $n+1$ [12].

Replacing $x(z) = \frac{z+z^{-1}}{2}$ in this last equation, we have

$$\int_{x(q^N)}^{x(z)} f\left(x(z)\right)\,d_q x(z) = \frac{1}{2}(1-q)\sum_{t=z}^{q^{N-1}} t\left(1 - \frac{1}{qt^2}\right)f\left(x\left(tq^{\frac{1}{2}}\right)\right). \tag{14}$$

Let us stop for a moment on the appropriate writing of (14) to verify in particular whether in these equations the integral on the left-hand side has a lower bound, actually lower than the upper bound. Note first that the function $x(z) = \frac{z+z^{-1}}{2}$, $z \in \mathbb{R}^+$, is decreasing for $0 < z < 1$ and increasing for $1 < z < \infty$.

Assume first that $0 < q < 1$. In this case, according to the construction $N \geq s \geq 0$, one will have $q^N \leq z = q^s$ and $x(z) \leq x(q^N)$, and the convenient writing of (14) is

$$\int_{x(z)}^{x(q^N)} f\big(x(z)\big)\, d_q x(z) = \frac{1}{2}(1-q) \sum_{t=z}^{q^{N-1}} t\left(\frac{1}{qt^2} - 1\right) f\big(x\big(tq^{\frac{1}{2}}\big)\big). \tag{15}$$

On the other hand, if $1 < q < \infty$, we will have $z \leq q^N$ and $x(z) \leq x(q^N)$ and (14) becomes (15) again. On the other side, in (15), the factor

$$h(t) = (1-q)t\left(\frac{1}{qt^2} - 1\right); \quad t = q^s, s \in \mathbb{Z}^+$$

is always positive regardless of whether $0 < q < 1$ or $1 < q < \infty$ (let us agree from now on that $0 < q < 1$ and therefore $0 \leq z \leq 1$). This leads us to the following fundamental *positivity* property of the integral in (15).

Property 2.1 *If $f(x(z)) \geq 0$ and $a = x(q^\alpha) \leq b = x(q^\beta)$, then*

$$\int_a^b f\big(x(z)\big)\, d_q x(z) \geq 0.$$

Corollary 2.1 *If $f(x(z)) \geq g(x(z))$ and $a = x(q^\alpha) \leq b = x(q^\beta)$, then*

$$\int_a^b f\big(x(z)\big)\, d_q x(z) \geq \int_a^b g\big(x(z)\big)\, d_q x(z).$$

2.1.2 Connection between the q-integral and q-non uniform integral

We have

$$\int_{x(z)}^{x(q^N)} f\big(x(z)\big)\, d_q x(z) = \frac{1}{2}\int_{q^N}^{z} \left(\frac{1}{qz^2} - 1\right) f\big(x\big(q^{\frac{1}{2}}z\big)\big)\, d_q z. \tag{16}$$

For $N \to \infty$,

$$\begin{aligned}\int_{x(z)}^{\infty} f\big(x(z)\big)\, d_q x(z) &= \frac{1}{2}\int_{0}^{z} \left(\frac{1}{qz^2} - 1\right) f\big(x\big(q^{\frac{1}{2}}z\big)\big)\, d_q z \\ &= \frac{1}{2}(1-q)z \sum_{i=0}^{\infty} \left(\frac{1}{q^{2i+1}z^2} - 1\right) f\big(x\big(q^{i+\frac{1}{2}}z\big)\big),\end{aligned} \tag{17}$$

and the integral with lower and upper finite bound can be written as follows:

$$\int_a^b f\big(x(z)\big)\, d_q x(z) = \int_a^{\infty} f\big(x(z)\big)\, d_q x(z) - \int_b^{\infty} f\big(x(z)\big)\, d_q x(z) \tag{18}$$

$a = x(q^\alpha) \leq b = x(q^\beta)$ (if the integrals on the r.h.s. of (18) exist).

A function $f(x(z))$ defined on $\mathbb{T}$ is said to be *q-non uniform integrable* on a finite interval $[x(z), x(q^N)]$ iff the sum on the r.h.s. of (15) exists and is finite. If the upper bound is infinite, the q-non uniform integrability means the convergence of the infinite series on the r.h.s. of (17). Every continuous function on $\mathbb{T}$ (in the topology of $\mathbb{R}$) is clearly q-non uniform integrable on any finite interval on that set.

Remark 2.1 It is not difficult to notice that to deal with the case where $s \geq N \geq 0$ it would suffice to replace in the preceding formulas $z = q^s$ by q^N and vice versa.

In this case, from (13), we obtain

$$\int_{x(z)}^{x(q^N)} f\bigl(x(z)\bigr)\, d_q x(z) \stackrel{\text{def}}{=} \sum_{x(t)=x(q^N)}^{x(zq^{-1})} \bigl[x(t) - x(qt)\bigr] f\bigl(x\bigl(q^{\frac{1}{2}} t\bigr)\bigr)$$

$$= \sum_{t=q^N}^{zq^{-1}} \bigl[x(t) - x(qt)\bigr] f\bigl(x\bigl(q^{\frac{1}{2}} t\bigr)\bigr) \tag{19}$$

or by replacing $x(z) = \frac{z+z^{-1}}{2}$ in (19)

$$\int_{x(z)}^{x(q^N)} f\bigl(x(z)\bigr)\, d_q x(z) = \frac{1}{2}(1-q) \sum_{t=q^N}^{zq^{-1}} t\left(1 - \frac{1}{qt^2}\right) f\bigl(x\bigl(q^{\frac{1}{2}} t\bigr)\bigr). \tag{20}$$

Passing to the q-integral, we will have

$$\int_{x(z)}^{x(q^N)} f\bigl(x(z)\bigr)\, d_q x(z) = \frac{1}{2} \int_{q^N}^{z} \left(\frac{1}{qt^2} - 1\right) f\bigl(x\bigl(q^{\frac{1}{2}} t\bigr)\bigr)\, d_q t. \tag{21}$$

For $N \to 0$,

$$\int_{1}^{x(z)} f\bigl(x(z)\bigr)\, d_q x(z) = \frac{1}{2}(q-1) \sum_{t=q^{-1}z}^{1} t\left(1 - \frac{1}{qt^2}\right) f\bigl(x\bigl(q^{\frac{1}{2}} t\bigr)\bigr)$$

$$= \frac{1}{2}(q-1) \sum_{i=0}^{s-1} q^i \left(1 - \frac{1}{q^{2i+1}}\right) f\bigl(x\bigl(q^{i+\frac{1}{2}}\bigr)\bigr)$$

$$= \frac{1}{2} \int_{1}^{z} \left(1 - \frac{1}{qt^2}\right) f\bigl(x\bigl(q^{\frac{1}{2}} t\bigr)\bigr)\, d_q t. \tag{22}$$

Finally, if $a = x(q^\alpha) \leq b = x(q^\beta)$,

$$\int_a^b f\bigl(x(z)\bigr)\, d_q x(z) \stackrel{\text{def}}{=} \int_1^b f\bigl(x(z)\bigr)\, d_q x(z) - \int_1^a f\bigl(x(z)\bigr)\, d_q x(z)$$

$$= \int_a^1 f\bigl(x(z)\bigr)\, d_q x(z) - \int_b^1 f\bigl(x(z)\bigr)\, d_q x(z). \tag{23}$$

2.1.3 *Fundamental principles of analysis*

(i) We can formulate the statement of the fundamental principle of analysis as follows: "The q-non uniform derivative of the integral of a function is this function itself". This corresponds to the formula

$$\mathcal{D}\left[\int_{x(q^N)}^{x(z)} f\bigl(x(z)\bigr)\, d_q x(z)\right] = f\bigl(x(z)\bigr). \tag{24}$$

(ii) This is the q-non uniform version of the Newton–Leibnitz formula

$$\int_{x(q^N)}^{x(z)} [\mathcal{D}f]\big(x(z)\big)\, d_q x(z) = f\big(x(z)\big) - f\big(x\big(q^N\big)\big). \tag{25}$$

