Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Strong convergence theorem for split monotone variational inclusion with constraints of variational inequalities and fixed point problems

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Abstract

In this paper, inspired by Jitsupa et al. (J. Comput. Appl. Math. 318:293–306, 2017), we propose a general iterative scheme for finding a solution of a split monotone variational inclusion with the constraints of a variational inequality and a fixed point problem of a finite family of strict pseudo-contractions in real Hilbert spaces. Under very mild conditions, we prove a strong convergence theorem for this iterative scheme. Our result improves and extends the corresponding ones announced by some others in the earlier and recent literature.

MSC: 47H05; 47H09; 47H10; 47J22

Keywords: Strong convergence; Iterative scheme; Split monotone variational inclusion; Variational inequality; Fixed point; *k*-strict pseudo-contractions; Hilbert spaces

1 Introduction

It is known that variational inequality, as a greatly important tool, has already been studied for a wide class of unilateral, obstacle, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Many numerical methods have been developed for solving variational inequalities and some related optimization problems; see [2–5] and the references therein.

The split monotone variational inclusion problem, which is the core of the modeling of many inverse problems arising in phase retrieval and other real-world problems, has been widely studied in sensor networks, intensity-modulated radiation therapy treatment planning, data compression, and computerized tomography in recent years; see, e.g., [6–10] and the references therein.

Split monotone variational inclusion problem (in short, SMVIP) was firstly introduced by Moudafi [11] as follows: find $x^* \in H_1$ such that

$$\begin{cases} 0 \in f_1 x^* + B_1 x^*, \\ y^* = Ax^* \in H_2 : 0 \in f_2 y^* + B_2 y^*, \end{cases} \tag{1.1}$$

where $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are two given single-valued mappings, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

If $f_1 = f_2 \equiv 0$, then problem (1.1) reduces to the following split variational inclusion problem (in short, SVIP): find $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1 x^*, \\ y^* = Ax^* \in H_2 : 0 \in B_2 y^*. \end{cases} \tag{1.2}$$

Also, if $f_1 \equiv 0$, then problem (1.1) reduces to the following split monotone variational inclusion problem (in short, SMVIP): find $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1 x^*, \\ y^* = Ax^* \in H_2 : 0 \in fy^* + B_2 y^*. \end{cases} \tag{1.3}$$

We denote the solution sets of variational inclusions $0 \in B_1 x^*$ and $0 \in fy^* + B_2 y^*$ by $\mathrm{SOLVIP}(B_1)$ and $\mathrm{SOLVIP}(f + B_2)$, respectively. Thus, the solution set of problem (1.3) can be denoted by $\Gamma = \{x^* \in H_1 : x^* \in \mathrm{SOLVIP}(B_1), Ax^* \in \mathrm{SOLVIP}(f + B_2)\}$.

In 2012, Byrne et al. [12] studied the following iterative scheme for SVIP (1.2): for given $x_0 \in H_1$ and $\lambda > 0$,

$$x_{n+1} = J_\lambda^{B_1}\bigl[x_n + \epsilon A^*\bigl(J_\lambda^{B_2} - I\bigr)Ax_n\bigr]. \tag{1.4}$$

Recently, Kazmi and Rivi [13] introduced a new iterative scheme for SVIP (1.2) and the fixed point problem of a nonexpansive mapping:

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \end{cases} \tag{1.5}$$

where A is a bounded linear operator, A^* is the adjoint of A, f is a contraction on H_1, T is a nonexpansive mapping of H_1. They obtained a strong convergence theorem under some mild restrictions on the parameters.

Very recently, Jitsupa et al. [1] modified algorithm (1.5) for SVIP (1.2) and the fixed point problem of a finite family of strict pseudo-contractions:

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n], \\ y_n = \beta_n u_n + (1 - \beta_n)\sum_{i=1}^N \eta_i^{(n)} T_i u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D)y_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where A is a bounded linear operator, A^* is the adjoint of A, $\{T_i\}_{i=1}^N$ is a finite family of k_i-strictly pseudo-contractions, f is a contraction, D is a strong positive linear bounded operator. They proved, under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\eta_i^{(n)}\}_{i=1}^N$, that $\{x_n\}$ defined by (1.6) converges strongly to a common solution of SVIP (1.2) and a fixed point of a finite family of k_i-strictly pseudo-contractions, which solves some variational inequality problem.

Remark 1.1

(1) We notice that Jitsupa et al. [1] did not define the domains and the ranges of B_1 and B_2 in the iteration process (1.6) and Theorem 3.1 of [1]. Certainly, it is easy to misunderstand that B_1 is defined on H_1 into 2^{H_1} and B_2 is defined on H_2 into 2^{H_2}. In that case, $\{u_n\}$ defined in (1.6) lies in H_1. However, the domain of T_i is C but not H_1, which makes the iteration process (1.6) not well-defined. Thus, it is necessary to give the definite domains and ranges of B_1 and B_2.

(2) Can the iterative scheme (1.6) be modified for solving more problems?

In this paper, we introduce a new general iterative scheme as follows:

$$\begin{cases} u_n = J_{\lambda_1}^{B_1}[x_n + \gamma A^*(J_{\lambda_2}^{B_2}(I - \lambda_2 f) - I)Ax_n], \\ v_n = P_C(u_n - \xi Du_n), \\ y_n = \beta_n v_n + (1 - \beta_n)\sum_{i=1}^N \eta_i^{(n)} T_i v_n, \\ x_{n+1} = P_C[\alpha_n \tau F(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n \mu V)y_n], \quad n \geq 1, \end{cases} \tag{1.7}$$

where $B_1 : C \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ are two multi-valued maximal monotone operators, $f : H_2 \to H_2$ is a ρ-inverse strongly monotone operator, $A : H_1 \to H_2$ is a bounded linear operator, and A^* is the adjoint of A, $D : C \to H_1$ is a δ-inverse strongly monotone operator, $\{T_i\}_{i=1}^N : C \to C$ is a finite family of k_i-strictly pseudo-contractions, P_C is the metric projection of H_1 onto the closed convex set C, F is L-Lipschitzian on H_1, and V is a η-strongly monotone and K-Lipschitzian operator. Under some suitable assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\eta_i^{(n)}\}_{i=1}^N$, we prove that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a common solution of SMVIP (1.3) with the constraints of a variational inequality and a fixed point problem of a finite family of strict pseudo-contractions, which solves the following variational inequality:

$$\langle \mu Vq - \tau Fq, q - p\rangle \leq 0, \quad \forall p \in \mathcal{F},$$

where $\mathcal{F}$ denotes the set of common solutions of SMVIP (1.3), a variational inequality, and a fixed point problem of a finite family of strict pseudo-contractions. Finally, we also provide a numerical example to support our strong convergence result.

2 Preliminaries

Throughout this paper, let H_1 and H_2 be two real Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H_1.

Recall that $S : H_1 \to H_1$ is said to be a nonexpansive mapping if $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in H_1$. It is also called firmly nonexpansive if $\langle Sx - Sy, x - y\rangle \geq \|Sx - Sy\|^2$, $\forall x, y \in H_1$. We can easily see that S is firmly nonexpansive if and only if S can be written as $S = \frac{1}{2}(I + T)$, where $T : H_1 \to H_1$ is nonexpansive.

Moreover, $S : H_1 \to H_1$ is called

(i) contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha\|x - y\|, \quad \forall x, y \in H_1; \tag{2.1}$$

(ii) L-Lipschitzian if there exists a positive constant L such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in H_1; \tag{2.2}$$

(iii) η-strongly monotone if there exists a positive constant η such that

$$\langle Sx - Sy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in H_1; \tag{2.3}$$

(iv) β-inverse strongly monotone (in short, β-*ism*) if there exists a positive constant β such that

$$\langle Sx - Sy, x - y\rangle \geq \beta\|Sx - Sy\|^2, \quad \forall x, y \in H_1; \tag{2.4}$$

(v) averaged if it can be expressed as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$S := (1 - \alpha)I + \alpha T, \tag{2.5}$$

where $\alpha \in (0, 1)$, I is the identity operator on H_1 and $T : H_1 \to H_1$ is nonexpansive.

It is easily seen that averaged mappings are nonexpansive. In the meantime, firmly nonexpansive mappings are averaged.

In addition, a mapping $S : H_1 \to H_1$ is called k-strict pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\big\|(I - S)x - (I - S)y\big\|^2, \quad \forall x, y \in H_1. \tag{2.6}$$

A linear operator D is said to be a strongly positive bounded linear operator on H_1 if there exists a positive constant $\overline{\tau}$ such that

$$\langle Dx, x\rangle \geq \overline{\tau}\|x\|^2, \quad \forall x \in H_1.$$

From the definition above, we obtain easily that a strongly positive bounded linear operator D is $\overline{\tau}$-strongly monotone and $\|D\|$-Lipschitzian.

A multi-valued mapping $M : D(M) \subseteq H_1 \to 2^{H_1}$ is called monotone if, for all $x, y \in D(M)$, $u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \geq 0.$$

A monotone mapping M is maximal if the Graph(M) is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for $x \in D(M)$, $u \in H_1$, $\langle x - y, u - v\rangle \geq 0$ for each $(y, v) \in \text{Graph}(M)$ implies that $u \in Mx$.

Let $M : D(M) \subseteq H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then the resolvent operator $J_\lambda^M : H_1 \to D(M)$ is defined by

$$J_\lambda^M x := (I + \lambda M)^{-1}(x), \quad \forall x \in H_1,$$

for $\forall\lambda > 0$, where I stands for the identity operator on H_1. We observe that J_λ^M is single-valued, nonexpansive, and firmly nonexpansive.

Let $D: C \to H_1$ be a nonlinear mapping. Then the variational inequality problem (VIP) is to find $u \in C$ such that

$$\langle Du, v - u\rangle \geq 0, \quad \forall v \in C. \tag{2.7}$$

We denote the solution set of VIP (2.7) by $\mathrm{VI}(C, D)$. Many different approaches have been studied for solving this problem; see, e.g., [14–17].

For each point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.8}$$

P_C is called the metric projection of H_1 onto C.

It is known that P_C is nonexpansive and satisfies the following inequalities:

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle, \quad \forall x, y \in H_1, \tag{2.9}$$

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H_1, y \in C. \tag{2.10}$$

We note that each nonexpansive mapping $S: H_1 \to H_1$ satisfies the following inequality (see Theorem 3 in [18] and Theorem 1 in [19]):

$$\bigl\langle (x - Sx) - (y - Sy), Sy - Sx\bigr\rangle \leq \frac{1}{2}\bigl\|(Sx - x) - (Sy - y)\bigr\|^2, \quad \forall x, y \in H_1, \tag{2.11}$$

particularly, for $\forall x \in H_1$, $y \in F(S)$,

$$\langle x - Sx, y - Sx\rangle \leq \frac{1}{2}\|Sx - x\|^2. \tag{2.12}$$

Proposition 2.1 ([11])

(i) *If $T = (1 - \alpha)S + \alpha V$, where $S: H_1 \to H_1$ is averaged, $V: H_1 \to H_1$ is nonexpansive, and $\alpha \in [0, 1]$, then T is averaged.*

(ii) *The composite of finitely many averaged mappings is averaged.*

(iii) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a nonempty common fixed point, then*

$$\bigcap_{i=1}^N F(T_i) = F(T_1 \circ T_2 \circ \cdots \circ T_N).$$

(iv) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*

(v) *T is averaged if and only if its complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$.*

Proposition 2.2 ([11]) *Let $\lambda > 0$, h be an α-ism operator, and B be a maximal monotone operator. If $\lambda \in (0, 2\alpha)$, then it is easily seen that the operator $J_\lambda^B(I - \lambda h)$ is averaged.*

Proposition 2.3 ([11]) *Let $\lambda > 0$ and B_1 be a maximal monotone operator. Then*

$$x^* \textit{ solves } (1.1) \quad \Leftrightarrow \quad x^* = J_\lambda^{B_1}(I - \lambda f_1)\bigl(x^*\bigr) \quad \textit{and} \quad Ax^* = J_\lambda^{B_2}(I - \lambda f_2)Ax^*.$$

Proposition 2.4 ([20]) *Let $D : C \to H_1$ be an inverse strongly monotone operator. Then*

$$u \in \mathrm{VI}(C,D) \quad \Leftrightarrow \quad u = P_C(u - \lambda Du), \quad \forall \lambda > 0.$$

Proposition 2.5 ([21]) *Let D be an inverse strongly-monotone mapping of C into H_1. Let $N_C v$ be the normal cone to C at $v \in C$, i.e.,*

$$N_C v = \bigl\{ w \in H_1 | \langle v - u, w\rangle \geq 0, \forall u \in C \bigr\},$$

and define

$$Tv = \begin{cases} Dv + N_C v, & v \in C, \\ \emptyset, & v \in H_1 \setminus C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C,D)$.

