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Oscillatory and asymptotic properties of third-order quasilinear delay differential equations

G.E. Chatzarakis[1], J. Džurina[2] and I. Jadlovská[2*]

*Correspondence:
irena.jadlovska@tuke.sk
[2]Department of Mathematics and Theoretical Informatics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Košice, Slovakia
Full list of author information is available at the end of the article

Abstract

In this paper, we consider a class of quasilinear third-order differential equations with a delayed argument. We establish new sufficient conditions for all solutions of such equations to be oscillatory or almost oscillatory. Those criteria improve, simplify and complement a number of existing results. The strength of the criteria obtained is tested on Euler type equations.

MSC: 34C10; 34K11

Keywords: Quasilinear differential equation; Delay; Third-order; Oscillation

1 Introduction

In this paper, we are concerned with the asymptotic and oscillatory behavior of solutions of quasilinear third-order delay differential equations of the form

$$\bigl(r(t)\bigl(y''(t)\bigr)^{\alpha}\bigr)' + q(t)y^{\alpha}\bigl(\tau(t)\bigr) = 0, \quad t \geq t_0 > 0. \tag{1.1}$$

Throughout the paper, without further mentioning, we will assume the following hypotheses:

(H$_0$) α is a quotient of odd positive integers;

(H$_1$) $r \in \mathcal{C}([t_0,\infty),\mathbb{R})$ is positive and satisfies

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\alpha}(t)} < \infty;$$

(H$_2$) $q \in \mathcal{C}([t_0,\infty),\mathbb{R})$ is non-negative and does not vanish eventually;

(H$_3$) the delay function $\tau \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$ is strictly increasing, $\tau(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \infty$.

By a solution of Eq. (1.1), we mean a nontrivial function $y \in \mathcal{C}^1([T_y,\infty),\mathbb{R})$ with $T_y \geq t_0$, which has the property $y', r(y'')^{\alpha} \in \mathcal{C}^1([T_y,\infty),\mathbb{R})$, and satisfies (1.1) on $[T_y,\infty)$. We only consider those solutions of (1.1) which exist on some half-line $[T_y,\infty)$ and satisfy the condition $\sup\{|y(t)| : T \leq t < \infty\} > 0$ for any $T \geq T_y$.

A solution y of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is *nonoscillatory*. The equation itself is termed *oscillatory* if all its solutions oscillate.

From the early years of the 18th century, differential equations of third-order have been used for modeling various phenomena in several areas of the applied sciences. The first step in this direction was taken by J. Bernoulli in 1696 who formulated the famous isoperimetric problem and, five years later, gave the solution that depends upon a third-order differential equation [8]. Since then, these equations have shown to be particularly important in the modeling of several physical phenomena, including the interactions between charged particles, in an external electromagnetic field [21], the entry-flow phenomenon [11], the propagation of action potentials in squid neurons [16] and others.

Although the importance of third-order equations in applications had been realized very early, the majority of the work on the qualitative behavior of those equations has been carried out only relatively recently, in the last three decades. For a review of key results up to 2014, we refer the reader to the recent monographs [17, 18].

The study of qualitative properties of differential equations of the form (1.1) and their particular cases or generalizations has become the subject of extensive research; see, for example, [1–6, 9, 10, 15, 19, 20] and the references cited therein. Mostly, Eq. (1.1) has been investigated under the assumption

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\alpha}(t)} = \infty. \tag{1.2}$$

In this case, by generalizing a familiar Kiguradze lemma, see [12, Lemma 1.1] or [13, Lemma 2], it follows that there are only two possibilities for a nonoscillatory, say positive, solution y of (1.1), namely cases (I) and (III) of Lemma 1 below. If, however, the integral in (1.2) is convergent, an additional case for nonoscillatory solutions must be considered.

For closely related results having in common that the function $r(t)$ satisfies condition (1.2), we refer the reader to [1, 3–6, 10, 19, 20].

The main objective of this work is to establish results for the solutions of (1.1) to be oscillatory or almost oscillatory under the crucial condition (H_1). We postulate new sufficient conditions for oscillations and/or property A (see Definition 1), which improve, simplify and complement some existing results reported in the literature. Finally, we test the strength of our criteria on Euler type equations.

2 Preliminaries, definitions and existing results

At first, we constrain the structure of possible nonoscillatory, let us say positive solutions of (1.1).

Lemma 1 *Let y be an eventually positive solution of* (1.1). *Then there exists $t_1 \in [t_0, \infty)$ such that y satisfies one of the following cases*:

(I) $y > 0$, $y' > 0$, $y'' > 0$,
(II) $y > 0$, $y' > 0$, $y'' < 0$,
(III) $y > 0$, $y' < 0$, $y'' > 0$,

for $t \ge t_1$.

Proof The proof is straightforward and hence we omit it. □

2.1 Notation and definitions

Throughout the paper, we will use the following notation:

$$\pi(t_0) = \int_{t_0}^{\infty} \frac{\mathrm{d}t}{r^{1/\alpha}(t)}, \qquad \widetilde{\pi}(t_0) = \int_{t_0}^{\infty} \pi(t)\,\mathrm{d}t,$$

and

$$R(v,u) = \int_u^v \int_x^v \frac{\mathrm{d}s}{r^{1/\alpha}(s)}\,\mathrm{d}x \quad \text{for } v \geq u.$$

Remark 1 All functional inequalities considered in the paper are supposed to hold eventually, that is, they are satisfied for all t large enough.

Remark 2 Note that if y is a solution of (1.1), then $x = -y$ is also a solution of (1.1). Thus, regarding nonoscillatory solutions of (1.1), we only need to consider the eventually positive ones.

Definition 1 We say that (1.1) has *property A* if any solution y of (1.1) is oscillatory or satisfies $\lim_{t\to\infty} y(t) = 0$. In such case, some authors say that Eq. (1.1) is *almost oscillatory*.

Definition 2 We say that (1.1) has *property P*, if any nonoscillatory, say positive, solution y of (1.1) satisfies case (III) of Lemma 1.