2.1.4 Integration by parts

By integrating the formula

$$f\big(x\big(q^{\frac{1}{2}}z\big)\big)\mathcal{D}g\big(x(z)\big) = \mathcal{D}[fg]\big(x(z)\big) - g\big(x\big(q^{-\frac{1}{2}}z\big)\big)\mathcal{D}f\big(x(z)\big) \tag{26}$$

and using the second fundamental principle of the analysis, one obtains

$$\begin{aligned}
&\int_{x(q^N)}^{x(z)} \big[f\big(x\big(q^{\frac{1}{2}}z\big)\big)\mathcal{D}g\big(x(z)\big)\big]\, d_q x(z) \\
&\quad = [fg]_{x(q^N)}^{x(z)} - \int_{x(q^N)}^{x(z)} g\big(x\big(q^{-\frac{1}{2}}z\big)\big)\mathcal{D}f\big(x(z)\big)\, d_q x(z).
\end{aligned} \tag{27}$$

2.2 Linear q-non uniform difference equations of first order

The general linear q-non uniform difference equation of first order is given by

$$\mathcal{D}y\big(x(z)\big) = a\big(x(z)\big)y\big(x\big(q^{-\frac{1}{2}}z\big)\big) + b\big(x(z)\big) \tag{28}$$

or

$$\mathcal{D}\tilde{y}\big(x(z)\big) = \tilde{a}\big(x(z)\big)\tilde{y}\big(x\big(q^{\frac{1}{2}}z\big)\big) + \tilde{b}\big(x(z)\big), \tag{29}$$

where $a(x(z))$, $b(x(z))$, $\tilde{a}(x(z))$, and $\tilde{b}(x(z))$ are known, while $y(x(z))$ and $\tilde{y}(x(z))$ are unknown functions to be determined.

Consider first the homogeneous equation corresponding to (28):

$$\mathcal{D}y_0\big(x(z)\big) = a\big(x(z)\big)y_0\big(x\big(q^{-\frac{1}{2}}z\big)\big). \tag{30}$$

Detailing, we get

$$\frac{y_0(x(q^{\frac{1}{2}}z)) - y_0(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)} = a\big(x(z)\big)y_0\big(x\big(q^{-\frac{1}{2}}z\big)\big),$$

which gives

$$y_0\big(x\big(q^{\frac{1}{2}}z\big)\big) = p\big(x(z)\big)y_0\big(x\big(q^{-\frac{1}{2}}z\big)\big),$$

where

$$\begin{aligned}
p\big(x(z)\big) &= 1 + \big(x\big(zq^{\frac{1}{2}}\big) - x\big(zq^{-\frac{1}{2}}\big)\big)a\big(x(z)\big) \\
&= 1 + \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{2}\big(z - z^{-1}\big)a\big(x(z)\big).
\end{aligned} \tag{31}$$

Using the recursion

$$y_0\big(x(z)\big) = \big[p\big(x\big(q^{\frac{1}{2}}z\big)\big)\big]^{-1} y_0\big(x(qz)\big),$$

we obtain

$$y_0\big(x(z)\big) = y_0\big(x(z_0)\big) \prod_{t=q^{-1}z_0}^{z} \big[p\big(x\big(tq^{\frac{1}{2}}\big)\big)\big]^{-1} \tag{32}$$

or

$$y_0\big(x(z)\big) = \prod_{i=0}^{N-1}\big[p\big(x\big(q^{i+\frac{1}{2}}z\big)\big)\big]^{-1} y_0\big(x\big(q^N z\big)\big). \tag{33}$$

For $N \longrightarrow \infty$ (or $z_0 \longrightarrow 0$), we obtain

$$y_0\big(x(z)\big) = \left(\prod_{i=0}^{\infty}\big[p\big(x\big(zq^{i+\frac{1}{2}}\big)\big)\big]^{-1}\right) y_0\big(x(0)\big). \tag{34}$$

Let us define the exponential function

$$E_{a,q^{\frac{1}{2}}}\big(x(z_0);x(z)\big) \overset{\text{def}}{=} \prod_{t=q^{-1}z_0}^{z} \big[p\big(x\big(tq^{\frac{1}{2}}\big)\big)\big]^{-1}. \tag{35}$$

It is clear that

$$E_{a,q^{-\frac{1}{2}}}\big(x(0);x(z)\big) = \prod_{i=0}^{\infty}\big[p\big(x\big(zq^{i+\frac{1}{2}}\big)\big)\big]^{-1} \tag{36}$$

and

$$E_{a,q^{-\frac{1}{2}}}\big(x(z_0);\infty\big) = \prod_{i=0}^{\infty}\big[p\big(x\big(q^{-i}z_0 q^{-\frac{1}{2}}\big)\big)\big]^{-1}. \tag{37}$$

Similarly, for the homogeneous equation corresponding to (29)

$$\mathcal{D}\tilde{y}_0\big(x(z)\big) = \tilde{a}\big(x(z)\big)\tilde{y}_0\big(x\big(q^{\frac{1}{2}}z\big)\big), \tag{38}$$

developing, we have

$$\tilde{y}_0\big(x\big(q^{\frac{1}{2}}z\big)\big)\tilde{p}\big(x(z)\big) = \tilde{y}_0\big(x\big(q^{-\frac{1}{2}}z\big)\big), \tag{39}$$

where

$$\begin{aligned}\tilde{p}\big(x(z)\big) &= 1 - \big(x\big(q^{\frac{1}{2}}z\big) - x\big(q^{-\frac{1}{2}}z\big)\big)\tilde{a}\big(x(z)\big) \\ &= 1 - \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{2}\big(z - z^{-1}\big)\tilde{a}\big(x(qz)\big).\end{aligned} \tag{40}$$

Using the recursion

$$\tilde{y}_0\big(x(z)\big) = \tilde{p}\big(x\big(q^{\frac{1}{2}}z\big)\big)\tilde{y}_0\big(x(qz)\big),$$

we obtain

$$\tilde{y}_0\big(x(z)\big) = \tilde{y}_0\big(x(z_0)\big) \prod_{t=q^{-1}z_0}^{z} \big[\tilde{p}\big(x\big(tq^{\frac{1}{2}}\big)\big)\big] \tag{41}$$

or

$$\tilde{y}_0\big(x(z)\big) = \prod_{i=0}^{N-1}\big[\tilde{p}\big(x\big(zq^{i+\frac{1}{2}}\big)\big)\big]\tilde{y}_0\big(q^N x\big). \tag{42}$$

For $N \longrightarrow \infty$ (or $z_0 \longrightarrow 0$), we have

$$\tilde{y}_0\big(x(z)\big) = \prod_{i=0}^{\infty}\big[\tilde{p}\big(x\big(zq^{i+\frac{1}{2}}\big)\big)\big]\tilde{y}_0\big(x(0)\big). \tag{43}$$

Let us now define the second exponential function

$$E_{\tilde{a};q^{\frac{1}{2}}}\big(x(z_0),x(z)\big) \stackrel{\text{def}}{=} \prod_{t=q^{-1}z_0}^{z} \big[\tilde{p}\big(x\big(tq^{\frac{1}{2}}\big)\big)\big]. \tag{44}$$

We have

$$E_{\tilde{a};q^{\frac{1}{2}}}\big(x(0),x(z)\big) = \prod_{i=0}^{\infty}\big[\tilde{p}\big(x\big(tq^{i+\frac{1}{2}}\big)\big)\big] \tag{45}$$

and

$$E_{\tilde{a},q^{\frac{1}{2}}}\big(x(z_0);\infty\big) = \prod_{i=0}^{\infty}\big[\tilde{p}\big(x\big(q^{i}z_0q^{\frac{1}{2}}\big)\big)\big]^{-1}. \tag{46}$$

From (35) and (44) we obtain that:

If $\tilde{a}(x(z)) = -a(x(z))$, then $\tilde{p}(x(z)) = p(x(z))$ and $y_0(x(z))\tilde{y}_0(x(z)) = y_0(x(z_0))\tilde{y}_0(x(z_0))$. Hence we have the following.