In order to prove our main results, we need the following lemmas.

Lemma 2.1 ([22]) *Let $T : C \to C$ be a k-strict pseudo-contraction. For $\lambda \in [k, 1)$, define $S : C \to C$ by $Sx = \lambda x + (1 - \lambda)Tx$ for each $x \in C$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

Lemma 2.2 ([23]) *If $T : C \to C$ is a k-strict pseudo-contraction, then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well-defined.*

Lemma 2.3 ([23]) *Let C be a nonempty closed convex subset of the Hilbert space H_1. Given an integer $N \geq 1$, assume that $\{T_i\}_{i=1}^N : C \to C$ is a finite family of k_i-strict pseudo-contractions. Suppose that $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i : C \to C$ is a k-strict pseudo-contraction with $k = \max\{k_i : 1 \leq i \leq N\}$ and $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.4 ([24]) *Let E be an inner product space. Then, for any $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.5 ([25]) *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying the property*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^\infty \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^\infty |\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.6 ([26]) *Assume that T is nonexpansive self-mapping of a closed convex subset C of a Hilbert space H_1. If T has a fixed point, then $I - T$ is demiclosed, i.e., whenever $\{x_n\}$*

weakly converges to some x and $\{(I-T)x_n\}$ converges strongly to y, it follows that $(I-T)x = y$. Here I is the identity mapping on H_1.

Lemma 2.7 ([27]) *Let V be a K-Lipschitzian and η-strongly monotone operator on a nonempty closed convex subset C of a Hilbert space H_1 with $0 < \eta \le K$ and $0 < t < 2\eta/K^2$. Then the mapping $S : C \to C$ defined by $S := (I - tV)$ is a contraction with coefficient $\tau_t = 1 - t(\eta - \frac{tK^2}{2})$.*

Lemma 2.8 ([28]) *Let C be a nonempty closed convex subset of a Hilbert space H_1 and P_C be the metric projection of H_1 onto C. Let $S : C \to C$ be a nonexpansive mapping with $F(S) \neq \emptyset$ and $F : C \to H_1$ be an L-Lipschitzian mapping with constant $L \ge 0$. Let $V : C \to H_1$ be an η-strongly monotone and K-Lipschitzian mapping. Suppose that $0 < \mu < 2\eta/K^2$ and $0 \le \tau L < \tau_0$, where $\tau_0 = 1 - \sqrt{1 - \mu(2\eta - \mu K^2)}$. Then the net $\{x_t\}_{t\in(0,1)}$ defined by $x_t = P_C[t\tau F x_t + (I - t\mu V)Sx_t]$ converges strongly as $t \to 0$ to a fixed point q of S which solves the variational inequality*

$$\langle(\mu V - \tau F)q, q - p\rangle \le 0, \quad \forall p \in F(S).$$

3 Main results

Lemma 3.1 *Let H_1 and H_2 be two real Hilbert spaces and C be a nonempty closed convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A, and r be the spectral radius of the operator A^*A. Let $f : H_2 \to H_2$ be a ρ-inverse strongly monotone operator and $B_1 : C \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be two multi-valued maximal monotone operators. Let $D : C \to H_1$ be a δ-inverse strongly monotone operator. Assume that $\{T_i\}_{i=1}^N : C \to C$ is a finite family of k_i-strict pseudo-contraction mappings such that $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \Gamma \cap \mathrm{VI}(C,D) \neq \emptyset$. Let P_C be the metric projection of H_1 onto C, and $F : C \to H_1$ be an L-Lipschitzian mapping with constant $L \ge 0$. Suppose that $V : C \to H_1$ is an η-strongly monotone and K-Lipschitzian mapping with $0 < \eta \le K$, $0 < \mu < 2\eta/K^2$ and $0 \le \tau L < \tau_0$, where $\tau_0 = 1 - \sqrt{1 - \mu(2\eta - \mu K^2)}$. For $x_1 \in C$, let $\{x_n\}$ be a sequence of C generated by (1.7). Assume that the following conditions hold*:

(i) $\lambda_1 > 0$, $0 < \lambda_2 < 2\rho$, $0 < \gamma < \frac{1}{r}$, $0 < \xi < 2\delta$;
(ii) $0 < \alpha_n < 1$, $\sum_{i=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\max_{1\le i\le N} k_i \le \beta_n \le l < 1$, $\lim_{n\to\infty} \beta_n = l$;
(iv) $\sum_{i=1}^N \eta_i^{(n)} = 1$, $0 < \gamma_n < 1$, $\lim_{n\to\infty} \gamma_n = 0$;
(v) $\sum_{n=1}^\infty (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n| + |\gamma_{n+1} - \gamma_n| + \sum_{i=1}^N |\eta_i^{(n+1)} - \eta_i^{(n)}|) < \infty$.

Then $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof Let $G_n := \sum_{i=1}^N \eta_i^{(n)} T_i$. By Lemma 2.3, we obtain that, for each $n \ge 1$, G_n is a k-strict pseudo-contraction on C and $F(G_n) = \bigcap_{i=1}^N F(T_i)$, where $k = \max\{k_i : 1 \le i \le N\}$. Let $U := J_{\lambda_2}^{B_2}(I - \lambda_2 f)$. Then the iterative scheme (1.7) can be rewritten as

$$\begin{cases} u_n = J_{\lambda_1}^{B_1}[x_n + \gamma A^*(U - I)Ax_n], \\ v_n = P_C(u_n - \xi D u_n), \\ y_n = \beta_n v_n + (1 - \beta_n) G_n v_n, \\ x_{n+1} = P_C[\alpha_n \tau F x_n + \gamma_n x_n + ((1 - \gamma_n)I - \alpha_n \mu V) y_n], \quad n \ge 1. \end{cases} \tag{3.1}$$

We divide the rest of the proof into two steps.

Step 1. We claim that the sequence $\{x_n\}$ is bounded.

Indeed, take $p \in \mathcal{F}$. Then $J_{\lambda_1}^{B_1} p = p$, $U(Ap) = Ap$, $G_n p = p$, $P_C(I - \xi D)p = p$, and it is easily seen that $Wp = p$, where $W := I + \gamma A^*(U - I)A$. From the definition of firm nonexpansion and Proposition 2.2, we have that $J_{\lambda_1}^{B_1}$ and U are averaged. Likewise W is also averaged because it is $\frac{\nu}{r}$-*ism* for some $\nu > \frac{1}{2}$. Actually, by (v) of Proposition 2.1, we know that $I - U$ is ν-*ism* with $\nu > \frac{1}{2}$. Hence, we have

$$\begin{aligned}\langle A^*(I-U)Ax - A^*(I-U)Ay, x - y\rangle &= \langle (I-U)Ax - (I-U)Ay, Ax - Ay\rangle \\ &\ge \nu \| (I-U)Ax - (I-U)Ay\|^2 \\ &\ge \frac{\nu}{r}\| A^*(I-U)Ax - A^*(I-U)Ay\|^2.\end{aligned}$$

Thus $\gamma A^*(I - U)A$ is $\frac{\nu}{\gamma r}$-*ism*. Due to the condition $0 < \gamma < \frac{1}{r}$, the complement $I - \gamma A^*(I - U)A$ is averaged, and so is $M := J_{\lambda_1}^{B_1}[I + \gamma A^*(U - I)A]$. Therefore, $J_{\lambda_1}^{B_1}$, U, W, and M are nonexpansive mappings.

From (3.1), we estimate

$$\begin{aligned}\|u_n - p\|^2 &= \| J_{\lambda_1}^{B_1}[x_n + \gamma A^*(U - I)Ax_n] - J_{\lambda_1}^{B_1} p\|^2 \\ &\le \| x_n + \gamma A^*(U-I)Ax_n - p\|^2 \\ &= \|x_n - p\|^2 + \gamma^2 \| A^*(U-I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^*(U-I)Ax_n\rangle.\end{aligned} \tag{3.2}$$

Thus, we get

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \gamma^2\langle (U-I)Ax_n, AA^*(U-I)Ax_n\rangle + 2\gamma\langle x_n - p, A^*(U-I)Ax_n\rangle. \tag{3.3}$$

Next, setting $\Lambda_1 := \gamma^2\langle (U - I)Ax_n, AA^*(U - I)Ax_n\rangle$, we estimate

$$\begin{aligned}\Lambda_1 &= \gamma^2\langle (U-I)Ax_n, AA^*(U-I)Ax_n\rangle \\ &\le r\gamma^2\langle (U-I)Ax_n, (U-I)Ax_n\rangle \\ &= r\gamma^2 \| (U-I)Ax_n\|^2.\end{aligned} \tag{3.4}$$

Setting $\Lambda_2 := 2\gamma\langle x_n - p, A^*(U - I)Ax_n\rangle$, we obtain from (2.12)

$$\begin{aligned}\Lambda_2 &= 2\gamma\langle x_n - p, A^*(U-I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - p), (U-I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - p) + (U-I)Ax_n - (U-I)Ax_n, (U-I)Ax_n\rangle \\ &= 2\gamma\big(\langle UAx_n - Ap, (U-I)Ax_n\rangle - \|(U-I)Ax_n\|^2\big) \\ &\le 2\gamma\left(\frac{1}{2}\|(U-I)Ax_n\|^2 - \|(U-I)Ax_n\|^2\right) \\ &\le -\gamma\|(U-I)Ax_n\|^2.\end{aligned} \tag{3.5}$$

In view of (3.3)–(3.5), we have

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \gamma(r\gamma - 1)\|(U-I)Ax_n\|^2. \tag{3.6}$$

From $0 < \gamma < \frac{1}{r}$, we obtain

$$\|u_n - p\| \le \|x_n - p\|. \tag{3.7}$$

Since D is δ-inverse strongly monotone and $0 < \xi < 2\delta$, we estimate

$$\begin{aligned}
\|v_n - p\|^2 &= \|P_C(I-\xi D)u_n - P_C(I-\xi D)p\|^2 \\
&\le \|(I-\xi D)u_n - (I-\xi D)p\|^2 \\
&= \|(u_n - p) - \xi(Du_n - Dp)\|^2 \\
&= \|u_n - p\|^2 - 2\xi\langle Du_n - Dp, u_n - p\rangle + \xi^2\|Du_n - Dp\|^2 \\
&\le \|u_n - p\|^2 - 2\xi\delta\|Du_n - Dp\|^2 + \xi^2\|Du_n - Dp\|^2 \\
&= \|u_n - p\|^2 + \xi(\xi - 2\delta)\|Du_n - Dp\|^2 \\
&\le \|u_n - p\|^2,
\end{aligned}$$

which implies

$$\|v_n - p\| \le \|u_n - p\|. \tag{3.8}$$

Define $S_n x := \beta_n x + (1-\beta_n)G_n x$, $\forall x \in C$. Using Lemma 2.1, we obtain that $S_n : C \to C$ is a nonexpansive mapping and $F(S_n) = F(G_n)$. It is clear that $S_n p = p$, and hence

$$\|y_n - p\| = \|S_n v_n - p\| = \|S_n v_n - S_n p\| \le \|v_n - p\|. \tag{3.9}$$