2.2 Motivation

In the sequel, we state and discuss in detail a triplet of related results for (1.1) under the assumptions (H_0)–(H_3), which are considered to be the primary motivation of the paper.

Grace et al. [9] studying the oscillatory behavior of (1.1) using comparison principles and established the following result, which we present below for the reader's convenience.

Theorem A (See [9, Theorem 3]) *Assume that there exist two functions $\xi(t)$ and $\eta(t) \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\xi'(t) \geq 0, \quad \eta'(t) \geq 0 \quad \textit{and} \quad \tau(t) < \xi(t) < \eta(t) < t \quad \textit{for } t \geq t_0.$$

If

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)\tau^{\alpha}(s)\left(\int_T^{\tau(s)} \frac{u}{r^{1/\alpha}(u)}\,\mathrm{d}u\right)^{\alpha} \mathrm{d}s > \frac{1}{\mathrm{e}} \tag{2.1}$$

for any $T \geq t_0$,

$$\int_{t_0}^{\infty} \left(\frac{1}{r(u)} \int_{t_0}^{u} q(s)\tau(s)\pi\big(\tau(s)\big)\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}u = \infty \tag{2.2}$$

and

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} q(s)\big(\xi(s)-\tau(s)\big)^{\alpha} \left(\int_{\xi(s)}^{\eta(s)} \frac{\mathrm{d}u}{r^{1/\alpha}(u)}\right)^{\alpha} \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.3}$$

then (1.1) *is oscillatory.*

It is useful to note that conditions (2.1), (2.2) and (2.3) eliminate solutions satisfying cases (I)–(III) of Lemma 1, respectively.

Making further use of comparison principles with first-order delay equations, Agarwal et al. [2] established the following oscillation result for (1.1) with $\alpha = 1$.

Theorem B (See [2, Corollary 1]) *Assume that $\alpha = 1$ and there exist two functions $\xi, \sigma \in C^1([t_0,\infty),\mathbb{R})$ such that $\xi(t) > t$, $\xi(t)$ is nondecreasing, $\tau(\xi(\xi(t))) < t$, $\sigma(t)$ is nondecreasing, and $\sigma(t) > t$. If for all $t_2 > t_1 \geq t_0$*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} q(s)\int_{t_2}^{\tau(s)}\int_{t_1}^{v}\frac{\mathrm{d}u}{r(u)}\,\mathrm{d}v\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.4}$$

$$\liminf_{t\to\infty}\int_{t}^{\sigma(t)} q(s)\big(\tau(s)-t_1\big)\pi\big(\sigma(s)\big)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.5}$$

and

$$\liminf_{t\to\infty}\int_{\tau(\xi(\xi(t)))}^{t}\int_{s}^{\xi(s)}\frac{1}{r(v)}\int_{v}^{\xi(v)} q(u)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.6}$$

then (1.1) is oscillatory.

In fact, both Theorems A and B strongly depend on the right choice of the auxiliary functions in lim inf-type conditions. Since there is no general rule for this choice, the application of such criteria may become difficult.

Using a different technique based on reducing the studied equation into a first-order Riccati-type inequality, which is generally considered as one of the most valuable tools in the oscillation theory, Li et al. [15] provided the following criterion for property A of (1.1).

Theorem C (See [15, Theorem 1]) *Assume that*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{r(u)}\int_{u}^{\infty} q(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u\,\mathrm{d}v = \infty \tag{2.7}$$

and

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\ell^{\alpha} s q(s)\left(\frac{\tau(s)-T_\ell}{2}\frac{\tau(s)}{s}\right)^{\alpha} - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)}{s^{\alpha}}\right)\mathrm{d}s = \infty \tag{2.8}$$

for some $\ell \in (0,1)$ and for sufficiently large $T_\ell \geq t_0$, $t_2 \geq T_\ell$. If, moreover,

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left(k^{\alpha} q(s)\tau^{\alpha}(s)\pi^{\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(s)r^{1/\alpha}(s)}\right)\mathrm{d}s = \infty \tag{2.9}$$

holds for some $k \in (0,1)$ and for sufficiently large $t_3 \geq t_0$, then (1.1) has property A.

Here, condition (2.7) works to ensure that any solution of type (III) converges to zero as t approaches infinity, while conditions (2.8) and (2.9) eliminate solutions of type (I) and (II), respectively.

It is well known that the Euler equation

$$\big(t^2 y''(t)\big)' + \frac{q_0}{t} y(t) = 0 \tag{2.10}$$

has property A if $q_0 > 2/(3\sqrt{3})$. However, Theorem C obviously fails to apply to (2.10) due to (2.7).

Even though the above-mentioned oscillation results were shown using different techniques, they all have in common that the desired property is ensured by means of three conditions independent from each other, eliminating solutions of particular cases. The aim in this paper is to provide new oscillation criteria for (1.1) that would significantly improve, complement, and simplify Theorems A–C. An advantage of our approach is that it reduces the number of conditions ensuring that all solutions of the studied equation oscillate. A similar issue has been considered recently in [7] for linear third-order equations of the form

$$\big(r_2(t)\big(r_1(t)y'(t)\big)'\big)' + q(t)y\big(\tau(t)\big) = 0$$

under the conditions

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r_1(t)} < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\mathrm{d}t}{r_2(t)} < \infty.$$

3 Main results

3.1 Nonexistence of solutions of type (I) and (II)

We start with a simple condition ensuring the nonexistence of solutions of type (I). As will be shown later, this condition is already included in those eliminating solutions of type (II).