Theorem 2.1 *If $y(x(z))$ and $\tilde{y}(x(z))$ are solutions of*

$$\mathcal{D}y\big(x(z)\big) = a\big(x(z)\big)y\big(x\big(q^{-\frac{1}{2}}z\big)\big)$$

and

$$\mathcal{D}\tilde{y}\big(x(z)\big) = -a\big(x(z)\big)\tilde{y}\big(x\big(q^{\frac{1}{2}}z\big)\big),$$

respectively, and satisfy $y(x(z_0))\tilde{y}(x(z_0)) = 1$, then $y(x(z))\tilde{y}(x(z)) = 1$.

Direct proof

$$\begin{aligned}\mathcal{D}(\tilde{y}y) &= y\big(x\big(q^{-\frac{1}{2}}z\big)\big)\mathcal{D}\tilde{y}\big(x(z)\big) + \tilde{y}\big(x\big(zq^{\frac{1}{2}}\big)\big)\mathcal{D}y\big(x(z)\big)\\ &= -a\big(x(z)\big)\tilde{y}\big(x\big(zq^{\frac{1}{2}}\big)\big)y\big(x\big(zq^{-\frac{1}{2}}\big)\big) + a\big(x(z)\big)\tilde{y}\big(x\big(zq^{\frac{1}{2}}\big)\big)y\big(x\big(zq^{-\frac{1}{2}}\big)\big) = 0\\ &\Rightarrow \mathcal{D}(\tilde{y}y) = 0; \quad \tilde{y}y = \text{const}.\end{aligned}$$

As well $y(x(z_0))\tilde{y}(x(z_0)) = 1$, then $y(x(z))\tilde{y}(x(z)) = 1$. □

Corollary 2.2

$$E_{a,q^{-\frac{1}{2}}}\big(x(z_0);x(z)\big).E_{-a,q^{\frac{1}{2}}}\big(x(z_0);x(z)\big) = 1 = E_{-a,q^{-\frac{1}{2}}}\big(x(z_0);x(z)\big).E_{a,q^{\frac{1}{2}}}\big(x(z_0);x(z)\big).$$

Let us consider now the non-homogeneous equations (28) and (29). To find the solution of the non-homogeneous equation

$$\mathcal{D}y\big(x(z)\big) = a\big(x(z)\big)y\big(x\big(zq^{-\frac{1}{2}}\big)\big) + b\big(x(z)\big), \tag{47}$$

we assume that $y_0(x(z))$ is the solution of the corresponding homogeneous equation

$$\mathcal{D}y\big(x(z)\big) = a\big(x(z)\big)y\big(x\big(zq^{-\frac{1}{2}}\big)\big), \tag{48}$$

and use the Lagrange method (variation of constants method) by taking

$$y\big(x(z)\big) = y_0\big(x(z)\big)c\big(x(z)\big) \tag{49}$$

as the solution of (47), where $c(x(z))$ is unknown. By placing (49) in (47), we have

$$\mathcal{D}\big[y_0\big(x(z)\big)c\big(x(z)\big)\big] = a\big(x(z)\big)c\big(x\big(zq^{-\frac{1}{2}}\big)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big) + b\big(x(z)\big)$$

or

$$\begin{aligned}&\mathcal{D}\big[c\big(x(z)\big)\big]y_0\big(x\big(zq^{\frac{1}{2}}\big)\big) + c\big(x\big(zq^{-\frac{1}{2}}\big)\big)\mathcal{D}\big[y_0\big(x(z)\big)\big]\\ &\quad = a\big(x(z)\big)c\big(x\big(zq^{-\frac{1}{2}}\big)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big) + b\big(x(z)\big).\end{aligned}$$

Then, since

$$\mathcal{D}y_0\big(x(z)\big) = a\big(x(z)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big),$$

we get

$$\begin{aligned}&\mathcal{D}\big[c\big(x(z)\big)\big]y_0\big(x\big(zq^{\frac{1}{2}}\big)\big) + c\big(x\big(zq^{-\frac{1}{2}}\big)\big)a\big(x(z)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big)\\ &\quad = a\big(x(z)\big)c\big(x\big(zq^{-\frac{1}{2}}\big)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big) + b\big(x(z)\big)\end{aligned}$$

or

$$\mathcal{D}\big[c\big(x(z)\big)\big]y_0\big(x\big(zq^{\frac{1}{2}}\big)\big) = b\big(x(z)\big), \tag{50}$$

then

$$\mathcal{D}\big[c\big(x(z)\big)\big] = y_0^{-1}\big(x\big(zq^{\frac{1}{2}}\big)\big)b\big(x(z)\big). \tag{51}$$

This gives us the following relation:

$$c\big(x(z)\big) = c\big(x(z_0)\big) + \int_{x(z_0)}^{x(z)} y_0^{-1}\big(x\big(tq^{\frac{1}{2}}\big)\big)b\big(x(t)\big)\,d_q x(t). \tag{52}$$

Placing (52) in (49), we obtain

$$y\big(x(z)\big) = y_0\big(x(z)\big)c\big(x(z_0)\big) + \int_{x(z_0)}^{x(z)} y_0\big(x(z)\big)y_0^{-1}\big(x\big(tq^{\frac{1}{2}}\big)\big)b\big(x(t)\big)\,d_q x(t), \tag{53}$$

hence

$$c\big(x(z_0)\big) = y_0^{-1}\big(x(z_0)\big)y\big(x(z_0)\big). \tag{54}$$

The relations (53)–(54) give

$$y\big(x(z)\big) = \Phi\big(x(z),x(z_0)\big)\left[y\big(x(z_0)\big) + \int_{x(z_0)}^{x(z)} \Phi\big(x(z_0),x\big(tq^{\frac{1}{2}}\big)\big)b\big(x(t)\big)\,d_q x(t)\right], \tag{55}$$

where

$$\Phi(a,b) = y_0(a)y_0^{-1}(b). \tag{56}$$

We do the same with the equation associated to (28), equation (29), using the Lagrange method with

$$\tilde{y}\big(x(z)\big) = \tilde{c}\big(x(z)\big)\tilde{y}_0\big(x(z)\big), \tag{57}$$

where $\tilde{c}(x(z))$ is an unknown function. Placing (57) in (29), we have

$$\tilde{c}\big(x(z)\big) = \tilde{c}\big(x(z_0)\big) + \int_{x(z_0)}^{x(z)} \tilde{y}_0^{-1}\big(x\big(tq^{-\frac{1}{2}}\big)\big)\tilde{b}\big(x(t)\big)\,d_q x(t). \tag{58}$$

Placing (58) in (57), we obtain

$$\tilde{y}\big(x(z)\big) = \tilde{y}_0\big(x(z)\big)\tilde{c}\big(x(z_0)\big) + \int_{x(z_0)}^{x(z)} \tilde{y}_0\big(x(z)\big)\tilde{y}_0^{-1}\big(x\big(tq^{-\frac{1}{2}}\big)\big)\tilde{b}\big(x(t)\big)\,d_q x(t), \tag{59}$$

hence

$$\tilde{c}\big(x(z_0)\big) = \tilde{y}_0^{-1}\big(x(z_0)\big)\tilde{y}\big(x(z_0)\big). \tag{60}$$

Now we can write the general solution of (29) like

$$\tilde{y}\big(x(z)\big) = \tilde{\Phi}\big(x(z),x(z_0)\big)\left[\tilde{y}\big(x(z_0)\big) + \int_{x(z_0)}^{x(z)} \tilde{\Phi}\big(x(z_0),x\big(tq^{-\frac{1}{2}}\big)\big)\tilde{b}\big(x(t)\big)\,d_q x(t)\right], \tag{61}$$

where

$$\tilde{\Phi}(a,b) = \tilde{y}_0(a)\tilde{y}_0^{-1}(b). \tag{62}$$

3 *q*-Non uniform difference integral inequalities

In this section, we will first establish q-non uniform versions of some integral inequalities of classical mathematical analysis such as the integral inequalities of Hölder, Cauchy–Schwarz, and Minkowski. It can be seen that the techniques used in classical analysis remain valid here. Then we establish q-non uniform versions of some other integral inequalities based on the linear q-non uniform difference equations of first order and the corresponding Lagrange resolution method: the inequalities of Grönwall, Bernoulli; and finally we prove the Lyapunov inequality for the solutions of the q-non uniform Sturm–Liouville equation.