By (3.7)–(3.9), we have

$$\|y_n - p\| \le \|v_n - p\| \le \|u_n - p\| \le \|x_n - p\|. \tag{3.10}$$

It follows from (3.1) and Lemma 2.7 that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \|P_C[\alpha_n\tau Fx_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu V)y_n] - P_C p\| \\
&\quad \le \|\alpha_n\tau Fx_n + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n\mu V)y_n - p\| \\
&\quad = \|\alpha_n(\tau Fx_n - \mu Vp) + \gamma_n(x_n - p) + [(1-\gamma_n)I - \alpha_n\mu V]y_n - [(1-\gamma_n)I - \alpha_n\mu V]p\| \\
&\quad \le \|[(1-\gamma_n)I - \alpha_n\mu V]y_n - [(1-\gamma_n)I - \alpha_n\mu V]p\| + \gamma_n\|x_n - p\| + \alpha_n\|\tau Fx_n - \mu Vp\| \\
&\quad \le \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|y_n - p\| + \gamma_n\|x_n - p\| + \alpha_n\|\tau Fx_n - \mu Vp\| \\
&\quad \le \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|x_n - p\| + \gamma_n\|x_n - p\| + \alpha_n\|\tau Fx_n - \mu Vp\|
\end{aligned}$$

$$
\begin{aligned}
&= \left[1-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|x_n-p\|+\alpha_n\|\tau Fx_n-\mu Vp\| \\
&\le \left[1-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|x_n-p\|+\alpha_n\bigl[\|\tau Fx_n-\tau Fp\|+\|\tau Fp-\mu Vp\|\bigr] \\
&\le \left[1-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|x_n-p\|+\alpha_n\tau L\|x_n-p\|+\alpha_n\|\tau Fp-\mu Vp\| \\
&= \left[1-\alpha_n\left(\mu\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}-\tau L\right)\right]\|x_n-p\|+\alpha_n\|\tau Fp-\mu Vp\|.
\end{aligned}
$$

By induction, we derive

$$
\|x_n-p\|\le \max\bigl\{\|x_0-p\|,M_1\bigr\},
$$

where $M_1=\sup_{n\ge1}\frac{\|\tau Fp-\mu Vp\|}{\mu\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}-\tau L}$. This shows that $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{v_n\}$, and $\{u_n\}$.

Step 2. We claim $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$.

Indeed, from (3.1), we have

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad= \bigl\|P_C\bigl[\alpha_n\tau Fx_n+\gamma_n x_n+\bigl((1-\gamma_n)I-\alpha_n\mu V\bigr)y_n\bigr] \\
&\qquad - P_C\bigl[\alpha_{n-1}\tau Fx_{n-1}+\gamma_{n-1}x_{n-1}+\bigl((1-\gamma_{n-1})I-\alpha_{n-1}\mu V\bigr)y_{n-1}\bigr]\bigr\| \\
&\quad\le \bigl\|\alpha_n\tau Fx_n+\gamma_n x_n+\bigl((1-\gamma_n)I-\alpha_n\mu V\bigr)y_n \\
&\qquad - \alpha_{n-1}\tau Fx_{n-1}-\gamma_{n-1}x_{n-1}-\bigl((1-\gamma_{n-1})I-\alpha_{n-1}\mu V\bigr)y_{n-1}\bigr\| \\
&\quad\le \bigl\|\bigl((1-\gamma_n)I-\alpha_n\mu V\bigr)y_n-\bigl((1-\gamma_{n-1})I-\alpha_{n-1}\mu V\bigr)y_{n-1}\bigr\|+\alpha_n\tau\|Fx_n-Fx_{n-1}\| \\
&\qquad + \|\gamma_n x_n-\gamma_{n-1}x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\tau Fx_{n-1}\| \\
&\quad\le \bigl\|\bigl((1-\gamma_n)I-\alpha_n\mu V\bigr)y_n-\bigl((1-\gamma_n)I-\alpha_n\mu V\bigr)y_{n-1}\bigr\|+\bigl\|\bigl((1-\gamma_n)I-\alpha_n\mu V\bigr)y_{n-1} \\
&\qquad - \bigl((1-\gamma_{n-1})I-\alpha_{n-1}\mu V\bigr)y_{n-1}\bigr\|+\alpha_n\tau L\|x_n-x_{n-1}\|+\|\gamma_n x_n-\gamma_n x_{n-1}\| \\
&\qquad + \|\gamma_n x_{n-1}-\gamma_{n-1}x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\tau Fx_{n-1}\| \\
&\quad\le \left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|y_n-y_{n-1}\| \\
&\qquad + |\gamma_n-\gamma_{n-1}|\|y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\mu Vy_{n-1}\|+\alpha_n\tau L\|x_n-x_{n-1}\| \\
&\qquad + \gamma_n\|x_n-x_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\tau Fx_{n-1}\| \\
&\quad= \left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|y_n-y_{n-1}\|+(\gamma_n+\alpha_n\tau L)\|x_n-x_{n-1}\| \\
&\qquad + |\alpha_n-\alpha_{n-1}|\bigl(\|\mu Vy_{n-1}\|+\|\tau Fx_{n-1}\|\bigr)+|\gamma_n-\gamma_{n-1}|\bigl(\|x_{n-1}\|+\|y_{n-1}\|\bigr) \\
&\quad\le \left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|y_n-y_{n-1}\|+(\gamma_n+\alpha_n\tau L)\|x_n-x_{n-1}\| \\
&\qquad + |\alpha_n-\alpha_{n-1}|M_2+|\gamma_n-\gamma_{n-1}|M_3,
\end{aligned}
\tag{3.11}
$$

where $M_2 = \sup_{n\geq 1}\{\|\mu V y_{n-1}\| + \|\tau F x_{n-1}\|\}$, $M_3 = \sup_{n\geq 1}\{\|x_{n-1}\| + \|y_{n-1}\|\}$. Furthermore, since $y_n = S_n v_n$, we have

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad = \|S_n v_n - S_{n-1} v_{n-1}\| \\
&\quad \leq \|S_n v_n - S_n v_{n-1}\| + \|S_n v_{n-1} - S_{n-1} v_{n-1}\| \\
&\quad \leq \|v_n - v_{n-1}\| + \big\|\beta_n v_{n-1} + (1-\beta_n) G_n v_{n-1} - \big[\beta_{n-1} v_{n-1} + (1-\beta_{n-1}) G_{n-1} v_{n-1}\big]\big\| \\
&\quad = \|v_n - v_{n-1}\| + \big\|(\beta_n - \beta_{n-1})(v_{n-1} - G_{n-1} v_{n-1}) + (1-\beta_n)(G_n v_{n-1} - G_{n-1} v_{n-1})\big\| \\
&\quad \leq \|v_n - v_{n-1}\| + |\beta_n - \beta_{n-1}| \|v_{n-1} - G_{n-1} v_{n-1}\| + (1-\beta_n)\|G_n v_{n-1} - G_{n-1} v_{n-1}\| \\
&\quad \leq \|v_n - v_{n-1}\| + |\beta_n - \beta_{n-1}| M_4 + \sum_{i=1}^{N} \big|\eta_i^{(n)} - \eta_i^{(n-1)}\big| \|T_i v_{n-1}\|,
\end{aligned} \tag{3.12}$$

where $M_4 = \sup_{n\geq 1} \|v_{n-1} - G_{n-1} v_{n-1}\|$.

By the nonexpansion of P_C and $I - \xi D$, we get

$$\begin{aligned}
\|v_n - v_{n-1}\| &= \big\|P_C(I-\xi D)u_n - P_C(I-\xi D)u_{n-1}\big\| \\
&\leq \big\|(I-\xi D)u_n - (I-\xi D)u_{n-1}\big\| = \|u_n - u_{n-1}\|.
\end{aligned} \tag{3.13}$$

Note that $M := J_{\lambda_1}^{B_1}[I + \gamma A^*(U-I)A]$ is nonexpansive, we have

$$\begin{aligned}
\|u_n - u_{n-1}\| &= \big\|J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U-I)A\big]x_n - J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U-I)A\big]x_{n-1}\big\| \\
&\leq \|x_n - x_{n-1}\|.
\end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) for (3.12), we have

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M_4 + \sum_{i=1}^{N} \big|\eta_i^{(n)} - \eta_i^{(n-1)}\big| \|T_i v_{n-1}\|. \tag{3.15}$$

This together with (3.11) leads to

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \leq \left[1 - \gamma_n - \alpha_n \mu\left(\eta - \frac{\alpha_n \mu K^2}{2(1-\gamma_n)}\right)\right]\Bigg[\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M_4 \\
&\qquad + \sum_{i=1}^{N} \big|\eta_i^{(n)} - \eta_i^{(n-1)}\big| \|T_i v_{n-1}\|\Bigg] + (\gamma_n + \alpha_n \tau L)\|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| M_2 + |\gamma_n - \gamma_{n-1}| M_3 \\
&\quad \leq \left[1 - \gamma_n - \alpha_n \mu\left(\eta - \frac{\alpha_n \mu K^2}{2(1-\gamma_n)}\right)\right]\|x_n - x_{n-1}\| + (\gamma_n + \alpha_n \tau L)\|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| M_2 + |\gamma_n - \gamma_{n-1}| M_3 + |\beta_n - \beta_{n-1}| M_4 + \sum_{i=1}^{N} \big|\eta_i^{(n)} - \eta_i^{(n-1)}\big| \|T_i v_{n-1}\|
\end{aligned}$$

$$= \left[1 - \alpha_n\left(\mu\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)} - \tau L\right)\right]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2$$
$$+ |\gamma_n - \gamma_{n-1}|M_3 + |\beta_n - \beta_{n-1}|M_4 + \sum_{i=1}^{N}\left|\eta_i^{(n)} - \eta_i^{(n-1)}\right|\|T_i v_{n-1}\|. \tag{3.16}$$

Noticing condition (v) and applying Lemma 2.5 to (3.16), we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.17}$$

This completes the proof. □

Lemma 3.2 *Let H_1 and H_2 be two real Hilbert spaces and C be a nonempty closed convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A, and r be the spectral radius of the operator A^*A. Let $f : H_2 \to H_2$ be a ρ-inverse strongly monotone operator and $B_1 : C \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be two multi-valued maximal monotone operators. Let $D : C \to H_1$ be a δ-inverse strongly monotone operator. Assume that $\{T_i\}_{i=1}^N : C \to C$ is a finite family of k_i-strict pseudo-contraction mappings such that $\mathcal{F} \neq \emptyset$. Let P_C be the metric projection of H_1 onto C, and $F : C \to H_1$ be an L-Lipschitzian mapping with constant $L \geq 0$. Suppose that $V : C \to H_1$ is an η-strongly monotone and K-Lipschitzian mapping, where η and μ satisfy the conditions of Lemma 3.1. For $x_1 \in C$, let $\{x_n\}$ be a sequence of C generated by* (1.7). *Assume that conditions* (i)–(v) *in Lemma* 3.1 *hold. Then $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the following variational inequality*:

$$\langle \mu Vq - \tau Fq, q - p\rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Proof The proof of the lemma is divided into four steps.