Lemma 2 *Let y be an eventually positive solution of* (1.1). *If*

$$\int_{t_0}^{\infty} q(t)\tau^{\alpha}(t)\,\mathrm{d}t = \infty, \tag{3.1}$$

then case (I) *in Lemma* 1 *cannot hold.*

Proof Assume for the sake of contradiction that y satisfies case (I) of Lemma 1 and pick $t_1 \in [t_0, \infty)$ such that $y(\tau(t)) > 0$ for $t \geq t_1$. Since y' is increasing, we have

$$y'(t) \geq y'(t_1) =: c \quad \text{on } [t_1, \infty).$$

Thus,

$$y\big(\tau(t)\big) \geq c\big(\tau(t) - t_1\big).$$

Clearly, there is $t_2 \geq t_1$ such that, for any $k \in (0,1)$ and $t \geq t_2$,

$$y\big(\tau(t)\big) \geq \tilde{c}\tau(t), \qquad \tilde{c} := ck. \tag{3.2}$$

Integrating (1.1) from t_2 to t and using (3.1) in the resulting inequality, we get

$$\begin{aligned} r(t)\big(y''(t)\big)^{\alpha} &= r(t_2)\big(y''(t_2)\big)^{\alpha} - \int_{t_2}^{t} q(s)y^{\alpha}\big(\tau(s)\big)\,\mathrm{d}s \\ &\le r(t_2)\big(y''(t_2)\big)^{\alpha} - \tilde{c}^{\alpha}\int_{t_2}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s \to -\infty \quad \text{as } t\to\infty, \end{aligned} \tag{3.3}$$

which leads to a contradiction. The proof is complete. □

Next, we state some useful properties of the type (II) solutions, which are useful when proving the main results.

Lemma 3 *Let y be an eventually positive increasing solution of* (1.1). *If*

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\left(\int_{t_0}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}t = \infty, \tag{3.4}$$

then y satisfies case (II) *in Lemma* 1 *for $t \ge t_1$ and, moreover,*

(a) *$y(t) \ge ty'(t)$ and $y(t)/t$ is decreasing for $t \ge t_2$, and $\lim_{t\to\infty} y(t)/t = y' = 0$,*
(b) *$y'(t) \ge -\pi(t)r^{1/\alpha}(t)y''(t)$ and $y'(t)/\pi(t)$ is increasing for $t \ge t_2$,*

where $t_2 \ge t_1$ is large enough.

Proof Since y is increasing, by Lemma 1, y satisfies either case (I) or case (II) for $t \ge t_1$, where $t_1 \in [t_0,\infty)$ is such that $y(\tau(t)) > 0$ for $t \ge t_1$.

At first, note that because of the assumption (H_1), condition (3.4) implies that (3.1) holds. Thus, by Lemma 2, y satisfies case (II) for $t \ge t_1$.

Since $y'(t)$ is decreasing, there exists a finite limit $\lim_{t\to\infty} y'(t) = \lambda \ge 0$. We claim that $\lambda = 0$. If not, then $y'(t) \ge \lambda > 0$ for $t \ge t_1$. Proceeding similarly as in the proof of Lemma 2, we obtain (3.3). From the fact that $r(t)(y''(t))^{\alpha}$ is negative, we get

$$y''(t) \le -\frac{\tilde{c}}{r^{1/\alpha}(t)}\left(\int_{t_2}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s\right)^{1/\alpha}.$$

Integrating the above inequality from t_2 to t, we have

$$y'(t) \le y'(t_2) - \tilde{c}\int_{t_2}^{t} \frac{1}{r^{1/\alpha}(u)}\left(\int_{t_2}^{u} q(s)\tau^{\alpha}(s)\,\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u \to -\infty \quad \text{as } t\to\infty,$$

which is a contradiction. Hence $\lambda = 0$. By l'Hospital's rule, we find

$$\lim_{t\to\infty}\frac{y(t)}{t} = y'(t) = 0. \tag{3.5}$$

On the other hand, it follows from the monotonicity of y' that

$$y(t) = y(t_1) + \int_{t_1}^{t} y'(s)\,\mathrm{d}s \ge y(t_1) + y'(t)(t-t_1).$$

In view of (3.5), there is a $t_2 \ge t_1$ such that

$$y(t_1) - y'(t)t_1 > 0$$

for $t \geq t_2$. So,

$$y(t) > ty'(t),$$

which implies that

$$\left(\frac{y(t)}{t}\right)' = \frac{ty' - y}{t^2} < 0.$$

To show case (b), it suffices to note that

$$y'(t) \geq -\int_t^\infty \frac{1}{r^{1/\alpha}(s)} r^{1/\alpha}(s) y''(s)\,\mathrm{d}s \geq -r^{1/\alpha}(t) y''(t)\pi(t),$$

in view of which

$$\left(\frac{y'(t)}{\pi(t)}\right)' = \frac{r^{1/\alpha}(t)y''(t)\pi(t) + y'(t)}{r^{1/\alpha}(t)\pi^2(t)} \geq 0.$$

The proof is complete. □

Now, we can proceed to present various simple criteria for property P for (1.1).

Theorem 1 *If*

$$\int_{t_0}^\infty \frac{1}{r^{1/\alpha}(t)} \left(\int_{t_0}^t q(s)\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}t = \infty, \tag{3.6}$$

then (1.1) *has property P.*

Proof Assume for the sake of contradiction that y satisfies case (I) or (II) of Lemma 1 for $t \geq t_1$. Since y is increasing, there exists a $t_2 \geq t_1$ such that $y(t) \geq y(t_1) =: \ell$ for $t \geq t_2$. Integrating (1.1) from t_2 to t, we get

$$\begin{aligned} r(t)\big(y''(t)\big)^\alpha &= r(t_2)\big(y''(t_2)\big)^\alpha - \int_{t_2}^t q(s)y^\alpha\big(\tau(s)\big) \\ &\leq r(t_2)\big(y''(t_2)\big)^\alpha - \ell^\alpha \int_{t_2}^t q(s)\,\mathrm{d}s. \end{aligned} \tag{3.7}$$

From (H_1) and (3.6), however, we see that

$$\int_{t_0}^\infty q(s)\,\mathrm{d}s = \infty. \tag{3.8}$$

If we assume that y is of (I)-type, then (3.8) contradicts the positivity of $r(t)(y''(t))^\alpha$.