3.1 *q*-Non uniform Hölder and Cauchy–Schwarz inequalities

Theorem 3.1 (q-non uniform Hölder inequality) *Let $a,b \in [1,\infty[\cap\mathbb{T}$. For all real-valued functions f, g, defined and q-non uniform integrable on $[a,b]$, we have*

$$\begin{aligned}&\int_a^b \left|f\bigl(x(z)\bigr)g\bigl(x(z)\bigr)\right| d_q x(z)\\ &\quad\leq \left(\int_a^b \left|f\bigl(x(z)\bigr)\right|^\alpha d_q x(z)\right)^{\frac{1}{\alpha}} \left(\int_a^b \left|g\bigl(x(z)\bigr)\right|^\beta d_q x(z)\right)^{\frac{1}{\beta}}\end{aligned} \tag{63}$$

with $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof Let us first show that for $A, B \in [1,\infty[$ we have

$$A^{\frac{1}{\alpha}} B^{\frac{1}{\beta}} \leq \frac{A}{\alpha} + \frac{B}{\beta}. \tag{64}$$

Indeed, since $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $A, B \in [1,\infty[$, $\frac{A^\alpha}{\alpha} + \frac{B^\beta}{\beta}$ runs through the segment $[A^\alpha, B^\beta]$, while $\frac{\log A^\alpha}{\alpha} + \frac{\log B^\beta}{\beta}$ runs through the segment linking the points $(A^\alpha, \log A^\alpha)$ and $(A^\beta, \log A^\beta)$. By concavity of the logarithm function, we conclude that $\log(\frac{A^\alpha}{\alpha} + \frac{B^\beta}{\beta}) \geq \frac{\log A^\alpha}{\alpha} + \frac{\log B^\beta}{\beta} = \log(AB)$. By applying the exponential to the two members of this inequality, we obtain (64).

Let us now take

$$A\bigl(x(z)\bigr) = \frac{|f(x(z))|^\alpha}{\int_a^b |f(x(z))|^\alpha \, d_q x(z)} \quad \text{and} \quad B\bigl(x(z)\bigr) = \frac{|g(x(z))|^\beta}{\int_a^b |g(x(z))|^\beta \, d_q x(z)},$$

considering that

$$\left(\int_a^b \left|f\bigl(x(z)\bigr)\right|^\alpha d_q x(z)\right)\left(\int_a^b \left|g\bigl(x(z)\bigr)\right|^\beta d_q x(z)\right) \neq 0 \tag{65}$$

(otherwise, clearly $f \equiv 0$ or $g \equiv 0$ and (63) becomes equality). Due to Property 2.1 and its Corollary on the positivity of the integral, by substituting $A(x(z))$ and $B(x(z))$ in (64) and integrating on $[a,b]$, we have

$$\begin{aligned}
&\int_a^b \frac{|f(x(z))|}{(\int_a^b |f(x(z))|^\alpha\, d_qx(z))^{\frac{1}{\alpha}}} \frac{|g(x(z))|}{(\int_a^b |g(x(z))|^\beta\, d_qx(z))^{\frac{1}{\beta}}}\, d_qx(z) \\
&\quad = \int_a^b A^{\frac{1}{\alpha}} B^{\frac{1}{\beta}}\, d_qx(z) \\
&\quad \le \int_a^b \left\{\frac{A}{\alpha} + \frac{B}{\beta}\right\} d_qx(z) \\
&\quad = \int_a^b \left\{\frac{1}{\alpha}\frac{|f(x(z))|^\alpha}{\int_a^b |f(x(z))|^\alpha\, d_qx(z)} + \frac{1}{\beta}\frac{|g(x(z))|^\beta}{\int_a^b |g(x(z))|^\beta\, d_qx(z)}\right\} d_qx(z) \\
&\quad = \frac{1}{\alpha}\int_a^b \left\{\frac{|f(x(z))|^\alpha}{\int_a^b |f(x(z))|^\alpha\, d_qx(z)}\right\} d_qx(z) + \frac{1}{\beta}\int_a^b \left\{\frac{|g(x(z))|^\beta}{\int_a^b |g(x(z))|^\beta\, d_qx(z)}\right\} d_qx(z) \\
&\quad = \frac{1}{\alpha} + \frac{1}{\beta} = 1,
\end{aligned}$$

which gives us directly the q-non uniform Hölder inequality

$$\int_a^b \left|f(x(z))g(x(z))\right| d_qx(z) \le \left(\int_a^b \left|f(x(z))\right|^\alpha d_qx(z)\right)^{\frac{1}{\alpha}} \left(\int_a^b \left|g(x(z))\right|^\beta d_qx(z)\right)^{\frac{1}{\beta}}. \qquad (66)$$

□

If we take $\alpha = \beta = 2$ in the q-non uniform Hölder inequality (63), we have the q-non uniform Cauchy–Schwarz inequality.

Corollary 3.1 (q-non uniform Cauchy–Schwarz inequality) *Let $a, b \in [1,\infty[\cap\mathbb{T}$. For all real-valued functions f, g, defined and q-non uniform integrable on $[a,b]$, we have*

$$\int_a^b \left|f(x(z))g(x(z))\right| d_qx(z) \le \sqrt{\left(\int_a^b \left|f(x(z))\right|^2 d_qx(z)\right)\left(\int_a^b \left|g(x(z))\right|^2 d_qx(z)\right)}. \qquad (67)$$

3.2 *q*-Non uniform Minkowski inequality

We can now use the q-non uniform Hölder inequality to deduce the q-non uniform Minkowski inequality.

Theorem 3.2 (q-non uniform Minkowski inequality) *Let $a, b \in [1,\infty[\cap\mathbb{T}$. For all real-valued functions f, g, defined and q-non uniform integrable on $[a,b]$, we have*

$$\begin{aligned}
\left(\int_a^b \left|(f+g)(x(z))\right|^\alpha d_qx(z)\right)^{\frac{1}{\alpha}} &\le \left(\int_a^b \left|f(x(z))\right|^\alpha d_qx(z)\right)^{\frac{1}{\alpha}} \\
&\quad + \left(\int_a^b \left|g(x(z))\right|^\alpha d_qx(z)\right)^{\frac{1}{\alpha}} \qquad (68)
\end{aligned}$$

with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, where $\alpha > 1$ and $\beta > 1$.

Proof We have

$$\begin{aligned}\int_a^b \left|(f+g)\big(x(z)\big)\right|^{\alpha} d_q x(z) &= \int_a^b \left|(f+g)\big(x(z)\big)\right|^{\alpha-1}\left|(f+g)\big(x(z)\big)\right| d_q x(z) \\ &\leq \int_a^b \left|(f+g)\big(x(z)\big)\right|^{\alpha-1}\left|f\big(x(z)\big)\right| d_q x(z) \\ &\quad + \int_a^b \left|(f+g)\big(x(z)\big)\right|^{\alpha-1}\left|g\big(x(z)\big)\right| d_q x(z). \end{aligned} \tag{69}$$

Using the q-non uniform Hölder inequality with $\beta = \frac{\alpha}{\alpha-1}$, we will obtain

$$\begin{aligned} &\int_a^b \left|(f+g)\big(x(z)\big)\right|^{\alpha} d_q x(z) \\ &\quad\leq \left\{\int_a^b \left|f\big(x(z)\big)\right|^{\alpha} d_q x(z)\right\}^{\frac{1}{\alpha}} \left\{\int_a^b \left|(f+g)\big(x(z)\big)\right|^{(\alpha-1)\beta} | d_q x(z)\right\}^{\frac{1}{\beta}} \\ &\qquad + \left\{\int_a^b \left|g\big(x(z)\big)\right|^{\alpha} d_q x(z)\right\}^{\frac{1}{\alpha}} \left\{\int_a^b \left|(f+g)\big(x(z)\big)\right|^{(\alpha-1)\beta} | d_q x(z)\right\}^{\frac{1}{\beta}} \\ &\quad= \left[\left\{\int_a^b \left|f\big(x(z)\big)\right|^{\alpha} d_q x(z)\right\}^{\frac{1}{\alpha}} + \left\{\int_a^b \left|g\big(x(z)\big)\right|^{\alpha} d_q x(z)\right\}^{\frac{1}{\alpha}}\right] \\ &\qquad \times \left\{\int_a^b \left|(f+g)\big(x(z)\big)\right|^{\alpha} | d_q x(z)\right\}^{\frac{1}{\beta}}. \end{aligned} \tag{70}$$