Step 1. We claim $\lim_{n\to\infty}\|x_n - G_n x_n\| = 0$.

Indeed, take $\forall p \in \mathcal{F}$. From (3.1) and (3.6), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \left\|P_C\left[\alpha_n\tau Fx_n + \gamma_n x_n + \left((1-\gamma_n)I - \alpha_n\mu V\right)y_n\right] - p\right\|^2 \\
&\quad \leq \left\|\alpha_n\tau Fx_n + \gamma_n x_n + \left((1-\gamma_n)I - \alpha_n\mu V\right)y_n - p\right\|^2 \\
&\quad = \left\|\alpha_n(\tau Fx_n - \mu Vp) + \gamma_n(x_n - p) + \left[(1-\gamma_n)I - \alpha_n\mu V\right]y_n - \left[(1-\gamma_n)I - \alpha_n\mu V\right]p\right\|^2 \\
&\quad \leq \left\|\alpha_n(\tau Fx_n - \mu Vp) + \left[(1-\gamma_n)I - \alpha_n\mu V\right]y_n - \left[(1-\gamma_n)I - \alpha_n\mu V\right]p\right\|^2 \\
&\qquad + \gamma_n^2\|x_n - p\|^2 + 2\gamma_n\|x_n - p\|\left\|\alpha_n(\tau Fx_n - \mu Vp)\right. \\
&\qquad \left. + \left[(1-\gamma_n)I - \alpha_n\mu V\right]y_n - \left[(1-\gamma_n)I - \alpha_n\mu V\right]p\right\| \\
&\quad \leq \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|y_n - p\|^2 + \alpha_n^2\|\tau Fx_n - \mu Vp\|^2 \\
&\qquad + 2\alpha_n\left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|\tau Fx_n - \mu Vp\|\|y_n - p\| \\
&\qquad + \gamma_n^2\|x_n - p\|^2 + 2\gamma_n\|x_n - p\|\left\{\alpha_n\|\tau Fx_n - \mu Vp\|\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \big\| \big[(1-\gamma_n)I - \alpha_n \mu V\big] y_n - \big[(1-\gamma_n)I - \alpha_n \mu V\big] p \big\| \big\} \\
&\le \left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2 \|u_n-p\|^2 + \alpha_n^2\|\tau Fx_n-\mu Vp\|^2 \\
&+ 2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\| + \gamma_n^2\|x_n-p\|^2 \\
&+ 2\gamma_n\|x_n-p\|\left\{\alpha_n\|\tau Fx_n-\mu Vp\| + \left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|y_n-p\|\right\} \\
&\le \left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2 \left[\|x_n-p\|^2 + \gamma(r\gamma-1)\big\|(U-I)Ax_n\big\|^2\right] \\
&+ \alpha_n^2\|\tau Fx_n-\mu Vp\|^2 + 2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\| + \gamma_n^2\|x_n-p\|^2 \\
&+ 2\gamma_n\|x_n-p\|\big(\alpha_n\|\tau Fx_n-\mu Vp\| + \|y_n-p\|\big),
\end{aligned}
\tag{3.18}
$$

which implies

$$
\begin{aligned}
&\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2 \gamma(1-r\gamma)\big\|(U-I)Ax_n\big\|^2 \\
&\quad\le \left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2 \|x_n-p\|^2 + \alpha_n^2\|\tau Fx_n-\mu Vp\|^2 \\
&\quad+ 2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\| + \gamma_n^2\|x_n-p\|^2 \\
&\quad+ 2\gamma_n\|x_n-p\|\big(\alpha_n\|\tau Fx_n-\mu Vp\| + \|y_n-p\|\big) - \|x_{n+1}-p\|^2 \\
&\quad\le \|x_n-p\|^2 + \left[\gamma_n+\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2 \|x_n-p\|^2 + \alpha_n^2\|\tau Fx_n-\mu Vp\|^2 \\
&\quad+ 2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\| + \gamma_n^2\|x_n-p\|^2 + 2\gamma_n\|x_n-p\|\big(\alpha_n\|\tau Fx_n-\mu Vp\| \\
&\quad+ \|y_n-p\|\big) - \|x_{n+1}-p\|^2 \\
&\quad\le \left[\gamma_n+\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2 \|x_n-p\|^2 + \alpha_n^2\|\tau Fx_n-\mu Vp\|^2 \\
&\quad+ 2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\| + \gamma_n^2\|x_n-p\|^2 \\
&\quad+ 2\gamma_n\|x_n-p\|\big(\alpha_n\|\tau Fx_n-\mu Vp\| + \|y_n-p\|\big) \\
&\quad+ \|x_n-x_{n+1}\|\big(\|x_n-p\| + \|x_{n+1}-p\|\big).
\end{aligned}
$$

Since $\gamma(1-r\gamma) > 0$, $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, and $\{x_n\}$, $\{y_n\}$ are bounded, from (3.17) we get

$$
\lim_{n\to\infty}\big\|(U-I)Ax_n\big\| = 0. \tag{3.19}
$$

In addition, by the firm nonexpansion of $J_{\lambda_1}^{B_1}$, (3.2), (3.6), and $\gamma\in(0,\frac{1}{r})$, we estimate

$$
\begin{aligned}
\|u_n-p\|^2 &= \big\|J_{\lambda_1}^{B_1}\big[x_n+\gamma A^*(U-I)Ax_n\big] - J_{\lambda_1}^{B_1}p\big\|^2 \\
&\le \big\langle J_{\lambda_1}^{B_1}\big[x_n+\gamma A^*(U-I)Ax_n\big] - J_{\lambda_1}^{B_1}p, x_n+\gamma A^*(U-I)Ax_n-p\big\rangle \\
&= \big\langle u_n-p, x_n+\gamma A^*(U-I)Ax_n-p\big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\big(\|u_n - p\|^2 + \big\|x_n + \gamma A^*(U-I)Ax_n - p\big\|^2 \\
&\quad - \big\|(u_n - p) - \big[x_n + \gamma A^*(U-I)Ax_n - p\big]\big\|^2\big) \\
&\le \frac{1}{2}\big[\|u_n - p\|^2 + \|x_n - p\|^2 + \gamma(r\gamma - 1)\big\|(U-I)Ax_n\big\|^2 \\
&\quad - \big\|u_n - x_n - \gamma A^*(U-I)Ax_n\big\|^2\big] \\
&\le \frac{1}{2}\big[\|u_n - p\|^2 + \|x_n - p\|^2 - \big\|u_n - x_n - \gamma A^*(U-I)Ax_n\big\|^2\big] \\
&= \frac{1}{2}\big[\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 - \gamma^2\big\|A^*(U-I)Ax_n\big\|^2 \\
&\quad + 2\gamma\langle u_n - x_n, A^*(U-I)Ax_n\rangle\big] \\
&\le \frac{1}{2}\big[\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\langle u_n - x_n, A^*(U-I)Ax_n\rangle\big] \\
&= \frac{1}{2}\big[\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\langle A(u_n - x_n), (U-I)Ax_n\rangle\big] \\
&\le \frac{1}{2}\big[\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\big\|A(u_n - x_n)\big\|\big\|(U-I)Ax_n\big\|\big],
\end{aligned}
$$

and hence

$$
\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\big\|A(u_n - x_n)\big\|\big\|(U-I)Ax_n\big\|. \tag{3.20}
$$

In view of (3.18) and (3.20), we obtain

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad\le \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|u_n - p\|^2 + \alpha_n^2\|\tau Fx_n - \mu Vp\|^2 \\
&\qquad + 2\alpha_n\|\tau Fx_n - \mu Vp\|\|y_n - p\| + \gamma_n^2\|x_n - p\|^2 \\
&\qquad + 2\gamma_n\|x_n - p\|\left\{\alpha_n\|\tau Fx_n - \mu Vp\| + \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|y_n - p\|\right\} \\
&\quad\le \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|u_n - p\|^2 + \alpha_n^2\|\tau Fx_n - \mu Vp\|^2 \\
&\qquad + 2\alpha_n\|\tau Fx_n - \mu Vp\|\|y_n - p\| + \gamma_n^2\|x_n - p\|^2 \\
&\qquad + 2\gamma_n\|x_n - p\|\big(\alpha_n\|\tau Fx_n - \mu Vp\| + \|y_n - p\|\big) \\
&\quad\le \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\big[\|x_n - p\|^2 - \|u_n - x_n\|^2 \\
&\qquad + 2\gamma\big\|A(u_n - x_n)\big\|\big\|(U-I)Ax_n\big\|\big] + \alpha_n^2\|\tau Fx_n - \mu Vp\|^2 \\
&\qquad + 2\alpha_n\|\tau Fx_n - \mu Vp\|\|y_n - p\| + \gamma_n^2\|x_n - p\|^2 \\
&\qquad + 2\gamma_n\|x_n - p\|\big(\alpha_n\|\tau Fx_n - \mu Vp\| + \|y_n - p\|\big) \\
&\quad\le \|x_n - p\|^2 + \left[\gamma_n + \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|x_n - p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|u_n-x_n\|^2 \\
&+ 2\gamma\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\big\|A(u_n-x_n)\big\|\big\|(U-I)Ax_n\big\| \\
&+ \alpha_n^2\|\tau Fx_n-Vp\|^2+2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\|+\gamma_n^2\|x_n-p\|^2 \\
&+ 2\gamma_n\|x_n-p\|\big(\alpha_n\|\tau Fx_n-\mu Vp\|+\|y_n-p\|\big),
\end{aligned}
$$

which hence implies that

$$
\begin{aligned}
&\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|u_n-x_n\|^2 \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\left[\gamma_n+\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|x_n-p\|^2 \\
&\qquad+ 2\gamma\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\big\|A(u_n-x_n)\big\|\big\|(U-I)Ax_n\big\| \\
&\qquad+ \alpha_n^2\|\tau Fx_n-\mu Vp\|^2+2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\|+\gamma_n^2\|x_n-p\|^2 \\
&\qquad+ 2\gamma_n\|x_n-p\|\big(\alpha_n\|\tau Fx_n-\mu Vp\|+\|y_n-p\|\big) \\
&\quad\le \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big)+\left[\gamma_n+\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|x_n-p\|^2 \\
&\qquad+ 2\gamma\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\big\|A(u_n-x_n)\big\|\big\|(U-I)Ax_n\big\| \\
&\qquad+ \alpha_n^2\|\tau Fx_n-\mu Vp\|^2+2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\|+\gamma_n^2\|x_n-p\|^2 \\
&\qquad+ 2\gamma_n\|x_n-p\|\big(\alpha_n\|\tau Fx_n-\mu Vp\|+\|y_n-p\|\big).
\end{aligned}
\tag{3.21}
$$