Assume now that y satisfies case (II). Using the fact that $r(t)(y''(t))^\alpha < 0$ in (3.7), we are led to

$$y''(t) \leq -\frac{\ell}{r^{1/\alpha}(t)} \left(\int_{t_2}^t q(s)\,\mathrm{d}s\right)^{1/\alpha}. \tag{3.9}$$

Integrating (3.9) from t_2 to t, we obtain

$$y'(t) \le y'(t_2) - \ell \int_{t_2}^{t} \frac{1}{r^{1/\alpha}(u)} \left(\int_{t_2}^{u} q(s)\,\mathrm{d}s \right)^{1/\alpha} \mathrm{d}u,$$

which, in view of (3.6), contradicts the positivity of $y'(t)$. The proof is complete. □

The next result is based on a comparison with a first-order delay inequality. This result, in connection with the results from Sect. 3.2, can be viewed as an improved and simplified alternative of Theorem B. In contrast to that theorem, we stress that the next theorem does not require the existence of auxiliary functions (as in condition (2.5)) and, moreover, the nonexistence of solutions of type (I) and (II) is ensured by means of only one condition.

Theorem 2 *If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{1}{r^{1/\alpha}(s)} \left(\int_{t_0}^{s} q(u)\tau^{\alpha}(u)\,\mathrm{d}u \right)^{1/\alpha} \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{3.10}$$

then (1.1) *has property P.*

Proof Assume for the sake of contradiction that y satisfies case (I) or (II) of Lemma 1 for $t \ge t_1$. Obviously, it is necessary for the validity of (3.10) that (3.4) holds. By Lemma 3, we conclude that y satisfies case (II) and the asymptotic properties (a) and (b) of the Lemma for $t \ge t_2 \ge t_1$. Therefore,

$$y\bigl(\tau(t)\bigr) \ge \tau(t) y'\bigl(\tau(t)\bigr)$$

for $t \ge t_2$. From (1.1), we get

$$-\bigl(r(t)\bigl(y''(t)\bigr)^{\alpha}\bigr)' = q(t)y^{\alpha}\bigl(\tau(t)\bigr) \ge q(t)\tau^{\alpha}(t)\bigl(y'\bigl(\tau(t)\bigr)\bigr)^{\alpha}.$$

Integrating the above inequality from t_2 to t and using the fact that y' is decreasing, we have

$$-r(t)\bigl(y''(t)\bigr)^{\alpha} \ge \int_{t_2}^{t} q(s)\tau^{\alpha}(s)\bigl(y'\bigl(\tau(s)\bigr)\bigr)^{\alpha}\,\mathrm{d}s \ge \bigl(y'\bigl(\tau(t)\bigr)\bigr)^{\alpha} \int_{t_2}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s, \tag{3.11}$$

that is,

$$x'(t) + \frac{1}{r^{1/\alpha}(t)} \left(\int_{t_2}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s \right)^{1/\alpha} x\bigl(\tau(t)\bigr) \le 0, \tag{3.12}$$

where we set $x(t) := y'(t) > 0$. However, by [14, Theorem 2.1.1], the inequality (3.12) does not possess a positive solution, which is a contradiction to our initial assumption. The proof is complete. □

A principle like the one we used in the proof of Theorem 2 always requires $\tau(t) < t$. The results presented in the sequel, however, apply also in the case when $\tau(t) = t$.

Theorem 3 *Assume that* (3.4) *holds. If*

$$\limsup_{t\to\infty} \pi^\alpha(t)\int_{t_0}^t q(s)\tau^\alpha(s)\,\mathrm{d}s > 1, \tag{3.13}$$

then (1.1) *has property P.*

Proof Assume for the sake of contradiction that y satisfies case (I) or (II) of Lemma 1 for $t \ge t_1$. At first, note that $\lim_{t\to\infty}\pi(t) = 0$ holds due to (H_1), which together with (3.13) implies (3.1). By Lemma 3, we conclude that y satisfies case (II) and the asymptotic properties (a) and (b) of the lemma for $t \ge t_2 \ge t_1$.

Proceeding as in the proof of Theorem 2, we arrive at (3.11). Using the monotonicity of $r(t)(y''(t))^\alpha$ and Lemma 3 (b) in (3.11), we find that

$$-r(t)\big(y''(t)\big)^\alpha \ge \big(y'(t)\big)^\alpha \int_{t_2}^t q(s)\tau^\alpha(s)\,\mathrm{d}s \ge -r(t)\big(y''(t)\big)^\alpha \pi^\alpha(t)\int_{t_2}^t q(s)\tau^\alpha(s)\,\mathrm{d}s,$$

or

$$1 \ge \pi^\alpha(t)\int_{t_2}^t q(s)\tau^\alpha(s)\,\mathrm{d}s.$$

However, the above inequality contradicts (3.13). The proof is complete. □

Theorem 4 *Assume that* (3.4) *holds. If there exists a nondecreasing function* $\rho \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left(\frac{\rho(u)}{\tau(u)r^{1/\alpha}(u)}\left(\int_{t_0}^u q(s)\tau^\alpha(s)\,\mathrm{d}s\right)^{1/\alpha} - \frac{(\rho'(u))^2}{4\tau'(u)\rho(u)}\right)\mathrm{d}u = \infty \tag{3.14}$$

for any $T \in [t_0,\infty)$, *then* (1.1) *has property P.*

Proof Assume for the sake of contradiction that y satisfies case (I) or (II) of Lemma 1 for $t \ge t_1$. By Lemma 3, we conclude that y satisfies case (II) and the asymptotic properties (a) and (b) of the lemma for $t \ge t_2 \ge t_1$.