Dividing both sides of this inequality by $\{\int_a^b |(f+g)(x(z))|^{\alpha} | d_q x(z)\}^{\frac{1}{\beta}}$, we have

$$\begin{aligned}\left(\int_a^b \left|(f+g)\big(x(z)\big)\right|^{\alpha} d_q x(z)\right)^{1-\frac{1}{\beta}} &\leq \left\{\int_a^b \left|f\big(x(z)\big)\right|^{\alpha} d_q x(z)\right\}^{\frac{1}{\alpha}} \\ &\quad + \left\{\int_a^b \left|g\big(x(z)\big)\right|^{\alpha} d_q x(z)\right\}^{\frac{1}{\alpha}}. \end{aligned} \tag{71}$$

As $1 - \frac{1}{\beta} = \frac{1}{\alpha}$, this gives us the q-non uniform Minkowski inequality. □

3.3 *q*-Non uniform Grönwall inequality

Let us first introduce the following inequalities based on the Lagrange method for the linear q-non uniform difference non-homogeneous equations.

Lemma 3.1 *Let y, f be real-valued functions defined and q-non uniform integrable on $[c,d]$, $\forall c,d \in [1,\infty[\cap \mathbb{T}$. Let $a(x(z))$ such that $p(x(z)) = 1 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(z - z^{-1})a(x(z)) > 0$. That means $a(x(z)) \geq 0$.*

Suppose that $y_0(x(z))$ is a solution of

$$\mathcal{D}y_0\big(x(z)\big) = a\big(x(z)\big)y_0\big(q^{-\frac{1}{2}}x(z)\big) \tag{72}$$

such that $y_0(x(z_0)) = 1$.

If

$$\mathcal{D}y\big(x(z)\big) \leq a\big(x(z)\big)y\big(x\big(zq^{-\frac{1}{2}}\big)\big) + f\big(x(z)\big) \quad \forall x(z) \in [1, \infty[, \tag{73}$$

then

$$y\big(x(z)\big) \leq y_0\big(x(z)\big)y\big(x(z_0)\big) + y_0\big(x(z)\big) \int_{x(z_0)}^{x(z)} y_0^{-1}\big(x\big(tq^{\frac{1}{2}}\big)\big)f\big(x(t)\big)\, d_q x(t). \tag{74}$$

Proof Let $y_0(x(z))$ be the solution of the homogeneous equation

$$\mathcal{D}y_0\big(x(z)\big) = a\big(x(z)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big) \tag{75}$$

such that $y_0(x(z_0)) = 1$. By looking for the function $y(x(z))$ satisfying (73) by the Lagrange method

$$y\big(x(z)\big) = c\big(x(z)\big)y_0\big(x(z)\big), \tag{76}$$

where $c(x(z))$ is indeterminate, we replace (76) in (73) and we have

$$\mathcal{D}\big[c\big(x(z)\big)y_0\big(x(z)\big)\big] \leq a\big(x(z)\big)\big[c\big(x\big(zq^{\frac{1}{2}}\big)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big)\big] + f\big(x(z)\big) \tag{77}$$

or

$$\begin{aligned} &y_0\big(x\big(zq^{\frac{1}{2}}\big)\big)\mathcal{D}c\big(x(z)\big) + c\big(x\big(zq^{-\frac{1}{2}}\big)\big)\mathcal{D}y_0\big(x(z)\big) \\ &\quad \leq a\big(x(z)\big)c\big(x(z)\big)y_0\big(x\big(zq^{\frac{1}{2}}\big)\big) + f\big(x(z)\big). \end{aligned} \tag{78}$$

Using (75), we have

$$\begin{aligned} &y_0\big(x\big(zq^{\frac{1}{2}}\big)\big)\mathcal{D}c\big(x(z)\big) + c\big(x\big(zq^{-\frac{1}{2}}\big)\big)a\big(x(z)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big) \\ &\quad \leq a\big(x(z)\big)c\big(x\big(zq^{-\frac{1}{2}}\big)\big)y_0\big(x\big(zq^{-\frac{1}{2}}\big)\big) + f\big(x(z)\big). \end{aligned} \tag{79}$$

Simplifying this gives

$$y_0\big(x\big(zq^{\frac{1}{2}}\big)\big)\mathcal{D}c\big(x(z)\big) \leq f\big(x(z)\big) \tag{80}$$

or

$$\mathcal{D}c\big(x(z)\big) \leq y_0^{-1}\big(x\big(zq^{\frac{1}{2}}\big)\big)f\big(x(z)\big), \tag{81}$$

since $a(x(z)) \in \mathbb{R}^+$ implies $y_0(x(z)) > 0$. By integrating the two members of the equality from $x(z_0)$ to $x(z)$, we will have

$$c\big(x(z)\big) - c\big(x(z_0)\big) \leq \int_{x(z_0)}^{x(z)} y_0^{-1}\big(x\big(tq^{\frac{1}{2}}\big)\big)f\big(x(t)\big)\, d_q x(t). \tag{82}$$

However, from (76) follows $c(x(z_0)) = y(x(z_0))$, given that $y_0(x(z_0)) = 1$. Hence

$$c\big(x(z)\big) \leq y\big(x(z_0)\big) + \int_{x(z_0)}^{x(z)} y_0^{-1}\big(x\big(tq^{\frac{1}{2}}\big)\big)f\big(x(t)\big)\,d_qx(t). \tag{83}$$

Placing (83) in (76), we get

$$\begin{aligned} y\big(x(z)\big) &= c\big(x(z)\big)y_0\big(x(z)\big) \\ &\leq y_0\big(x(z)\big)\Big[y\big(x(z_0)\big) + \int_{x(z_0)}^{x(z)} y_0^{-1}\big(x\big(tq^{\frac{1}{2}}\big)\big)f\big(x(t)\big)\,d_qx(t)\Big], \end{aligned} \tag{84}$$

which gives the desired result

$$y\big(x(z)\big) \leq y_0\big(x(z)\big)y\big(x(z_0)\big) + y_0\big(x(z)\big)\int_{x(z_0)}^{x(z)} y_0^{-1}\big(x\big(tq^{\frac{1}{2}}\big)\big)f\big(x(t)\big)\,d_qx(t). \tag{85}$$

□

Taking account of Corollary 2.2 and the fact that by the definition $E_{a,q^{-\frac{1}{2}}}(x(z_0);x(z_0)) = E_{a,q^{\frac{1}{2}}}(x(z_0);x(z_0)) = 1$, we obtain the following.

Corollary 3.2 *If y, f, and a are functions satisfying the conditions of Lemma 3.1, then*

$$\begin{aligned} y\big(x(z)\big) &\leq E_{a;q^{-\frac{1}{2}}}\big(x(z_0), x(z)\big)y\big(x(z_0)\big) \\ &\quad + E_{a;q^{-\frac{1}{2}}}\big(x(z_0), x(z)\big)\int_{x(z_0)}^{x(z)} E_{-a;q^{\frac{1}{2}}}\big(x(z_0), x\big(tq^{\frac{1}{2}}\big)\big)f\big(x(t)\big)\,d_qx(t). \end{aligned} \tag{86}$$

Lemma 3.2 *Let y, f be real-valued functions defined and q-non uniform integrable on $[c,d]$, $\forall c,d \in [1,\infty[\cap \mathbb{T}$. Let $a(x(z))$ such that*

$$p\big(x(z)\big) = 1 - \big(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\big)\big(z - z^{-1}\big)a\big(x(z)\big) > 0.$$

This means that $a(x(z)) \leq 0$.

Suppose that $y_0(x(z))$ is a solution of

$$\mathcal{D}y_0\big(x(z)\big) = a\big(x(z)\big)y_0\big(x\big(zq^{\frac{1}{2}}\big)\big)$$

such that $y_0(x(z_0)) = 1$.