From conditions (ii), (iv), (3.17), and (3.19), we get

$$
\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.22}
$$

According to (3.1) and (3.10), we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \big\|P_C\big[\alpha_n\tau Fx_n+\gamma_nx_n+\big((1-\gamma_n)I-\alpha_n\mu V\big)y_n\big]-p\big\|^2 \\
&\le \big\|\alpha_n\tau Fx_n+\gamma_nx_n+\big((1-\gamma_n)I-\alpha_n\mu V\big)y_n-p\big\|^2 \\
&= \big\|\alpha_n(\tau Fx_n-\mu Vy_n)+\gamma_n(x_n-y_n)+y_n-p\big\|^2 \\
&= \|y_n-p\|^2+\big\|\alpha_n(\tau Fx_n-\mu Vy_n)+\gamma_n(x_n-y_n)\big\|^2 \\
&\quad+ 2\big\langle\alpha_n(\tau Fx_n-\mu Vy_n)+\gamma_n(x_n-y_n),y_n-p\big\rangle \\
&\le \|v_n-p\|^2+\big\|\alpha_n(\tau Fx_n-\mu Vy_n)+\gamma_n(x_n-y_n)\big\|^2 \\
&\quad+ 2\big\langle\alpha_n(\tau Fx_n-\mu Vy_n)+\gamma_n(x_n-y_n),y_n-p\big\rangle \\
&\le \|u_n-p\|^2+\xi(\xi-2\delta)\|Du_n-Dp\|^2+\big\|\alpha_n(\tau Fx_n-\mu Vy_n)+\gamma_n(x_n-y_n)\big\|^2 \\
&\quad+ 2\big\langle\alpha_n(\tau Fx_n-\mu Vy_n)+\gamma_n(x_n-y_n),y_n-p\big\rangle
\end{aligned}
$$

$$\begin{aligned}&\leq \|x_n - p\|^2 + \xi(\xi - 2\delta)\|Du_n - Dp\|^2 + \bigl\|\alpha_n(\tau Fx_n - \mu Vy_n) + \gamma_n(x_n - y_n)\bigr\|^2 \\ &\quad + 2\bigl\langle \alpha_n(\tau Fx_n - \mu Vy_n) + \gamma_n(x_n - y_n), y_n - p\bigr\rangle,\end{aligned}$$

and hence

$$\begin{aligned}&\xi(2\delta - \xi)\|Du_n - Dp\|^2 \\ &\quad\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \bigl\|\alpha_n(\tau Fx_n - \mu Vy_n) + \gamma_n(x_n - y_n)\bigr\|^2 \\ &\qquad + 2\bigl\langle \alpha_n(\tau Fx_n - \mu Vy_n) + \gamma_n(x_n - y_n), y_n - p\bigr\rangle \\ &\quad\leq \|x_n - x_{n+1}\|\bigl(\|x_n - p\| + \|x_{n+1} - p\|\bigr) + \bigl\|\alpha_n(\tau Fx_n - \mu Vy_n) + \gamma_n(x_n - y_n)\bigr\|^2 \\ &\qquad + 2\bigl\langle \alpha_n(\tau Fx_n - \mu Vy_n) + \gamma_n(x_n - y_n), y_n - p\bigr\rangle \\ &\quad\leq \|x_n - x_{n+1}\|\bigl(\|x_n - p\| + \|x_{n+1} - p\|\bigr) + \bigl(\alpha_n\|\tau Fx_n - \mu Vy_n\| + \gamma_n\|x_n - y_n\|\bigr)^2 \\ &\qquad + 2\bigl(\alpha_n\|\tau Fx_n - \mu Vy_n\| + \gamma_n\|x_n - y_n\|\bigr)\|y_n - p\|.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, and $\{x_n\}$, $\{y_n\}$ are bounded, by (3.17), we obtain

$$\lim_{n\to\infty}\|Du_n - Dp\| = 0. \tag{3.23}$$

It follows from (2.9), (3.1), and (3.10) that

$$\begin{aligned}&\|v_n - p\|^2 \\ &\quad= \bigl\|P_C(I - \xi D)u_n - P_C(I - \xi D)p\bigr\|^2 \\ &\quad\leq \bigl\langle P_C(I - \xi D)u_n - P_C(I - \xi D)p, (I - \xi D)u_n - (I - \xi D)p\bigr\rangle \\ &\quad= \bigl\langle v_n - p, (I - \xi D)u_n - (I - \xi D)p\bigr\rangle \\ &\quad= \frac{1}{2}\Bigl\{\|v_n - p\|^2 + \bigl\|u_n - p - \xi(Du_n - Dp)\bigr\|^2 \\ &\qquad - \bigl\|(v_n - p) - \bigl[(I - \xi D)u_n - (I - \xi D)p\bigr]\bigr\|^2\Bigr\} \\ &\quad\leq \frac{1}{2}\Bigl\{\|v_n - p\|^2 + \|u_n - p\|^2 + \xi(\xi - 2\delta)\|Du_n - Dp\|^2 - \bigl\|(v_n - u_n) + \xi(Du_n - Dp)\bigr\|^2\Bigr\} \\ &\quad\leq \frac{1}{2}\Bigl\{\|v_n - p\|^2 + \|u_n - p\|^2 + \xi(\xi - 2\delta)\|Du_n - Dp\|^2 \\ &\qquad - \|v_n - u_n\|^2 - \xi^2\|Du_n - Dp\|^2 - 2\xi\langle v_n - u_n, Du_n - Dp\rangle\Bigr\} \\ &\quad= \frac{1}{2}\Bigl\{\|v_n - p\|^2 + \|u_n - p\|^2 - 2\xi\delta\|Du_n - Dp\|^2 \\ &\qquad - \|v_n - u_n\|^2 + 2\xi\langle u_n - v_n, Du_n - Dp\rangle\Bigr\} \\ &\quad\leq \frac{1}{2}\bigl(\|v_n - p\|^2 + \|u_n - p\|^2 - \|v_n - u_n\|^2 + 2\xi\langle u_n - v_n, Du_n - Dp\rangle\bigr) \\ &\quad\leq \frac{1}{2}\bigl(\|v_n - p\|^2 + \|x_n - p\|^2 - \|v_n - u_n\|^2 + 2\xi\|u_n - v_n\|\|Du_n - Dp\|\bigr),\end{aligned}$$

which implies

$$\|v_n - p\|^2 \leq \|x_n - p\|^2 - \|v_n - u_n\|^2 + 2\xi\|u_n - v_n\|\|Du_n - Dp\|. \tag{3.24}$$

From (3.18) and (3.24), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad\le\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|y_n-p\|^2+\alpha_n^2\|\tau Fx_n-\mu Vp\|^2\\
&\qquad+2\alpha_n\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|\tau Fx_n-\mu Vp\|\|y_n-p\|\\
&\qquad+\gamma_n^2\|x_n-p\|^2+2\gamma_n\|x_n-p\|\bigl\{\alpha_n\|\tau Fx_n-\mu Vp\|\\
&\qquad+\bigl\|\bigl[(1-\gamma_n)I-\alpha_n\mu V\bigr]y_n-\bigl[(1-\gamma_n)I-\alpha_n\mu V\bigr]p\bigr\|\bigr\}\\
&\quad\le\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|v_n-p\|^2+\alpha_n^2\|\tau Fx_n-\mu Vp\|^2\\
&\qquad+2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\|+\gamma_n^2\|x_n-p\|^2\\
&\qquad+2\gamma_n\|x_n-p\|\left\{\alpha_n\|\tau Fx_n-\mu Vp\|+\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|y_n-p\|\right\}\\
&\quad\le\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\\
&\qquad\times\bigl(\|x_n-p\|^2-\|v_n-u_n\|^2+2\xi\|u_n-v_n\|\|Du_n-Dp\|\bigr)\\
&\qquad+\alpha_n^2\|\tau Fx_n-\mu Vp\|^2+2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\|+\gamma_n^2\|x_n-p\|^2\\
&\qquad+2\gamma_n\|x_n-p\|\bigl(\alpha_n\|\tau Fx_n-\mu Vp\|+\|y_n-p\|\bigr)\\
&\quad\le\|x_n-p\|^2+\left[\gamma_n+\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|x_n-p\|^2\\
&\qquad-\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|v_n-u_n\|^2\\
&\qquad+2\xi\|u_n-v_n\|\|Du_n-Dp\|+\alpha_n^2\|\tau Fx_n-\mu Vp\|^2+2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\|\\
&\qquad+\gamma_n^2\|x_n-p\|^2+2\gamma_n\|x_n-p\|\bigl(\alpha_n\|\tau Fx_n-\mu Vp\|+\|y_n-p\|\bigr),
\end{aligned}$$

and hence

$$\begin{aligned}
&\left[1-\gamma_n-\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|v_n-u_n\|^2\\
&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\left[\gamma_n+\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|x_n-p\|^2\\
&\qquad+2\xi\|u_n-v_n\|\|Du_n-Dp\|+\alpha_n^2\|\tau Fx_n-\mu Vp\|^2+2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\|\\
&\qquad+\gamma_n^2\|x_n-p\|^2+2\gamma_n\|x_n-p\|\bigl(\alpha_n\|\tau Fx_n-\mu Vp\|+\|y_n-p\|\bigr)\\
&\quad\le\|x_n-x_{n+1}\|\bigl(\|x_n-p\|-\|x_{n+1}-p\|\bigr)+\left[\gamma_n+\alpha_n\mu\left(\eta-\frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]^2\|x_n-p\|^2\\
&\qquad+2\xi\|u_n-v_n\|\|Du_n-Dp\|+\alpha_n^2\|\tau Fx_n-\mu Vp\|^2+2\alpha_n\|\tau Fx_n-\mu Vp\|\|y_n-p\|\\
&\qquad+\gamma_n^2\|x_n-p\|^2+2\gamma_n\|x_n-p\|\bigl(\alpha_n\|\tau Fx_n-\mu Vp\|+\|y_n-p\|\bigr).
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, and $\{x_n\}$, $\{y_n\}$ are bounded, we obtain from (3.17) and (3.23)

$$\lim_{n\to\infty}\|v_n - u_n\| = 0. \tag{3.25}$$

Combining (3.22) with (3.25), we get

$$\|v_n - x_n\| \le \|v_n - u_n\| + \|u_n - x_n\| \to 0 \quad \text{as } n\to\infty. \tag{3.26}$$

By (3.1) and the nonexpansion of S_n, we obtain

$$\begin{aligned}\|x_n - S_nx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - S_nx_n\| \\ &\le \|x_n - x_{n+1}\| + \big\|\alpha_n\tau Fx_n + \gamma_nx_n + \big((1-\gamma_n)I - \alpha_n\mu V\big)y_n - S_nx_n\big\| \\ &= \|x_n - x_{n+1}\| + \big\|\alpha_n(\tau Fx_n - \mu Vy_n) + y_n - S_nx_n + \gamma_n(x_n - y_n)\big\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\tau Fx_n - \mu Vy_n\| + \|S_nv_n - S_nx_n\| + \gamma_n\|x_n - y_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\tau Fx_n - \mu Vy_n\| + \|v_n - x_n\| + \gamma_n\|x_n - y_n\|.\end{aligned}$$

It follows from $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, (3.17) and (3.26) that

$$\lim_{n\to\infty}\|x_n - S_nx_n\| = 0. \tag{3.27}$$

In the meantime, observe that

$$\begin{aligned}\|x_n - S_nx_n\| &= \big\|\beta_nx_n + (1-\beta_n)G_nx_n - x_n\big\| \\ &= \big\|\beta_nx_n + (1-\beta_n)G_nx_n - \beta_nx_n - (1-\beta_n)x_n\big\| \\ &= (1-\beta_n)\|x_n - G_nx_n\|.\end{aligned}$$

From condition (iii), we have

$$\lim_{n\to\infty}\|x_n - G_nx_n\| = 0. \tag{3.28}$$

Step 2. We claim that $q \in \mathcal{F}$, for q any weak cluster point of $\{x_n\}$.