Let us define the Riccati-type function

$$w(t) := \rho(t)\frac{y'(t)}{y(\tau(t))} > 0 \quad \text{on } [t_2,\infty).$$

By differentiating $w(t)$ and using the monotonicity of y', we see that

$$\begin{aligned} w'(t) &= \frac{\rho'(t)}{\rho(t)}w(t) + \frac{\rho(t)y''(t)}{y(\tau(t))} - \frac{\rho(t)y'(t)y'(\tau(t))\tau'(t)}{y^2(\tau(t))} \\ &\le \frac{\rho'(t)}{\rho(t)}w(t) + \frac{\rho(t)y''(t)}{y(\tau(t))} - \frac{\tau'(t)}{\rho(t)}w^2(t). \end{aligned} \tag{3.15}$$

Integrating (1.1) from t_2 to t and using the fact that $y(\tau(t))/\tau(t)$ is decreasing, we get

$$\begin{aligned} -r(t)\bigl(y''(t)\bigr)^{\alpha} &\geq -r(t_2)\bigl(y''(t_2)\bigr)^{\alpha} + \int_{t_2}^{t} q(s)y^{\alpha}\bigl(\tau(s)\bigr)\,\mathrm{d}s \\ &\geq -r(t_2)\bigl(y''(t_2)\bigr)^{\alpha} + \left(\frac{y(\tau(t))}{\tau(t)}\right)^{\alpha} \int_{t_2}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s. \end{aligned} \tag{3.16}$$

Since $\lim_{t\to\infty} y(t)/t = 0$, there is $t_3 > t_2$ such that

$$-r(t_2)\bigl(y''(t_2)\bigr)^{\alpha} - \left(\frac{y(\tau(t))}{\tau(t)}\right)^{\alpha} \int_{t_0}^{t_2} q(s)\tau^{\alpha}(s)\,\mathrm{d}s > 0 \tag{3.17}$$

for $t \geq t_3$. Combining (3.16) and (3.17), we arrive at

$$\begin{aligned} -r(t)\bigl(y''(t)\bigr)^{\alpha} \geq {}& -r(t_2)\bigl(y''(t_2)\bigr)^{\alpha} + \left(\frac{y(\tau(t))}{\tau(t)}\right)^{\alpha} \int_{t_0}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s \\ & - \left(\frac{y(\tau(t))}{\tau(t)}\right)^{\alpha} \int_{t_0}^{t_2} q(s)\tau^{\alpha}(s)\,\mathrm{d}s \\ \geq {}& \left(\frac{y(\tau(t))}{\tau(t)}\right)^{\alpha} \int_{t_0}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s, \end{aligned}$$

or

$$\frac{y''(t)}{y(\tau(t))} \leq -\frac{1}{r^{1/\alpha}(t)\tau(t)} \left(\int_{t_0}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s\right)^{1/\alpha} \tag{3.18}$$

for $t \geq t_3$. Combining (3.18) and (3.15), we obtain

$$\begin{aligned} w'(t) &\leq \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\rho(t)(\int_{t_0}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s)^{1/\alpha}}{r^{1/\alpha}(t)\tau(t)} - \frac{\tau'(t)}{\rho(t)} w^2(t) \\ &= -\frac{\rho(t)(\int_{t_0}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s)^{1/\alpha}}{r^{1/\alpha}(t)\tau(t)} - \frac{\tau'(t)}{\rho(t)}\left(w(t) - \frac{\rho'(t)}{2\tau'(t)}\right)^2 + \frac{(\rho'(t))^2}{4\rho(t)\tau'(t)} \\ &\leq -\frac{\rho(t)(\int_{t_0}^{t} q(s)\tau^{\alpha}(s)\,\mathrm{d}s)^{1/\alpha}}{r^{1/\alpha}(t)\tau(t)} + \frac{(\rho'(t))^2}{4\rho(t)\tau'(t)}. \end{aligned}$$

Integrating the above inequality from t_3 to t, we get

$$w(t) \leq w(t_3) - \int_{t_3}^{t} \left(\frac{\rho(u)(\int_{t_0}^{u} q(s)\tau^{\alpha}(s)\,\mathrm{d}s)^{1/\alpha}}{r^{1/\alpha}(u)\tau(u)} - \frac{(\rho'(u))^2}{4\rho(u)\tau'(u)}\right)\mathrm{d}u,$$

a contradiction. The proof is complete. □

Letting $\rho(t) = 1/\pi(t)$, the following consequence is immediate.

Corollary 1 *Assume that* (3.4) *holds. If*

$$\limsup_{t\to\infty} \int_{T}^{t} \left(\frac{1}{\pi(u)\tau(u)r^{1/\alpha}(u)} \left(\int_{t_0}^{u} q(s)\tau^{\alpha}(s)\,\mathrm{d}s\right)^{1/\alpha} - \frac{1}{4\pi^3(u)r^{2/\alpha}(u)\tau'(u)}\right)\mathrm{d}u = \infty$$

for any $T \in [t_0, \infty)$, *then* (1.1) *has property P.*

Theorem 5 *Assume that* (3.4) *holds. If there exists a function* $\delta \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\left\{\frac{\pi^\alpha(t)\tau^\alpha(t)}{\delta(t)}\int_T^t\left(\delta(s)q(s)-\frac{(\delta'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^\alpha(s)\pi^\alpha(s)(\tau'(s))^\alpha}\right)\mathrm{d}s\right\}>1, \tag{3.19}$$

for any $T \in [t_0,\infty)$, *then* (1.1) *has property P.*

Proof Assume for the sake of contradiction that y satisfies case (I) or (II) of Lemma 1 for $t \ge t_1$. By Lemma 3, we conclude that y satisfies case (II) and the asymptotic properties (a) and (b) of the lemma for $t \ge t_2 \ge t_1$.

Define the Riccati-type function as

$$w(t) := \delta(t)\left(\frac{r(t)(y''(t))^\alpha}{y^\alpha(\tau(t))}+\frac{1}{\pi^\alpha\tau^\alpha(t)}\right)\quad\text{on } [t_2,\infty). \tag{3.20}$$

From Lemma 3, we have

$$y\bigl(\tau(t)\bigr)\ge\tau(t)y'\bigl(\tau(t)\bigr)\ge\tau(t)y'(t)\ge-\tau(t)\pi(t)r^{1/\alpha}(t)y''(t), \tag{3.21}$$

which implies that $w \ge 0$ on $[t_1,\infty)$. Differentiating (3.20) and using (1.1) and the definition of w, we obtain

$$\begin{aligned}
w'(t) &= \frac{\delta'(t)}{\delta(t)}w(t)+\frac{\delta(t)(r(t)(y''(t))^\alpha)'}{y^\alpha(\tau(t))}-\frac{\alpha r(t)(y''(t))^\alpha y'(t)\tau'(t)}{y^{\alpha+1}(\tau(t))}\\
&\quad+\frac{\alpha\rho(t)}{(\pi(t)\tau(t))^{\alpha+1}}\left(\frac{\tau(t)}{r^{1/\alpha}(t)}-\tau'(t)\pi(t)\right)\\
&= \frac{\delta'(t)}{\delta(t)}w(t)-\delta(t)q(t)\\
&\quad-\frac{\alpha y'(t)\tau'(t)}{\delta^{1/\alpha}(t)(-r^{1/\alpha}(t)y''(t))}\left(w(t)-\frac{\delta(t)}{\pi^\alpha(t)\tau^\alpha(t)}\right)^{1+1/\alpha}\\
&\quad+\frac{\alpha\rho(t)}{(\pi(t)\tau(t))^{\alpha+1}}\left(\frac{\tau(t)}{r^{1/\alpha}(t)}-\tau'(t)\pi(t)\right).
\end{aligned}$$