In that case, if

$$\mathcal{D}y\big(x(z)\big) \leq a\big(x(z)\big)y\big(\big(x\big(zq^{\frac{1}{2}}\big)\big) + f\big(x(z)\big), \quad \forall x(z) \in [1,\infty[, \tag{87}$$

then, for all $x(z) \in [1,\infty[$, we have

$$y\big(x(z)\big) \leq y_0\big(x(z)\big)y\big(x(z_0)\big) + y_0\big(x(z)\big)\int_{x(z_0)}^{x(z)} y_0^{-1}\big(x\big(tq^{-\frac{1}{2}}\big)\big)f\big(x(t)\big)\,d_qx(t). \tag{88}$$

Proof Let $y_0(x(z))$ be a solution of the homogeneous equation

$$\mathcal{D}y_0\bigl(x(z)\bigr) = a\bigl(x(z)\bigr)y_0\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr), \quad y_0\bigl(x(z_0)\bigr) = 1. \tag{89}$$

Looking for the function $y(x(z))$ satisfying (87) by the Lagrange method

$$y\bigl(x(z)\bigr) = c\bigl(x(z)\bigr)y_0\bigl(x(z)\bigr), \tag{90}$$

where $c(x(z))$ is indeterminate, we replace the function $y(x(z))$ in (87) and we have

$$\mathcal{D}\bigl[c\bigl(x(z)\bigr)y_0\bigl(x(z)\bigr)\bigr] \le a\bigl(x(z)\bigr)c\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr)y_0\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr) + f\bigl(x(z)\bigr)$$

or

$$\begin{aligned} &c\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr)\mathcal{D}y_0\bigl(x(z)\bigr) + y_0\bigl(x\bigl(zq^{-\frac{1}{2}}\bigr)\bigr)\mathcal{D}c\bigl(x(z)\bigr) \\ &\quad \le a\bigl(x(z)\bigr)c\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr)y_0\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr) + f\bigl(x(z)\bigr). \end{aligned}$$

Using (89), we have

$$\begin{aligned} &c\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr)a\bigl(x(z)\bigr)y_0\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr) + y_0\bigl(x\bigl(zq^{-\frac{1}{2}}\bigr)\bigr)\mathcal{D}c\bigl(x(z)\bigr) \\ &\quad \le a\bigl(x(z)\bigr)c\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr)y_0\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr) + f\bigl(x(z)\bigr). \end{aligned} \tag{91}$$

Simplifying this gives

$$y_0\bigl(x\bigl(zq^{-\frac{1}{2}}\bigr)\bigr)\mathcal{D}c\bigl(x(z)\bigr) \le f\bigl(x(z)\bigr) \tag{92}$$

or

$$\mathcal{D}c\bigl(x(z)\bigr) \le y_0^{-1}\bigl(x\bigl(zq^{-\frac{1}{2}}\bigr)\bigr)f\bigl(x(z)\bigr) \tag{93}$$

since $a(x(z)) \le 0$ implies $y_0(x(z)) > 0$. By integrating the two members of the equality above from $x(z_0)$ to $x(z)$, we will have

$$c\bigl(x(z)\bigr) - c\bigl(x(z_0)\bigr) \le \int_{x(z_0)}^{x(z)} y_0^{-1}\bigl(x\bigl(tq^{-\frac{1}{2}}\bigr)\bigr)f\bigl(x(t)\bigr)\,d_qx(t). \tag{94}$$

However, from (90) follows $c(x(z_0)) = y(x(z_0))$, given that $y_0(x(z_0)) = 1$. This gives

$$c\bigl(x(z)\bigr) \le y\bigl(x(z_0)\bigr) + \int_{x(z_0)}^{x(z)} y_0^{-1}\bigl(x\bigl(tq^{-\frac{1}{2}}\bigr)\bigr)f\bigl(x(t)\bigr)\,d_qx(t). \tag{95}$$

Then we obtain

$$\begin{aligned} y\bigl(x(z)\bigr) &= c\bigl(x(z)\bigr)y_0\bigl(x(z)\bigr) \\ &\le y_0\bigl(x(z)\bigr)\biggl[y\bigl(x(z_0)\bigr) + \int_{x(z_0)}^{x(z)} y_0^{-1}\bigl(x\bigl(tq^{-\frac{1}{2}}\bigr)\bigr)f\bigl(x(t)\bigr)\,d_qx(t)\biggr], \end{aligned} \tag{96}$$

which gives the desired result

$$y\big(x(z)\big) \le y_0\big(x(z)\big)y\big(x(z_0)\big) + y_0\big(x(z)\big)\int_{x(z_0)}^{x(z)} y_0^{-1}\big(x\big(tq^{-\frac{1}{2}}\big)\big)f\big(x(t)\big)\,d_qx(t). \tag{97}$$

□

As well as for Corollary 3.2, we deduce the following.

Corollary 3.3 *If y, f, and a satisfy the conditions of Lemma 3.2, then*

$$\begin{aligned} y\big(x(z)\big) \le{}& E_{a;q^{\frac{1}{2}}}\big(x(z_0),x(z)\big)y\big(x(z_0)\big) \\ &+ E_{a;q^{\frac{1}{2}}}\big(x(z_0),x(z)\big) \\ &\times \int_{x(z_0)}^{x(z)} E_{-a;q^{-\frac{1}{2}}}\big(x(z_0),x\big(tq^{-\frac{1}{2}}\big)\big)f\big(x(t)\big)\,d_qx(t). \end{aligned} \tag{98}$$

Theorem 3.3 (q-non uniform Grönwall inequality) *Let y, f be real-valued and q-non uniform integrable functions on $[c,d]$, $\forall c,d \in [1,\infty[\cap\mathbb{T}$, and $a \ge 0$. If*

$$y\big(x(z)\big) \le f\big(x(z)\big) + \int_{x(z_0)}^{x(z)} y\big(x\big(tq^{-\frac{1}{2}}\big)\big)a\big(x(t)\big)\,d_qx(t), \tag{99}$$

then

$$\begin{aligned} y\big(x(z)\big) \le{}& f\big(x(z)\big) \\ &+ E_{a,q^{-\frac{1}{2}}}\big(x(z_0),x(z)\big) \\ &\times \int_{x(z_0)}^{x(z)} a\big(x(t)\big)f\big(x\big(tq^{-\frac{1}{2}}\big)\big)E_{-a,q^{\frac{1}{2}}}\big(x(z_0);x\big(tq^{\frac{1}{2}}\big)\big)\,d_qx(t). \end{aligned} \tag{100}$$

Proof Let us define

$$\nu\big(x(z)\big) = \int_{x(z_0)}^{x(z)} y\big(x\big(tq^{-\frac{1}{2}}\big)\big)a\big(x(t)\big)\,d_qx(t). \tag{101}$$

Then $\nu(x(z_0)) = 0$ and $\mathcal{D}\nu = y(x(zq^{-\frac{1}{2}}))a(x(z))$.

Hence hypothesis (99) gives

$$y\big(x(z)\big) \le f\big(x(z)\big) + \nu\big(x(z)\big) \tag{102}$$

and

$$\mathcal{D}\nu\big(x(z)\big) = y\big(x\big(zq^{-\frac{1}{2}}\big)\big)a\big(x(z)\big) \le \big[f\big(x\big(zq^{-\frac{1}{2}}\big)\big) + \nu\big(x\big(zq^{-\frac{1}{2}}\big)\big)\big]a\big(x(z)\big), \tag{103}$$

or

$$\mathcal{D}\nu\big(x(z)\big) \le f\big(x\big(zq^{-\frac{1}{2}}\big)\big)a\big(x(z)\big) + \nu\big(x\big(zq^{-\frac{1}{2}}\big)\big)a\big(x(z)\big). \tag{104}$$

From Lemma 3.1, inequality (104) gives

$$\begin{aligned} \nu\bigl(x(z)\bigr) \le{}& \nu\bigl(x(z_0)\bigr)E_{a,q^{-\frac{1}{2}}}\bigl(x(z_0);x(z)\bigr) \\ &+ E_{a,q^{-\frac{1}{2}}}\bigl(x(z_0);x(z)\bigr) \\ &\times \int_{x(z_0)}^{x(z)} a\bigl(x(t)\bigr)f\bigl(x\bigl(tq^{-\frac{1}{2}}\bigr)\bigr)E_{-a,q^{\frac{1}{2}}}\bigl(x(z_0);x\bigl(tq^{\frac{1}{2}}\bigr)\bigr)\,d_qx(t), \end{aligned} \tag{105}$$

and inequality (102) implies that (with $\nu(x(z_0)) = 0$)

$$\begin{aligned} y\bigl(x(z)\bigr) \le{}& f\bigl(x(z)\bigr) \\ &+ E_{a,q^{-\frac{1}{2}}}\bigl(x(z_0);x(z)\bigr) \\ &\times \int_{x(z_0)}^{x(z)} a\bigl(x(z)\bigr)f\bigl(x\bigl(tq^{-\frac{1}{2}}\bigr)\bigr)E_{-a,q^{\frac{1}{2}}}\bigl(x(z_0);x\bigl(tq^{\frac{1}{2}}\bigr)\bigr)\,d_qx(t), \end{aligned} \tag{106}$$

which is the q-non uniform Grönwall inequality. □

As a direct consequence, we get the following results.