Indeed, by condition (v), we know that $\lim_{n\to\infty}\eta_i^{(n)} = \eta_i$ for every $1 \le i \le N$. It is easy to see that each $\eta_i > 0$ and $\sum_{i=1}^N \eta_i = 1$. Define $G := \sum_{i=1}^N \eta_iT_i$. Then it follows from Lemma 2.3 that $G : C \to C$ is a k-strict pseudo-contraction and $F(G) = F(G_n) = \bigcap_{i=1}^N F(T_i)$. Furthermore, $G_nx \to Gx$ as $n\to\infty$ for all $x \in C$. In addition, $S : C \to C$ is defined as $Sx := lx + (1-l)Gx$. Then S is nonexpansive and $F(S) = F(G)$ by Lemma 2.1. Observe that

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - S_nx_n\| + \|S_nx_n - Sx_n\| \\ &= \|x_n - S_nx_n\| + \big\|\beta_nx_n + (1-\beta_n)G_nx_n - lx_n - (1-l)Gx_n\big\| \\ &\le \|x_n - S_nx_n\| + |\beta_n - l|\|x_n - G_nx_n\| + (1-\beta_n)\|G_nx_n - Gx_n\| \\ &\le \|x_n - S_nx_n\| + |\beta_n - l|\|x_n - G_nx_n\| + (1-\beta_n)\sum_{i=1}^N\big|\eta_i^{(n)} - \eta_i\big|\|T_ix_n\|.\end{aligned}$$

From (3.27) and (3.28), we obtain

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.29}$$

Since $\{x_n\}$ is bounded, we may assume that q is any weak cluster point of $\{x_n\}$. Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges weakly to q. Now, since S is nonexpansive, by (3.29) and Lemma 2.6, we obtain that $q \in F(S)$. Thus, we have $q \in F(G) = F(G_n) = \bigcap_{i=1}^N F(T_i)$.

In addition, we rewrite $u_{n_k} = J_{\lambda_1}^{B_1}[x_{n_k} + \gamma A^*(U-I)Ax_{n_k}]$ as

$$\frac{x_{n_k} - u_{n_k} + \gamma A^*(U-I)Ax_{n_k}}{\lambda_1} \in B_1 u_{n_k}. \tag{3.30}$$

Letting $k \to \infty$ in (3.30) and using (3.19), (3.22) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we have $0 \in B_1 q$, i.e., $q \in \mathrm{SOLVIP}(B_1)$. Furthermore, since x_n and u_n have the same asymptotical behavior, Ax_{n_k} weakly converges to Aq. It follows from (3.19), the nonexpansion of U, and Lemma 2.6 that $(I-U)Aq = 0$. Thus, by Proposition 2.3, we have $0 \in f(Aq) + B_2(Aq)$, i.e., $Aq \in \mathrm{SOLVIP}(B_2)$. As a result, $q \in \Gamma$.

Moreover, it follows from (3.25) that v_{n_k} weakly converges to q. Define

$$\mathcal{H}v = \begin{cases} Dv + N_C v, & v \in C, \\ \emptyset, & v \in H_1 \setminus C. \end{cases}$$

Then $\mathcal{H}$ is maximal monotone by Proposition 2.5. Take $\forall(v,w) \in \mathrm{Graph}(\mathcal{H})$. It is easy to see that $w - Dv \in N_C v$. Since $v_n \in C$, we have

$$\langle v - v_n, w - Dv\rangle \geq 0. \tag{3.31}$$

Combining (2.10) with $v_n = P_C(u_n - \xi Du_n)$, we get

$$\langle u_n - \xi Du_n - v_n, v_n - v\rangle \geq 0, \tag{3.32}$$

and hence

$$\left\langle v - v_n, \frac{v_n - u_n}{\xi} + Du_n \right\rangle \geq 0. \tag{3.33}$$

Thus, from (3.31) and (3.33), we obtain

$$\begin{aligned} \langle v - v_{n_k}, w\rangle &\geq \langle v - v_{n_k}, Dv\rangle \\ &\geq \langle v - v_{n_k}, Dv\rangle - \left\langle v - v_{n_k}, Du_{n_k} + \frac{v_{n_k} - u_{n_k}}{\xi}\right\rangle \\ &= \langle v - v_{n_k}, Dv - Dv_{n_k}\rangle + \langle v - v_{n_k}, Dv_{n_k} - Du_{n_k}\rangle - \left\langle v - v_{n_k}, \frac{v_{n_k} - u_{n_k}}{\xi}\right\rangle \end{aligned}$$

$$\geq \delta\|Dv - Dv_{n_k}\|^2 + \langle v - v_{n_k}, Dv_{n_k} - Du_{n_k}\rangle - \left\langle v - v_{n_k}, \frac{v_{n_k} - u_{n_k}}{\xi}\right\rangle$$
$$\geq \langle v - v_{n_k}, Dv_{n_k} - Du_{n_k}\rangle - \left\langle v - v_{n_k}, \frac{v_{n_k} - u_{n_k}}{\xi}\right\rangle.$$

Letting $k \to \infty$, we have $\langle v - q, w\rangle \geq 0$ as $k \to \infty$. Since $\mathcal{H}$ is maximal monotone, we get $q \in \mathcal{H}^{-1}0$. So it follows from Proposition 2.5 that $q \in \mathrm{VI}(C, D)$. Therefore, $q \in \bigcap_{i=1}^{N} F(T_i) \cap \Gamma \cap \mathrm{VI}(C, D) = \mathcal{F}$.

Step 3. We claim that

$$\limsup_{n\to\infty}\langle(\mu V - \tau F)q, q - x_n\rangle \leq 0,$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction Ψ_t on C defined by

$$\Psi_t x := P_C\bigl[t\tau Fx + (I - t\mu V)Tx\bigr], \quad \forall x \in C,$$

here $t \in (0, 2\eta/K^2)$ and $Tx := SP_C(I - \xi D)J_\lambda^{B_1}[I + \gamma A^*(U - I)A]x$, $\forall x \in C$.

Indeed, first, for each $x, y \in C$, note that

$$\begin{aligned}
&\|Tx - Ty\| \\
&\quad = \bigl\|SP_C(I - \xi D)J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]x - SP_C(I - \xi D)J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]y\bigr\| \\
&\quad \leq \bigl\|P_C(I - \xi D)J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]x - P_C(I - \xi D)J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]y\bigr\| \\
&\quad \leq \bigl\|(I - \xi D)J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]x - (I - \xi D)J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]y\bigr\| \\
&\quad \leq \bigl\|J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]x - J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]y\bigr\| \\
&\quad \leq \|x - y\|,
\end{aligned}$$

which implies that T is nonexpansive. Further, we estimate

$$\begin{aligned}
&\|Tx_n - x_n\| \\
&\quad = \bigl\|SP_C(I - \xi D)J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]x_n - x_n\bigr\| \\
&\quad = \bigl\|SP_C(I - \xi D)u_n - x_n\bigr\| \\
&\quad = \|Sv_n - x_n\| \\
&\quad \leq \|Sv_n - S_n v_n\| + \|S_n v_n - x_n\| \\
&\quad = \bigl\|\beta_n v_n + (1 - \beta_n)G_n v_n - l v_n - (1 - l)Gv_n\bigr\| + \|S_n v_n - S_n x_n + S_n x_n - x_n\| \\
&\quad \leq |\beta_n - l|\|v_n - Gv_n\| + (1 - \beta_n)\|G_n v_n - Gv_n\| + \|S_n v_n - S_n x_n\| + \|S_n x_n - x_n\| \\
&\quad \leq |\beta_n - l|\|v_n - Gv_n\| + (1 - \beta_n)\sum_{i=1}^{N}\bigl|\eta_i^{(n)} - \eta_i\bigr|\|T_i v_n\| + \|v_n - x_n\| + \|S_n x_n - x_n\|.
\end{aligned}$$

From condition (iii), (3.26), and (3.27), we obtain

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{3.34}$$

Also, for each $x, y \in C$, it follows from Lemma 2.8 that Ψ_t has a unique fixed point $x_t \in C$ such that $x_t = P_C[t\tau Fx + (I - t\mu V)Tx_t]$, and the net $\{x_t\}_{t\in(0,1)}$ converges strongly as $t \to 0$ to a fixed point q of T which solves the variational inequality $\langle(\mu V - \tau F)q, q - p\rangle \le 0$, $\forall p \in F(T)$.

Next, from the above arguments, we know that $F(S) \cap \Gamma \cap \mathrm{VI}(C,D) = \bigcap_{i=1}^N F(T_i) \cap \Gamma \cap \mathrm{VI}(C,D) = \mathcal{F}$. Further, for $\forall q_1 \in F(T) = F(SP_C(I - \xi D)J_{\lambda_1}^{B_1}[I + \gamma A^*(U - I)A])$ and $\forall q_2 \in F(S) \cap \Gamma \cap \mathrm{VI}(C,D)$. Then we have $q_2 = J_{\lambda_1}^{B_1} q_2$, $Aq_2 = UAq_2$, $q_2 = J_{\lambda_1}^{B_1}[I + \gamma A^*(U - I)A]q_2$, and $q_2 = P_C(I - \xi D)q_2$. By the nonexpansion of S, $P_C(I - \xi D)$ and $J_{\lambda_1}^{B_1}$, we get

$$\begin{aligned}
&\|q_1 - q_2\|^2 \\
&\quad = \big\|SP_C(I - \xi D)J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_1 - SP_C(I - \xi D)J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_2\big\|^2 \\
&\quad \le \big\|P_C(I - \xi D)J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_1 - P_C(I - \xi D)J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_2\big\|^2 \\
&\quad \le \big\|J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_1 - J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_2\big\|^2 \\
&\quad \le \big\|\big[I + \gamma A^*(U - I)A\big]q_1 - \big[I + \gamma A^*(U - I)A\big]q_2\big\|^2 \\
&\quad = \big\|q_1 + \gamma A^*(U - I)Aq_1 - q_2\big\|^2 \\
&\quad \le \|q_1 - q_2\|^2 + \gamma(r\gamma - 1)\big\|(U - I)Aq_1\big\|^2.
\end{aligned}$$

Since $\gamma \in (0, \frac{1}{r})$, we infer that

$$(U - I)Aq_1 = 0, \tag{3.35}$$

it follows from Proposition 2.3 that $Aq_1 \in \mathrm{SOLVIP}(B_2)$. In addition, since $J_{\lambda_1}^{B_1}$ is firmly nonexpansive, from (3.35) we estimate

$$\begin{aligned}
&\|q_1 - q_2\|^2 \\
&\quad \le \big\|J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_1 - J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_2\big\|^2 \\
&\quad \le \big\langle J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_1 - J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_2, \\
&\qquad \big[I + \gamma A^*(U - I)A\big]q_1 - \big[I + \gamma A^*(U - I)A\big]q_2\big\rangle \\
&\quad = \big\langle J_{\lambda_1}^{B_1}\big[I + \gamma A^*(U - I)A\big]q_1 - q_2, \big[I + \gamma A^*(U - I)A\big]q_1 - \big[I + \gamma A^*(U - I)A\big]q_2\big\rangle \\
&\quad = \big\langle J_{\lambda_1}^{B_1}q_1 - q_2, q_1 - q_2\big\rangle \le \big\|J_{\lambda_1}^{B_1}q_1 - q_2\big\|\|q_1 - q_2\| \\
&\quad = \big\|J_{\lambda_1}^{B_1}q_1 - J_{\lambda_1}^{B_1}q_2\big\|\|q_1 - q_2\| \le \|q_1 - q_2\|^2,
\end{aligned}$$

which implies

$$\|q_1 - q_2\|^2 = \big\langle J_{\lambda_1}^{B_1}q_1 - q_2, q_1 - q_2\big\rangle, \tag{3.36}$$

hence

$$\big\langle J_{\lambda_1}^{B_1}q_1 - q_1, q_1 - q_2\big\rangle = 0. \tag{3.37}$$