Using that $y'(t) \ge -\pi(t)r^{1/\alpha}(t)y''(t)$ in the above inequality, we arrive at

$$\begin{aligned}
w'(t) &\le \frac{\delta'(t)}{\delta(t)}w(t)-\delta(t)q(t)-\frac{\alpha\pi(t)\tau'(t)}{\delta^{1/\alpha}(t)}\left(w(t)-\frac{\delta(t)}{\pi^\alpha(t)\tau^\alpha(t)}\right)^{1+1/\alpha}\\
&\quad+\frac{\alpha\rho(t)}{(\pi(t)\tau(t))^{\alpha+1}}\left(\frac{\tau(t)}{r^{1/\alpha}(t)}-\tau'(t)\pi(t)\right).
\end{aligned}$$

Then using the inequality stated in [22, Lemma 2.3], namely,

$$Au-B(u-C)^{(\alpha+1)/\alpha}\le AC+\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{A^{\alpha+1}}{B^\alpha},\quad B>0, A, C\in\mathbb{R}, \tag{3.22}$$

with

$$A := \frac{\delta'(t)}{\delta(t)},\qquad B := \frac{\alpha\pi(t)\tau'(t)}{\delta^{1/\alpha}(t)},\qquad C := \frac{\delta(t)}{\pi^\alpha(t)\tau^\alpha(t)},\quad\text{and}\quad u = w(t),$$

we find that

$$\begin{aligned} w'(t) &\le -\delta(t)q(t) + \frac{\delta'(t)}{\pi^\alpha(t)\tau^\alpha(t)} + \frac{(\delta'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^\alpha(t)\pi^\alpha(t)(\tau'(t))^\alpha} \\ &\quad + \frac{\alpha\rho(t)}{(\pi(t)\tau(t))^{\alpha+1}}\left(\frac{\tau(t)}{r^{1/\alpha}(t)} - \tau'(t)\pi(t)\right) \\ &= -\delta(t)q(t) + \left(\frac{\delta(t)}{\pi^\alpha(t)\tau^\alpha(t)}\right)' + \frac{(\delta'(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^\alpha(t)\pi^\alpha(t)(\tau'(t))^\alpha}. \end{aligned} \tag{3.23}$$

Integrating (3.23) from t_2 to t, we are led to

$$\begin{aligned} &\int_{t_2}^{t}\left(\delta(s)q(s) - \frac{(\delta'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^\alpha(s)\pi^\alpha(s)(\tau'(s))^\alpha}\right)\mathrm{d}s - \frac{\delta(t)}{\pi^\alpha(t)\tau^\alpha(t)} + \frac{\delta(t_2)}{\pi^\alpha(t_2)\tau^\alpha(t_2)} \\ &\quad \le w(t_2) - w(t). \end{aligned}$$

Using (3.20) in the last inequality, we have

$$\begin{aligned} &\int_{t_2}^{t}\left(\delta(s)q(s) - \frac{(\delta'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^\alpha(s)\pi^\alpha(s)(\tau'(s))^\alpha}\right)\mathrm{d}s \\ &\quad \le \delta(t_2)\frac{r(t_2)(y''(t_2))^\alpha}{y^\alpha(\tau(t_2))} - \delta(t)\frac{r(t)(y''(t))^\alpha}{y^\alpha(\tau(t))}. \end{aligned} \tag{3.24}$$

On the other hand, from (3.21), it follows that

$$-\frac{\delta(t)}{\pi^\alpha(t)\tau^\alpha(t)} \le \delta(t)\frac{r(t)(y''(t))^\alpha}{y^\alpha(\tau(t))} \le 0.$$

Substituting the above estimate into (3.24), we obtain

$$\int_{t_2}^{t}\left(\delta(s)q(s) - \frac{(\delta'(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\delta^\alpha(s)\pi^\alpha(s)(\tau'(s))^\alpha}\right)\mathrm{d}s \le \frac{\delta(t)}{\pi^\alpha(t)\tau^\alpha(t)}. \tag{3.25}$$

Multiplying (3.25) by $\pi^\alpha(t)\tau^\alpha(t)/\delta(t)$ and taking the limsup on both sides of the resulting inequality, we arrive at contradiction to (3.19). The proof is complete. □

Finally, we turn our attention to the existing result presented in the introductory section, namely Theorem C. By careful observation, it is easy to show that condition (2.8) is redundant.

Theorem 6 *If* (2.9) *holds for some* $k \in (0,1)$ *and for sufficiently large* $t_3 \ge t_0$, *then* (1.1) *has property P.*

Proof Following the proof of [15, Theorem 1], we remark that (2.8) eliminates solutions of type (I) and (2.9) those of type (II). It is enough to note that it is necessary for the validity of (2.9) that

$$\int_{t_0}^{\infty} q(s)\tau^\alpha(s)\pi^\alpha(s)\,\mathrm{d}s = \infty, \tag{3.26}$$

which in view of (H_1) implies (3.1). By Lemma 2, solutions of type (I) do not exist.

The proof is complete. □

Remark 3 We note that in the proof of Theorem C, a weaker version of (a) of Lemma 3 was used for solutions of type (II), namely, $y(t) \geq ky'(t)$ for $k \in (0,1)$ and t large enough. Assuming that condition (3.4) holds, one can easily provide a stronger version of the above theorem with $k = 1$.