Corollary 3.4 *Let y, f be real-valued functions defined and q-non uniform integrable on $[c,d]$, $\forall c,d \in [1,\infty[\cap\mathbb{T}$ and $a(x(z))$ such that $1 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(z - z^{-1})a(x(z)) > 0$. That means $a(x(z)) \ge 0$. Then*

$$y\bigl(x(z)\bigr) \le \int_{x(z_0)}^{x(z)} y\bigl(x\bigl(tq^{-\frac{1}{2}}\bigr)\bigr)a\bigl(x(t)\bigr)\,d_qx(t) \tag{107}$$

for all $x(z)$ implies that $y(x(z)) \le 0$.

Proof This is due to Theorem 3.3 with $f(x(z)) \equiv 0$ □

Corollary 3.5 *Let $a \ge 0$ and $\alpha \in \mathbb{R}$. If*

$$y\bigl(x(z)\bigr) \le \alpha + \int_{x_0}^{x(z)} y\bigl(x\bigl(tq^{-\frac{1}{2}}\bigr)\bigr)a\bigl(x(t)\bigr)\,d_qx(t) \tag{108}$$

for all $x_0 = x(z_0) > 0$, then $y(x(z)) \le \alpha E_{a,q^{-\frac{1}{2}}}(x_0, x(z))$.

Proof By the q-non uniform Grönwall integral inequality (100), if we take $f(x(z)) = \alpha$, then

$$\begin{aligned} y\bigl(x(z)\bigr) &\le \alpha + E_{a,q^{-\frac{1}{2}}}\bigl(x_0, x(z)\bigr)\int_{x_0}^{x(z)} \alpha a\bigl(x(t)\bigr)E_{-a,q^{\frac{1}{2}}}\bigl(x_0, x\bigl(tq^{\frac{1}{2}}\bigr)\bigr)\,d_qx(t) \\ &= \alpha\left(1 - E_{a,q^{-\frac{1}{2}}}\bigl(x_0, x(z)\bigr)\int_{x_0}^{x(z)} \mathcal{D}E_{-a,q^{\frac{1}{2}}}\bigl(x_0, x(z)\bigr)\,d_qx(t)\right) \\ &= \alpha\bigl(1 - E_{a,q^{-\frac{1}{2}}}\bigl(x_0, x(z)\bigr)\bigl(E_{-a,q^{\frac{1}{2}}}\bigl(x_0, x(z)\bigr) - E_{-a,q^{\frac{1}{2}}}(x_0, x_0)\bigr)\bigr) \\ &= \alpha - \alpha E_{a,q^{-\frac{1}{2}}}\bigl(x_0, x(z)\bigr)E_{-a;q^{\frac{1}{2}}}\bigl(x_0; x(z)\bigr) + \alpha E_{a,q^{-\frac{1}{2}}}\bigl(x_0, x(z)\bigr) \\ &= \alpha E_{a,q^{-\frac{1}{2}}}\bigl(x_0, x(z)\bigr). \end{aligned}$$

□

3.4 *q*-Non uniform Bernoulli inequality

Theorem 3.4 (q-non uniform Bernoulli inequality) *Let* $\alpha \in \mathbb{R}$. *Then* $\forall x(z)$, $x_0 = x(z_0) \in [1, \infty[$ *with* $x(zq^{-\frac{1}{2}}) > x_0$, *we have*

$$E_{a,q^{-\frac{1}{2}}}\big(x_0, x(z)\big) \geq 1 + \alpha\big(x(z) - x_0\big). \tag{109}$$

Proof Let us take $y(x(z)) = \alpha(x(z) - x_0)$, $x(zq^{-\frac{1}{2}}) > x_0$, $\forall z$. Then $\mathcal{D}y(x(z)) = \alpha$ and we have

$$\alpha y\big(x\big(zq^{-\frac{1}{2}}\big)\big) + \alpha = \alpha^2\big(x\big(zq^{-\frac{1}{2}}\big) - x_0\big) + \alpha \geq \alpha = \mathcal{D}y\big(x(z)\big),$$

which implies that $\mathcal{D}y(x(z)) \leq \alpha y(x(zq^{-\frac{1}{2}})) + \alpha$.

On the other hand, by Lemma 3.1, we get (with $y(x_0) = 0$)

$$\begin{aligned} y\big(x(z)\big) &\leq y(x_0)E_{a,q^{-\frac{1}{2}}}\big(x_0, x(z)\big) + E_{a,q^{-\frac{1}{2}}}\big(x_0, x(z)\big)\int_{x_0}^{x(z)} aE_{-a,q^{\frac{1}{2}}}\big(x_0, x\big(tq^{\frac{1}{2}}\big)\big)\,d_qx(t) \\ &= E_{a,q^{-\frac{1}{2}}}\big(x_0, x(z)\big)\int_{x_0}^{x(z)} (-)\mathcal{D}E_{-a,q^{\frac{1}{2}}}\big(x_0, x(t)\big)\,d_qx(t) \\ &= -E_{a,q^{-\frac{1}{2}}}\big(x_0, x(z)\big)\big(E_{-a,q^{\frac{1}{2}}}\big(x_0, x(z)\big) - 1\big) \\ &= -1 + E_{a,q^{-\frac{1}{2}}}\big(x_0, x(z)\big). \end{aligned}$$

That is why $E_{a,q^{-\frac{1}{2}}}(x_0, x(z)) \geq 1 + y(x(z)) = 1 + \alpha(x(z) - x_0)$. □

3.5 *q*-Non uniform Lyapunov inequality

Let us now turn to q-non uniform Lyapunov inequality regarding the derivative and integral introduced. For that, consider the following q-non uniform Sturm–Liouville equation:

$$\mathcal{D}\mathcal{D}^{+}u\big(x(z)\big) + f\big(x(z)\big)u\big(x\big(zq^{\frac{1}{2}}\big)\big) = 0, \tag{110}$$

where D^{+} is defined by $\mathcal{D}^{+}f(x(z)) = \frac{f(x(zq)) - f(x(z))}{x(zq) - x(z)}$.