Meanwhile, by (3.35) and (3.37), we have

$$\begin{aligned}\|q_1 - q_2\|^2 &\geq \bigl\|J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]q_1 - J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]q_2\bigr\|^2 \\ &= \bigl\|J_{\lambda_1}^{B_1}\bigl[I + \gamma A^*(U - I)A\bigr]q_1 - q_2\bigr\|^2 \\ &= \bigl\|J_{\lambda_1}^{B_1}q_1 - q_1 + q_1 - q_2\bigr\|^2 \\ &= \bigl\|J_{\lambda_1}^{B_1}q_1 - q_1\bigr\|^2 + \|q_1 - q_2\|^2 + 2\bigl\langle J_{\lambda_1}^{B_1}q_1 - q_1, q_1 - q_2\bigr\rangle \\ &= \bigl\|J_{\lambda_1}^{B_1}q_1 - q_1\bigr\|^2 + \|q_1 - q_2\|^2,\end{aligned}$$

and hence $J_{\lambda_1}^{B_1}q_1 = q_1$. Thus, $0 \in B_1q_1$, i.e., $q_1 \in \mathrm{SOLVIP}(B_1)$. As a result, we get $q_1 \in \Gamma$. By the assumption $q_1 = Tq_1 = SP_C(I - \xi D)J_{\lambda_1}^{B_1}[I + \gamma A^*(U - I)A]q_1$, we have $q_1 = SP_C(I - \xi D)q_1$. Moreover, from the above arguments, we get

$$\begin{aligned}\|q_1 - q_2\|^2 &= \bigl\|SP_C(I - \xi D)q_1 - SP_C(I - \xi D)q_2\bigr\|^2 \\ &\leq \bigl\|P_C(I - \xi D)q_1 - P_C(I - \xi D)q_2\bigr\|^2 \\ &\leq \bigl\|(I - \xi D)q_1 - (I - \xi D)q_2\bigr\|^2 \\ &= \bigl\|q_1 - q_2 - \xi(Dq_1 - Dq_2)\bigr\|^2 \\ &\leq \|q_1 - q_2\|^2 + \xi(\xi - 2\delta)\|Dq_1 - Dq_2\|^2 \\ &\leq \|q_1 - q_2\|^2,\end{aligned}$$

thus, we have

$$Dq_1 - Dq_2 = 0. \tag{3.38}$$

From (3.38), we obtain

$$\begin{aligned}\|q_1 - q_2\|^2 &= \bigl\|SP_C(I - \xi D)q_1 - SP_C(I - \xi D)q_2\bigr\|^2 \\ &\leq \bigl\|P_C(I - \xi D)q_1 - P_C(I - \xi D)q_2\bigr\|^2 \\ &\leq \bigl\langle P_C(I - \xi D)q_1 - P_C(I - \xi D)q_2, (I - \xi D)q_1 - (I - \xi D)q_2\bigr\rangle \\ &= \bigl\langle P_C(I - \xi D)q_1 - q_2, q_1 - q_2 - \xi(Dq_1 - Dq_2)\bigr\rangle \\ &= \bigl\langle P_C(I - \xi D)q_1 - q_2, q_1 - q_2\bigr\rangle \\ &\leq \bigl\|P_C(I - \xi D)q_1 - q_2\bigr\|\|q_1 - q_2\| \\ &= \bigl\|P_C(I - \xi D)q_1 - P_C(I - \xi D)q_2\bigr\|\|q_1 - q_2\| \\ &\leq \|q_1 - q_2\|^2,\end{aligned}$$

and hence

$$\|q_1 - q_2\|^2 = \bigl\langle P_C(I - \xi D)q_1 - q_2, q_1 - q_2\bigr\rangle, \tag{3.39}$$

that is,

$$\langle P_C(I-\xi D)q_1 - q_1, q_1 - q_2\rangle = 0. \tag{3.40}$$

Meanwhile, from (3.40), we get

$$\begin{aligned}
\|q_1-q_2\|^2 &\geq \|P_C(I-\xi D)q_1 - P_C(I-\xi D)q_2\|^2 \\
&= \|P_C(I-\xi D)q_1 - q_2\|^2 \\
&= \|P_C(I-\xi D)q_1 - q_1 + q_1 - q_2\|^2 \\
&= \|P_C(I-\xi D)q_1 - q_1\|^2 + \|q_1-q_2\|^2 + 2\langle P_C(I-\xi D)q_1 - q_1, q_1 - q_2\rangle \\
&= \|P_C(I-\xi D)q_1 - q_1\|^2 + \|q_1-q_2\|^2 \\
&\geq \|q_1-q_2\|^2,
\end{aligned}$$

which immediately implies $P_C(I-\xi D)q_1 = q_1$, and so $q_1 \in \mathrm{VI}(C,D)$. It follows from $q_1 = SP_C(I-\xi D)q_1$ that $q_1 = Sq_1$, i.e., $q_1 \in F(S)$. Thus, $q_1 \in F(S)\cap \mathrm{VI}(C,D)$. Since $q_1 \in \Gamma$, we obtain that $q_1 \in F(S)\cap\Gamma\cap \mathrm{VI}(C,D)$, which implies that $F(T)\subset F(S)\cap\Gamma\cap\mathrm{VI}(C,D)$. In addition, it is easy to see that $F(S)\cap\Gamma\cap\mathrm{VI}(C,D)\subset F(T)$. Therefore, $F(T) = F(S)\cap\Gamma\cap \mathrm{VI}(C,D) = \bigcap_{i=1}^N F(T_i)\cap\Gamma\cap\mathrm{VI}(C,D) = \mathcal{F}$.

Finally, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, assume that $x_{n_k} \rightharpoonup \omega$, where $\omega \in F(T) = \mathcal{F}$. By using Lemma 2.6 and (3.34), we have

$$\limsup_{n\to\infty}\langle(\mu V - \tau F)q, q - x_n\rangle = \limsup_{k\to\infty}\langle(\mu V - \tau F)q, q - x_{n_k}\rangle = \langle(\mu V - \tau F)q, q - \omega\rangle \leq 0.$$

Step 4. We claim $\lim_{n\to\infty}\|x_n - q\| = 0$.

Indeed, we put

$$z_n = \alpha_n\tau F x_n + \gamma_n x_n + \big[(1-\gamma_n)I - \alpha_n\mu V\big]y_n. \tag{3.41}$$

From (2.10), (3.1), (3.9), and (3.41), we obtain

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\quad= \langle P_C z_n - z_n, x_{n+1}-q\rangle + \langle z_n - q, x_{n+1}-q\rangle \\
&\quad= \langle P_C z_n - z_n, P_C z_n - q\rangle + \langle z_n - q, x_{n+1}-q\rangle \\
&\quad\leq \langle z_n - q, x_{n+1}-q\rangle \\
&\quad= \langle \alpha_n\tau F x_n + \gamma_n x_n + \big[(1-\gamma_n)I - \alpha_n\mu V\big]y_n - q, x_{n+1}-q\rangle \\
&\quad= \langle \big[(1-\gamma_n)I - \alpha_n\mu V\big]y_n - \big[(1-\gamma_n)I - \alpha_n\mu V\big]q \\
&\qquad + \alpha_n(\tau F x_n - \mu V q) + \gamma_n(x_n - q), x_{n+1}-q\rangle \\
&\quad= \langle \big[(1-\gamma_n)I - \alpha_n\mu V\big]y_n - \big[(1-\gamma_n)I - \alpha_n\mu V\big]q, x_{n+1}-q\rangle \\
&\qquad + \langle \alpha_n(\tau F x_n - \tau F q), x_{n+1}-q\rangle + \alpha_n\langle \tau F q - \mu V q, x_{n+1}-q\rangle + \gamma_n\langle x_n - q, x_{n+1}-q\rangle \\
&\quad\leq \big\|\big[(1-\gamma_n)I - \alpha_n\mu V\big]y_n - \big[(1-\gamma_n)I - \alpha_n\mu V\big]q\big\|\|x_{n+1}-q\|
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n \tau L\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\langle \tau Fq - \mu Vq, x_{n+1} - q\rangle + \gamma_n\|x_n - q\|\|x_{n+1} - q\| \\
&\le \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|y_n - q\|\|x_{n+1} - q\| + \alpha_n\tau L\|x_n - q\|\|x_{n+1} - q\| \\
&+ \alpha_n\langle \tau Fq - \mu Vq, x_{n+1} - q\rangle + \gamma_n\|x_n - q\|\|x_{n+1} - q\| \\
&\le \left[1 - \gamma_n - \alpha_n\mu\left(\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)}\right)\right]\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\tau L\|x_n - q\|\|x_{n+1} - q\| \\
&+ \alpha_n\langle \tau Fq - \mu Vq, x_{n+1} - q\rangle + \gamma_n\|x_n - q\|\|x_{n+1} - q\| \\
&= \left[1 - \alpha_n\left(\mu\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)} - \tau L\right)\right]\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\langle \tau Fq - \mu Vq, x_{n+1} - q\rangle \\
&\le \frac{1}{2}\left[1 - \alpha_n\left(\mu\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)} - \tau L\right)\right]\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) \\
&+ \alpha_n\langle \tau Fq - \mu Vq, x_{n+1} - q\rangle \\
&\le \frac{1}{2}\left[1 - \alpha_n\left(\mu\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)} - \tau L\right)\right]\|x_n - q\|^2 + \frac{1}{2}\|x_{n+1} - q\|^2 \\
&+ \alpha_n\langle \tau Fq - \mu Vq, x_{n+1} - q\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 \le{}& \left[1 - \alpha_n\left(\mu\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)} - \tau L\right)\right]\|x_n - q\|^2 \\
&+ 2\alpha_n\langle \tau Fq - \mu Vq, x_{n+1} - q\rangle.
\end{aligned}
\tag{3.42}
$$

Put $a_n = \alpha_n(\mu\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)} - \tau L)$ and $c_n = \frac{2\langle \tau Fq - \mu Vq, x_{n+1} - q\rangle}{\mu\eta - \frac{\alpha_n\mu K^2}{2(1-\gamma_n)} - \tau L}$. Applying Lemma 2.5 to (3.42), we obtain $\lim_{n\to\infty}\|x_n - q\| = 0$. This completes the proof. □

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces and C be a nonempty closed convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A, and r be the spectral radius of the operator A^*A. Let $f : H_2 \to H_2$ be a ρ-inverse strongly monotone operator and $B_1 : C \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be two multi-valued maximal monotone operators. Let $D : C \to H_1$ be a δ-inverse strongly monotone operator. Assume that $\{T_i\}_{i=1}^N : C \to C$ is a finite family of k_i-strict pseudo-contraction mappings such that $\mathcal{F} \neq \emptyset$. Let P_C be the metric projection of H_1 onto C, and $F : C \to H_1$ be an L-Lipschitzian mapping with constant $L \ge 0$. Suppose that $V : C \to H_1$ is an η-strongly monotone and K-Lipschitzian mapping, where η and μ satisfy the conditions of Lemma 3.1. For $x_1 \in C$, let $\{x_n\}$ be a sequence of C generated by* (1.7). *Assume that conditions* (i)–(v) *in Lemma 3.1 hold. Then $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the following variational inequality*:

$$\langle \mu Vq - \tau Fq, q - p\rangle \le 0, \quad \forall p \in \mathcal{F}.$$

Proof Combining the proof of Lemma 3.1 with the proof of Lemma 3.2, we can obtain the conclusion. □

Remark 3.1 Compared with Theorem 3.1 of Jitsupa et al. [1], our result is different from it in the following aspects:

(i) We not only change the parameter λ of resolvent operators $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ into different parameters λ_1 and λ_2, but also change the resolvent operator $J_\lambda^{B_2}$ into $J_{\lambda_2}^{B_2}(I-\lambda_2 f)$ which is more general than $J_\lambda^{B_2}$. It is worth stressing that the parameter λ of resolvent operators $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ in many results is the same λ; see, e.g., [1, 11–13]. Thus our result improves and extends these results and other related results.