3.2 Convergence to zero and/or nonexistence of solutions of type (III)

Lemma 4 *Let y be a solution of* (1.1) *satisfying case* (III) *of Lemma* 1. *If*

$$\int_{t_0}^{\infty} q(s)\,\mathrm{d}s = \infty \tag{3.27}$$

or

$$\int_{t_0}^{\infty}\int_t^{\infty} \frac{1}{r^{1/\alpha}(s)}\left(\int_s^{\infty} q(u)\,\mathrm{d}u\right)^{1/\alpha}\mathrm{d}s\,\mathrm{d}t = \infty, \tag{3.28}$$

then $\lim_{t\to\infty} y(t) = 0$.

Proof Pick $t_1 \in [t_0,\infty)$ such that $y(\tau(t)) > 0$ for $t \geq t_1$. Since y is a positive decreasing solution, there exists a finite limit $\lim_{t\to\infty} y(t) = \lambda \geq 0$. Assume for the sake of contradiction that $\lambda > 0$. Integrating (1.1) from t_1 to t and taking into account that (3.27) holds, we have

$$\begin{aligned} r(t)\big(y''(t)\big)^{\alpha} &= r(t_1)\big(y''(t_1)\big)^{\alpha} - \int_{t_1}^{t} q(s)y^{\alpha}\big(\tau(s)\big)\,\mathrm{d}s, \\ &\leq r(t_1)\big(y''(t_1)\big)^{\alpha} - \lambda^{\alpha}\int_{t_1}^{t} q(s)\,\mathrm{d}s \to -\infty \quad \text{as } t\to\infty, \end{aligned}$$

which is a contradiction. Thus $\lim_{t\to\infty} y(t) = 0$. To show that the same conclusion holds in the case where

$$\int_{t_0}^{\infty} q(s)\,\mathrm{d}s < \infty,$$

we refer the reader to [15, Theorem 1]. The proof is complete. □

Theorem 7 *Let y be an eventually positive solution of* (1.1). *If*

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} q(s)R^{\alpha}\big(\tau(t),\tau(s)\big)\,\mathrm{d}s > 1, \tag{3.29}$$

then case (III) *in Lemma* 1 *is impossible.*

Proof Pick $t_1 \in [t_0,\infty)$ such that $\tau(t) \geq t_1$ for $t \geq t_1$. It follows from the monotonicity of $r(t)(y''(t))^{\alpha}$ that, for $v \geq u$,

$$-y'(u) \geq \int_u^{v} \frac{1}{r^{1/\alpha}(s)} r^{1/\alpha}(s)y''(s)\,\mathrm{d}s \geq r^{1/\alpha}(v)y''(v)\int_u^{v}\frac{\mathrm{d}s}{r^{1/\alpha}(s)}.$$

Integrating the last inequality again from u to $v \geq u$ in u, we get

$$y(u) \geq r^{1/\alpha}(v)y''(v) \int_u^v \int_x^v \frac{\mathrm{d}s}{r^{1/\alpha}(s)}\,\mathrm{d}x =: r^{1/\alpha}(v)y''(v)R(v,u). \tag{3.30}$$

Integrating (1.1) from $\tau(t)$ to t and using (3.30) with $u = \tau(s)$ and $v = \tau(t)$, we get

$$\begin{aligned} r\big(\tau(t)\big)\big(y''\big(\tau(t)\big)\big)^\alpha &\geq \int_{\tau(t)}^t q(s)y^\alpha\big(\tau(s)\big)\,\mathrm{d}s \\ &\geq r\big(\tau(t)\big)\big(y''\big(\tau(t)\big)\big)^\alpha \int_{\tau(t)}^t q(s)R^\alpha\big(\tau(t),\tau(s)\big)\,\mathrm{d}s. \end{aligned}$$

Dividing the above inequality by $r(\tau(t))(y''(\tau(t)))^\alpha$ and taking the lim sup on both sides of the resulting inequality as $t \to \infty$, we are led to a contradiction. The proof is complete. □

3.3 Applications

3.3.1 Property A

Combining Theorems 1–5 with Lemma 4, one can easily provide fundamentally new criteria for property A of (1.1).

Theorem 8 *If* (3.6) *holds, then* (1.1) *has property A.*

Proof It is enough to note that (H_1) along with (3.6) implies (3.27). □

Theorem 9 *If* (3.10) *and either* (3.27) *or* (3.28) *hold, then* (1.1) *has property A.*

Theorem 10 *If* (3.4), (3.13) *and either* (3.27) *or* (3.28) *hold, then* (1.1) *has property A.*

Theorem 11 *If* (3.4) *holds and there exists a nondecreasing function* $\rho \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that* (3.14) *and either* (3.27) *or* (3.28) *hold, then* (1.1) *has property A.*

Theorem 12 *If* (3.4) *holds and there exists a function* $\delta \in \mathcal{C}^1([t_0,\infty),(0,\infty))$ *such that* (3.19) *and either* (3.27) *or* (3.28) *hold, then* (1.1) *has property A.*

3.3.2 Oscillation

We are now interested in the situation in which all solutions of Eq. (1.1) are oscillatory. To attain this goal, we combine Theorems 1–6 with Theorem 7.

Theorem 13 *Assume that all assumptions of Theorem* 1 (2, 3, 4, 5, 6) *and* (3.29) *hold. Then* (1.1) *is oscillatory.*

4 Examples

Example 1 Let us consider the Euler type equation

$$\big(t^2y''(t)\big)' + \frac{q_0}{t}y(\lambda t) = 0, \quad t \geq 1, \tag{E_x}$$

where $\lambda \in (0,1)$ and $q_0 > 0$.

First, we present criteria for Property P of (E_x) following from Theorems 1–6.

