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On a new discrete Mulholland-type inequality in the whole plane

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Abstract

A new discrete Mulholland-type inequality in the whole plane with a best possible constant factor is presented by introducing multi-parameters, applying weight coefficients, and using Hermite–Hadamard's inequality. Moreover, the equivalent forms, some particular cases, and the operator expressions are considered.

MSC: 26D15; 47A07

Keywords: Mulholland-type inequality; Parameter; Weight coefficient; Equivalent form; Operator expression

1 Introduction

Assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, Hardy–Hilbert's inequality is provided as follows (cf. [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where $\frac{\pi}{\sin(\pi/p)}$ is the best possible constant factor. By Theorem 343 in [1] (replacing $\frac{a_m}{m}$ and $\frac{b_n}{n}$ by a_m and b_n, respectively), it yields the following Mulholland's inequality:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n}\right)^{\frac{1}{q}}. \tag{2}$$

Equations (1) and (2) are important inequalities in analysis and its applications (cf. [1, 2]).

In 2007, Yang [3] firstly provided the following Hilbert-type integral inequality in the whole plane:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(1+e^{x+y})^{\lambda}}\,dx\,dy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_{-\infty}^{\infty}e^{-\lambda x}f^2(x)\,dx\int_{-\infty}^{\infty}e^{-\lambda y}g^2(y)\,dy\right)^{\frac{1}{2}}, \tag{3}$$

where $B(\frac{\lambda}{2},\frac{\lambda}{2})$ ($\lambda > 0$) is the best possible constant factor. Various extensions of (1)–(3) have been presented since then (cf. [4–15]).

Recently, Yang and Chen [16] presented an extension of (1) in the whole plane as follows:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{a_m b_n}{(|m-\xi|+|n-\eta|)^{\lambda}} < 2B(\lambda_1,\lambda_2)\left[\sum_{|m|=1}^{\infty}|m-\xi|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n-\eta|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \quad (4)$$

where $2B(\lambda_1,\lambda_2)$ ($0<\lambda_1,\lambda_2\le 1$, $\lambda_1+\lambda_2=\lambda$, $\xi,\eta\in[0,\frac{1}{2}]$) is the best possible constant factor. In addition, Yang et al. [17, 18] also carried out a few similar works.

In this paper, we present a new discrete Mulholland-type inequality in the whole plane with a best possible constant factor that is similar to that in (4) via introducing multi-parameters, applying weight coefficients, and using Hermite–Hadamard's inequality. Moreover, the equivalent forms, some particular cases, and the operator expressions are considered.

2 An example and two lemmas

In what follows, we assume that $0<\lambda_1,\lambda_2<1$, $\lambda_1+\lambda_2=\lambda\le 1$, $\xi,\eta\in[0,\frac{1}{2}]$, $\alpha,\beta\in[\arccos\frac{1}{3},\frac{\pi}{2}]$, and

$$k_\gamma(\lambda_1):=\frac{2\pi^2\csc^2\gamma}{\lambda^2\sin^2(\frac{\pi\lambda_1}{\lambda})}\quad(\gamma=\alpha,\beta). \quad (5)$$

Remark 1 In view of the assumptions that $\xi,\eta\in[0,\frac{1}{2}]$, $\alpha,\beta\in[\arccos\frac{1}{3},\frac{\pi}{2}]$, it follows that

$$\left(\frac{3}{2}\pm\eta\right)(1\mp\cos\beta)\ge 1\quad\text{and}\quad\left(\frac{3}{2}\pm\xi\right)(1\mp\cos\alpha)\ge 1.$$

Example 1 For $u>0$, we set $g(u):=\frac{\ln u}{u-1}$ ($u>0$), $g(1):=\lim_{u\to 1}g(u)=1$. Then we have $g(u)>0$, $g'(u)<0$, $g''(u)>0$ ($u>0$). In fact, we find

$$g(u)=\frac{\ln[1+(u-1)]}{u-1}=\sum_{k=0}^{\infty}(-1)^k\frac{(u-1)^k}{k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k k!}{k+1}\frac{(u-1)^k}{k!}\quad(-1<u-1\le 1),$$

and then $g^{(k)}(1)=\frac{(-1)^k k!}{k+1}$ ($k=0,1,2,\ldots$). Hence, $g^{(0)}(1)=g(1)$, $g'(1)=-\frac{1}{2}$, $g''(1)=\frac{2}{3}$. It is evident that $g(u)>0$. We obtain $g'(u)=\frac{h(u)}{u(u-1)^2}$, $h(u):=u-1-u\ln u$. Since