(ii) We improve and extend Theorem 3.1 of Jitsupa et al. [1]. Especially, we use the Lipschitzian instead of the contraction, and also use the η-strongly monotone and K-Lipschitzian operator instead of the strong positive linear bounded operator to construct our iteration process.

(iii) It is worth mentioning here that our result in this paper is more applicable and efficient than the result of Jitsupa et al. [1]. We give the definite domains and ranges of B_1 and B_2 to make the iterative scheme (1.6) well-defined. We also modify the iterative scheme (1.6) by adding the projection operator. As a result, our result can be applied to finding a common solution of SMVIP (1.3) and VIP (2.7) and fixed point problem of a finite family of strict pseudo-contraction mappings instead of SVIP (1.2) and fixed point problem of a finite family of strict pseudo-contraction mappings.

In Theorem 3.1, if $\lambda_1=\lambda_2$, $f=D\equiv 0$, $\gamma_n=0$, F is a contraction mapping, and V is a strongly positive bounded linear operator, then we get the following corollary immediately.

Corollary 3.1 *Let H_1 and H_2 be two real Hilbert spaces and C be a nonempty closed convex subset of H_1. Let $A:H_1\to H_2$ be a bounded linear operator, A^* be the adjoint of A, and r be the spectral radius of the operator A^*A. Let $B_1:C\to 2^{H_1}$, $B_2:H_2\to 2^{H_2}$ be two multi-valued maximal monotone operators. Assume that $\{T_i\}_{i=1}^N:C\to C$ is a finite family of k_i-strict pseudo-contraction mappings such that $\widetilde{\mathcal{F}}:=\bigcap_{i=1}^N F(T_i)\cap\Gamma\neq\emptyset$. Let $f:C\to H_1$ be a contraction mapping with constant $\rho\in(0,1)$ and $D:C\to H_1$ be a strongly positive bounded linear operator with coefficient $\overline{\tau}>0$. For $x_1\in C$, let $\{x_n\}$ be a sequence generated by the following scheme*:

$$\begin{cases} u_n=J_\lambda^{B_1}[x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n],\\ y_n=\beta_n u_n+(1-\beta_n)\sum_{i=1}^N\eta_i^{(n)}T_iu_n,\\ x_{n+1}=\alpha_n\tau f(x_n)+(I-\alpha_n D)y_n,\quad n\geq 1.\end{cases}$$

Assume that conditions (ii), (iii) *in Lemma* 3.1 *and the following conditions hold*:

(i) $\lambda>0$, $0<\gamma<\frac{1}{r}$;

(ii) $\sum_{i=1}^N\eta_i^{(n)}=1$, $\sum_{n=1}^\infty(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|+\sum_{i=1}^N|\eta_i^{(n+1)}-\eta_i^{(n)}|)<\infty$.

Then the sequence $\{x_n\}$ converges strongly to $q\in\widetilde{\mathcal{F}}$, which solves the following variational inequality:

$$\langle Dq-\tau fq,q-p\rangle\leq 0,\quad\forall p\in\widetilde{\mathcal{F}}.$$

4 Numerical examples

The purpose of this section is to give an example and numerical results to support Theorem 3.1.

Example 4.1 Let $H_1 = H_2 = \mathbb{R}^3$ and $C = [0,+\infty) \times [0,+\infty) \times [0,+\infty)$. Let the inner product $\langle \cdot,\cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by $\langle x,y \rangle = x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$ and the usual norm $\| \cdot \| : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Let two operators of matrix multiplication $B_1 : C \to \mathbb{R}^3$, $B_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then we can define the resolvent operators $J_{\lambda_1}^{B_1}$ and $J_{\lambda_2}^{B_2}$ on $\mathbb{R}^3$ associated with B_1 and B_2 where $\lambda_1, \lambda_2 > 0$. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

be a singular matrix operator and A^* be the adjoint of A. It is easy to calculate that

$$A^* = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The mappings $T_i : C \to C$ defined by $T_1x = (\frac{x_1}{10(1+x_1)}, \frac{x_2}{10(1+x_2)}, \frac{x_3}{10(1+x_3)})$, $T_2x = (\frac{|\sin x_1|}{20(1+x_1)}, \frac{|\sin x_2|}{20(1+x_2)}, \frac{|\sin x_3|}{20(1+x_3)})$, and $T_3x = (\frac{x_1}{30+x_1}, \frac{x_2}{30+x_2}, \frac{x_3}{30+x_3})$ are k_i-strict pseudo-contractions for $i = 1,2,3$ (see [29]). Let $fx = \frac{1}{2}x$ ($\forall x \in \mathbb{R}^3$), $Dx = \frac{1}{3}x$ ($\forall x \in C$), $Vx = \frac{1}{2}x$ ($\forall x \in C$), and $Fx = \frac{3}{2}x$ ($\forall x \in C$). Now, we present the following algorithm.

Algorithm 4.2

Step 0. Choose the initial point $x_1 = (2,3,4) \in C$. Put $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$, $\gamma = \frac{1}{2}$, $\xi = \frac{1}{2}$, $\beta_n = \frac{1}{10}$, $\eta_1^n = \eta_2^n = \eta_3^n = \frac{1}{3}$, $\alpha_n = \frac{1}{8n}$, $\tau = \frac{1}{6}$, $\gamma_n = \frac{1}{10n}$, $\mu = \frac{2}{3}$ which satisfy the all assumed conditions of Theorem 3.1, and let $n = 1$.

Step 1. Given $x_n \in C$, compute $x_{n+1} \in C$ as follows:

$$\begin{cases} u_n = J_{\frac{1}{2}}^{B_1}[x_n + \frac{1}{2}A^*(J_{\frac{1}{3}}^{B_2}(I - \frac{1}{3}f) - I)Ax_n], \\ v_n = P_C(u_n - \frac{1}{2}Du_n), \\ y_n = \frac{1}{10}v_n + \frac{9}{10}\sum_{i=1}^3 \frac{1}{3}T_iv_n, \\ x_{n+1} = P_C[\frac{1}{8n}\frac{1}{6}Fx_n + \frac{1}{10n}x_n + ((1 - \frac{1}{10n})I - \frac{1}{8n}\frac{2}{3}V)y_n], \quad n \geq 1. \end{cases}$$

Step 2. Put $n := n + 1$ and go to Step 1.

Setting $\|x_{n+1} - x_n\| \leq 10^{-8}$ as a stop criterion, we get the numerical results of Algorithm 4.2.

Table 1 shows the values of the components of sequence x_n and $\|x_{n+1} - x_n\|$.

Figure 1 shows the convergence of the iterative sequence of Algorithm 4.2.

Solution: We can see from both Table 1 and Fig. 1 that the sequence $\{x_n\}$ converges to $(0,0,0)$, that is, $(0,0,0)$ is the solution in Example 4.1. In addition, it is also easy to

Table 1 Values of the components of x_n and $\|x_{n+1} - x_n\|$

n	x_n^1	x_n^2	x_n^3	$\|x_{n+1} - x_n\|$
1	2.0000	3.0000	4.0000	4.5762
2	3.3463×10^{-1}	4.4135×10^{-1}	5.9093×10^{-1}	6.8947×10^{-1}
3	5.4164×10^{-2}	6.4209×10^{-2}	8.6473×10^{-2}	1.0297×10^{-1}
4	8.4481×10^{-3}	9.1707×10^{-3}	1.2416×10^{-2}	1.5088×10^{-2}
5	1.2664×10^{-3}	1.2838×10^{-3}	1.7454×10^{-3}	2.1608×10^{-3}
6	1.8212×10^{-4}	1.7606×10^{-4}	2.4004×10^{-4}	3.0176×10^{-4}
7	2.5079×10^{-5}	2.3641×10^{-5}	3.2281×10^{-5}	4.1013×10^{-5}
8	3.3006×10^{-6}	3.1068×10^{-6}	4.2426×10^{-6}	5.4163×10^{-6}
9	4.1426×10^{-7}	3.9941×10^{-7}	5.4467×10^{-7}	6.9425×10^{-7}
10	4.9466×10^{-8}	5.0206×10^{-8}	6.8263×10^{-8}	8.6332×10^{-8}
11	5.6050×10^{-9}	6.1673×10^{-9}	8.3472×10^{-9}	1.0419×10^{-8}
12	6.0092×10^{-10}	7.3997×10^{-10}	9.9524×10^{-10}	1.2216×10^{-9}

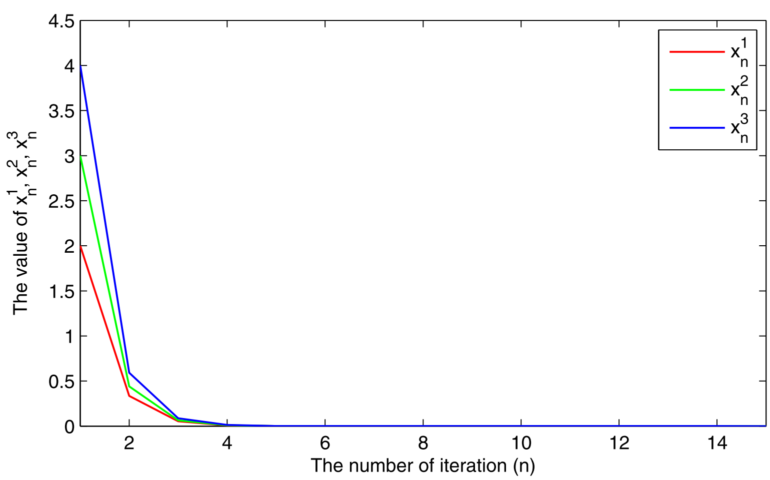

Figure 1 The convergence of x_n with initial $x_1 = (2, 3, 4)$

check from Example 4.1 that $\bigcap_{i=1}^{N} F(T_i) \cap \Gamma \cap \mathrm{VI}(C, D) = \{(0, 0, 0)\}$. Therefore, the iterative algorithm of Theorem 3.1 is well-defined and efficient.

5 Results and discussion

In this paper, we propose a new iterative scheme for finding a solution of SMVIP (1.3) with the constraints of a variational inequality and a fixed point problem of a finite family of strict pseudo-contractions in real Hilbert spaces. Moreover, we prove a strong convergence theorem for this iterative scheme.

In our main result, we not only give the definite domains and ranges of B_1 and B_2 to make sure our iterative scheme (1.7) well-defined, but also modify the iterative scheme (1.6) of Jitsupa et al. by adding the projection operator. Our result can be applied to finding a common solution of SMVIP (1.3), VIP (2.7), and fixed point problem of a finite family of strict pseudo-contraction mappings instead of SVIP (1.2) and fixed point problem of a finite family of strict pseudo-contraction mappings. Thus, our result improves and extends the result in [1].

6 Conclusions

In this paper, we first propose a modified iterative scheme (1.7) and then prove the strong convergence of the sequence $\{x_n\}$ generated by (1.7) to a common solution of SMVIP (1.3), VIP (2.7), and a fixed point problem under suitable conditions. Finally, we give a numerical example to support our strong convergence result. As a result, our result includes, improves, and enriches the corresponding ones announced by some others, see, e.g., [1, 12, 13].

7 Experimental

A numerical experiment is provided to support our iterative scheme in Algorithm 4.2, Table 1 and Fig. 1 above indicate the strong convergence of Algorithm 4.2. Therefore, our the iterative algorithm of Theorem 3.1 is well-defined and valid.

Funding
his research was supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of the Ministry of Education of China (20123127110002), and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100).

Competing interests
The authors declare that there is no conflict of interests regarding this manuscript.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 2 February 2018 Accepted: 6 November 2018 Published online: 15 November 2018

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