– Since (3.6) fails, Theorem 1 does not apply.
– Theorem 2 requires

$$q_0\lambda \ln\left(\frac{1}{\lambda}\right) > \frac{1}{\mathrm{e}}. \tag{4.1}$$

– Theorem 3 or Theorem 5 with $\rho(t) = \pi(t)\tau(t)$ requires

$$q_0\lambda > 1. \tag{4.2}$$

– Theorem 4 (Corollary 1) or Theorem 6 (Remark 3) requires

$$q_0\lambda > \frac{1}{4}. \tag{4.3}$$

Among conditions (4.1)–(4.3), we remark that (4.1) is more efficient for small values of λ, while (4.3) for larger ones. Since Lemma 4 is satisfied, we conclude that (E_x) has property A if any of conditions (4.1)–(4.3) hold. Note that Theorem C does not apply due to (2.7).

Second, we apply Theorem 7 for the nonexistence of positive decreasing solutions of (E_x), which requires

$$q_0 > \frac{1}{1 - \lambda + \ln\lambda + \frac{1}{2}\ln^2\lambda}. \tag{4.4}$$

Finally, by Theorem 13, we conclude that (E_x) is oscillatory if any of conditions (4.1)–(4.3) and (4.4) hold.

5 Conclusions

In the present paper, several new oscillation results for Eq. (1.1) have been presented, which further improve, complement and simplify existing criteria introduced in the paper as Theorems A–C.

In Sect. 3.1, we provided various criteria for the nonexistence of solutions of type (I) and (II). In particular, Theorem 1 serves as a single condition alternative to Theorem A, while Theorem 2 offers a single condition criterion, which is based on similar principles (compared with first-order delay equations) as Theorem B, but does not require the existence of auxiliary functions. By a simple refinement in the proof of Theorem C, we have shown that (2.8) is unnecessary and can be removed. We have also pointed out how a stronger version with $k = 1$ can be attained. Using different substitutions as in the proof of Theorem C, we have presented more general results for the nonexistence of solutions of type (I) and (II).

In Sect. 3.2, we were dealing with the asymptotic properties and nonexistence of solutions of type (III) of Lemma 1. In that section, we extended (2.7) from Theorem C to be applied on (2.10). Furthermore, we provided a new criterion for the nonexistence of such solutions.

Finally, we have combined the results from Sects. 3.1 and 3.2 to obtain new results for oscillation and/or property A of (1.1).

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Author details
[1]Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), Athens, Greece. [2]Department of Mathematics and Theoretical Informatics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Košice, Slovakia.

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References

1. Agarwal, R., Grace, S., Smith, T.: Oscillation of certain third-order functional differential equations. Adv. Math. Sci. Appl. **16**(1), 69–94 (2006)
2. Agarwal, R.P., Bohner, M., Li, T., Zhang, C.: Oscillation of third-order nonlinear delay differential equations. Taiwan. J. Math. **17**(2), 545–558 (2013)
3. Agarwal, R.P., Bohner, M., Li, T., Zhang, C.: A philos-type theorem for third-order nonlinear retarded dynamic equations. Appl. Math. Comput. **249**, 527–531 (2014)
4. Agarwal, R.P., Grace, S.R., O'Regan, D.: On the oscillation of certain functional differential equations via comparison methods. J. Math. Anal. Appl. **286**(2), 577–600 (2003)
5. Baculíková, B., Džurina, J.: Oscillation of third-order nonlinear differential equations. Appl. Math. Lett. **24**(4), 466–470 (2011)
6. Chatzarakis, G.E., Grace, S.R., Jadlovská, I.: Oscillation criteria for third-order delay differential equations. Adv. Differ. Equ. **2017**(1), 330 (2017)
7. Džurina, J., Jadlovská, I.: Oscillation of third-order differential equations with noncanonical operators. Appl. Math. Comput. **336**, 394–402 (2018)
8. Fraser, C.G.: Isoperimetric problems in the variational calculus of Euler and Lagrange. Hist. Math. **19**(1), 4–23 (1992)
9. Grace, S.R., Agarwal, R.P., Pavani, R., Thandapani, E.: On the oscillation of certain third-order nonlinear functional differential equations. Appl. Math. Comput. **202**(1), 102–112 (2008)
10. Hassan, T.S.: Oscillation of third-order nonlinear delay dynamic equations on time scales. Math. Comput. Model. **49**(7–8), 1573–1586 (2009)
11. Jayaraman, G., Padmanabhan, N., Mehrotra, R.: Entry flow into a circular tube of slowly varying cross-section. Fluid Dyn. Res. **1**(2), 131–144 (1986)
12. Kiguradze, I.T., Chanturia, T.A.: Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. Mathematics and Its Applications (Soviet Series), vol. 89. Kluwer Academic, Dordrecht (1993) Translated from the 1985 Russian original
13. Kusano, T., Naito, M.: Comparison theorems for functional-differential equations with deviating arguments. J. Math. Soc. Jpn. **33**(3), 509–532 (1981)
14. Ladde, G.S., Lakshmikantham, V., Zhang, B.G.: Oscillation Theory of Differential Equations with Deviating Arguments. Monographs and Textbooks in Pure and Applied Mathematics, vol. 110. Dekker, New York (1987)
15. Li, T., Zhang, C., Baculíková, B., Džurina, J.: On the oscillation of third-order quasi-linear delay differential equations. Tatra Mt. Math. Publ. **48**, 117–123 (2011)
16. McKean, H.P.: Nagumo's equation. Adv. Math. **4**(3), 209–223 (1970)
17. Padhi, S., Pati, S.: Theory of Third-Order Differential Equations. Springer, New Delhi (2014)
18. Saker, S.: Oscillation Theory of Delay Differential and Difference Equations: Second and Third Orders. LAP Lambert Academic Publishing (2010)
19. Saker, S., Džurina, J.: On the oscillation of certain class of third-order nonlinear delay differential equations. Math. Bohem. **135**(3), 225–237 (2010)

20. Şenel, M.T., Utku, N.: Oscillation criteria for third-order neutral dynamic equations with continuously distributed delay. Adv. Differ. Equ. **2014**(1), 220 (2014)
21. Vreeke, S.A., Sandquist, G.M.: Phase space analysis of reactor kinetics. Nucl. Sci. Eng. **42**(3), 295–305 (1970)
22. Wu, H., Erbe, L., Peterson, A.: Oscillation of solution to second-order half-linear delay dynamic equations on time scales. Electron. J. Differ. Equ. **2016**, 71 (2016)