Let us define the function F by

$$F(y) = \int_a^b \big[\big(\mathcal{D}y\big(x(z)\big)\big)^2 - f\big(x(z)\big)\big(y\big(x\big(zq^{\frac{1}{2}}\big)\big)\big)^2\big]\,d_qx(z). \tag{111}$$

Lemma 3.3 *Let* u *be a non-trivial solution of the* q*-non uniform Sturm–Liouville equation* (110). *Then, for all* y *belonging to the domain of* F, *the following equality remains verified*:

$$F(y) - F(u) - F(y - u) = 2(y - u)(b)\mathcal{D}^{+}u(b) - 2(y - u)(a)\mathcal{D}^{+}u(a). \tag{112}$$

Proof We have

$$\begin{aligned} &F(y) - F(u) - F(y - u) \\ &\quad = \int_a^b \big\{\big(\mathcal{D}y\big(x(z)\big)\big)^2 - f\big(x(z)\big)\big(y\big(x\big(q^{\frac{1}{2}}z\big)\big)\big)^2 \end{aligned}$$

$$
\begin{aligned}
&- \big(\mathcal{D}u\big(x(z)\big)\big)^2 - f\big(x(z)\big)\big(y\big(x\big(q^{\frac{1}{2}}z\big)\big)\big) - \big(\mathcal{D}(y-u)\big(x(z)\big)\big)^2 \\
&- f\big(x(z)\big)\big((y-u)\big(x\big(q^{\frac{1}{2}}z\big)\big)\big)^2\big\}\, d_qx(z) \\
={}& 2\int_a^b \big\{-\big(\mathcal{D}u\big(x(z)\big)\big)^2 + f\big(x(z)\big)\big(u\big(x\big(zq^{\frac{1}{2}}\big)\big)\big)^2 + \mathcal{D}y\big(x(z)\big)\mathcal{D}u\big(x(z)\big) \\
&- f\big(x(z)\big)y\big(x\big(zq^{\frac{1}{2}}\big)\big)u\big(x\big(zq^{\frac{1}{2}}\big)\big)\big\}\, d_qx(z) \\
={}& 2\int_a^b \big\{-\big(\mathcal{D}u\big(x(z)\big)\big)^2 - u\big(x\big(zq^{\frac{1}{2}}\big)\big)\mathcal{D}\mathcal{D}^+u\big(x(z)\big) + \mathcal{D}y\big(x(z)\big)\mathcal{D}u\big(x(z)\big) \\
&+ y\big(x\big(zq^{\frac{1}{2}}\big)\big)\mathcal{D}\mathcal{D}^+u\big(x(z)\big)\big\}\, d_qx(z) \\
={}& 2\int_a^b \big\{\mathcal{D}y\big(x(z)\big)\mathcal{D}u\big(x(z)\big) + y\big(x\big(zq^{\frac{1}{2}}\big)\big)\mathcal{D}\mathcal{D}^+u\big(x(z)\big) \\
&- \big[\big(\mathcal{D}u\big(x(z)\big)\big)^2 + u\big(x\big(zq^{\frac{1}{2}}\big)\big)\mathcal{D}\mathcal{D}^+u\big(x(z)\big)\big]\big\}\, d_qx(z),
\end{aligned}
$$

(using $f(x(z))u(x(zq^{\frac{1}{2}})) = -\mathcal{D}\mathcal{D}^+u(x(z))$), or

$$
\begin{aligned}
&F(y) - F(u) - F(y-u) \\
&\quad = 2\int_a^b \big\{\mathcal{D}\big[y\big(x(z)\big)\mathcal{D}^+u\big(x(z)\big)\big] - \mathcal{D}\big[u\big(x(z)\big)\mathcal{D}^+u\big(x(z)\big)\big]\big\}\, d_qx(z) \\
&\quad = 2\int_a^b \mathcal{D}\big\{\big(y\big(x(z)\big) - u\big(x(z)\big)\big)\mathcal{D}^+u\big(x(z)\big)\big\}\, d_qx(z) \\
&\quad = 2\big(y(b) - u(b)\big)\mathcal{D}^+u(b) - 2\big(y(a) - u(a)\big)\mathcal{D}^+u(a). \qquad \square
\end{aligned}
$$

Lemma 3.4 *Let y be in the domain of F, then $\forall a, b \in [1, \infty[\cap \mathbb{T}$ and $c, d \in [a, b] \cap \mathbb{T}$ such that $a \le c \le d \le b$, we have*

$$
\int_c^d \big(\mathcal{D}y\big(x(z)\big)\big)^2\, d_qx(z) \ge \frac{(y(d) - y(c))^2}{d - c}. \tag{113}
$$

Proof Let us take

$$
u\big(x(z)\big) = \frac{y(d) - y(c)}{d - c}x(z) + \frac{dy(c) - cy(d)}{d - c}.
$$

Then $\mathcal{D}^+u(x(z)) = \frac{y(d)-y(c)}{d-c}$ and $\mathcal{D}\mathcal{D}^+u(x(z)) = 0$, which proves that $u(x(z))$ is a solution of (110) with $f(x(z)) = 0$, $\forall x(z) \in [1, \infty[\cap \mathbb{T}$ and

$$
F(y) = \int_a^b \big(\mathcal{D}y\big(x(z)\big)\big)^2\, d_qx(z),
$$

$\forall y(x(z))$ from the domain of F. From Lemma 3.3, we get

$$
F(y) - F(u) - F(y-u) = 2(y-u)(b)\mathcal{D}^+u(b) - 2(y-u)(a)\mathcal{D}^+u(a) = 0.
$$

Consequently,

$$
F(y) = F(u) + F(y-u) \ge F(u).
$$

This leads us to the following result:

$$\begin{aligned}\int_c^d \bigl(\mathcal{D}y\bigl(x(z)\bigr)\bigr)^2\, d_qx(z) &\geq \int_c^d \bigl(\mathcal{D}u\bigl(x(z)\bigr)\bigr)^2\, d_qx(z)\\ &= \int_c^d \biggl(\frac{y(d)-y(c)}{d-c}\biggr)^2\, d_qx(z)\\ &= \frac{(y(d)-y(c))^2}{d-c}.\end{aligned}$$

□

Theorem 3.5 (q-non uniform Lyapunov inequality) *Let f be a real-valued function defined and q-non uniform differentiable on $[a,b]$, $\forall a,b \in [1,\infty[\cap \mathbb{T}$, $a<b$, and let $u(x(z))$ be a non-trivial solution of equation (110) with $u(a)=u(b)=0$, then*

$$\int_a^b f\bigl(x(z)\bigr)\, d_qx(z) = -\int_a^b \frac{\mathcal{D}\mathcal{D}^{+}u(x(z))}{u(x(zq^{\frac{1}{2}}))}\, d_qx(z) \geq \frac{4}{b-a}. \tag{114}$$

Proof From Lemma 3.3 with $y=0$ and $u(a)=u(b)=0$, we have

$$F(u) = \int_a^b \bigl[\bigl(\mathcal{D}u\bigl(x(z)\bigr)\bigr)^2 - f\bigl(x(z)\bigr)\bigl(u\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr)\bigr)^2\bigr]\, d_qx(z) = 0. \tag{115}$$

Let $M = \max\{u^2(x(zq^{\frac{1}{2}})); x(q^{\frac{1}{2}}z) \in [a,b]\cap\mathbb{T}\}$ and $c\in[a,b]\cap\mathbb{T}$ such that $u^2(c)=M$. Then

$$M = u^2(c) \geq u^2\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr) > 0,$$

and

$$\begin{aligned}M\int_a^b f\bigl(x(z)\bigr)\, d_qx(z) &\geq \int_a^b f\bigl(x(z)\bigr)u^2\bigl(x\bigl(zq^{\frac{1}{2}}\bigr)\bigr)\, d_qx(z)\\ &= \int_a^b \bigl(\mathcal{D}u\bigl(x(z)\bigr)\bigr)^2\, d_qx(z)\\ &= \int_a^c \bigl(\mathcal{D}u\bigl(x(z)\bigr)\bigr)^2\, d_qx(z) + \int_c^b \bigl(\mathcal{D}u\bigl(x(z)\bigr)\bigr)^2\, d_qx(z)\\ &\geq \frac{(u(c)-u(a))^2}{c-a} + \frac{(u(b)-u(c))^2}{b-c}\\ &= M\biggl[\frac{1}{c-a} + \frac{1}{b-c}\biggr]\bigl(u(a)=u(b)=0 \text{ and } u^2(c)=M\bigr)\\ &= M\biggl[\frac{(a+b-2c)^2}{(c-a)(b-c)(b-a)} + \frac{4}{b-a}\biggr]\\ &\geq M\frac{4}{b-a},\end{aligned}$$

which implies that $\int_a^b f(x(z))\, d_qx(z) \geq M\frac{4}{b-a}$.

The q-non uniform Lyapunov inequality is thus proved. □

4 Conclusion

In this paper, q-non uniform difference versions of the integral inequalities of Hölder, Cauchy–Schwarz, and Minkowski, and also the integral inequalities of Grönwall and

Bernoulli based on the Lagrange method of linear q-non uniform difference equations of first order were established. Finally, the Lyapunov inequality for the solutions of the q-non uniform Sturm–Liouville equation was proved.

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