$$h'(u)=-\ln u>0\quad(0<u<1);\qquad h'(u)<0\quad(u>1),$$

it follows that $h_{\max}=h(1)=0$ and $h(u)<0$ ($u\ne 1$). Then we have $g'(u)<0$ ($u\ne 1$). In view of $g'(1)=-\frac{1}{2}<0$, it follows that $g'(u)<0$ ($u>0$). We find

$$g''(u)=\frac{J(u)}{u^2(u-1)^3},\qquad J(u):=-(u-1)^2-2u(u-1)+2u^2\ln u,$$

$J'(u)=-4(u-1)+4u\ln u$, and

$$J''(u)=4\ln u<0\quad(0<u<1);\qquad J''(u)>0\quad(u>1).$$

It follows that $J'_{\min} = J'(1) = 0$, $J'(u) > 0$ $(u \neq 1)$ and $J(u)$ is strictly increasing. In view of $J(1) = 0$, we have

$$J(u) < 0 \quad (0 < u < 1); \qquad J(u) > 0 \quad (u > 1),$$

and $g''(u) > 0$ $(u \neq 1)$. Since $g''(1) = \frac{2}{3} > 0$, we find $g''(u) > 0$ $(u > 0)$.

For $0 < \lambda \leq 1$, $0 < \lambda_2 < 1$, setting $G(u) := g(u^\lambda)u^{\lambda_2 - 1}$ $(u > 0)$, we still have $G(u) > 0$, $G'(u) = \lambda g'(u^\lambda)u^{\lambda+\lambda_2-2} + (\lambda_2 - 1)g(u^\lambda)u^{\lambda_2-2} < 0$, and

$$\begin{aligned} G''(u) &= \lambda^2 g''\bigl(u^\lambda\bigr)u^{2\lambda+\lambda_2-3} + \lambda(\lambda + \lambda_2 - 2)g'\bigl(u^\lambda\bigr)u^{\lambda+\lambda_2-3} \\ &\quad + \lambda(\lambda_2 - 1)g'\bigl(u^\lambda\bigr)u^{\lambda+\lambda_2-3} + (\lambda_2 - 1)(\lambda_2 - 2)g\bigl(u^\lambda\bigr)u^{\lambda_2-3} > 0. \end{aligned}$$

We set $F(x,y) := \frac{\ln(x/y)}{x^\lambda - y^\lambda}(\frac{y}{x})^{\lambda_2-1}$ $(x, y > 0)$. Since $F(x,y) = \frac{1}{x^\lambda}G(\frac{y}{x})$, we have

$$F(x,y) > 0, \qquad \frac{\partial}{\partial y}F(x,y) < 0, \qquad \frac{\partial^2}{\partial y^2}F(x,y) > 0.$$

Hence, for $x, y > 1$, we still have

$$\frac{1}{y}F(\ln x, \ln y) > 0, \qquad \frac{\partial}{\partial y}\left(\frac{1}{y}F(\ln x, \ln y)\right) < 0, \qquad \frac{\partial^2}{\partial y^2}\left(\frac{1}{y}F(\ln x, \ln y)\right) > 0.$$

Lemma 1 (cf. [19]) *If $f(u) > 0$, $f'(u) < 0$, $f''(u) > 0$ $(u > \frac{3}{2})$ and $\int_{\frac{3}{2}}^\infty f(u)\,du < \infty$, then we have the following Hermite–Hadamard's inequality*:

$$\int_k^{k+1} f(u)\,du < f(k) < \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(u)\,du \quad \bigl(k \in \mathbf{N} \setminus \{1\}\bigr),$$

and then

$$\int_2^\infty f(u)\,du < \sum_{k=2}^\infty f(k) < \int_{\frac{3}{2}}^\infty f(u)\,du. \tag{6}$$

For $|x|, |y| \geq \frac{3}{2}$, let the functions

$$A_{\xi,\alpha}(x) := |x - \xi| + (x - \xi)\cos\alpha,$$

$A_{\eta,\beta}(y) = |y - \eta| + (y - \eta)\cos\beta$, and

$$k(x,y) := \frac{\ln(\ln A_{\xi,\alpha}(x)/\ln A_{\eta,\beta}(y))}{\ln^\lambda A_{\xi,\alpha}(x) - \ln^\lambda A_{\eta,\beta}(y)}. \tag{7}$$

We define two weight coefficients as follows:

$$\omega(\lambda_2, m) := \sum_{|n|=2}^\infty \frac{k(m,n)}{A_{\eta,\beta}(n)} \cdot \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2} A_{\eta,\beta}(n)}, \quad |m| \in \mathbf{N} \setminus \{1\}, \tag{8}$$

$$\varpi(\lambda_1,n) := \sum_{|m|=2}^{\infty} \frac{k(m,n)}{A_{\xi,\alpha}(m)} \cdot \frac{\ln^{\lambda_2} A_{\eta,\beta}(n)}{\ln^{1-\lambda_1} A_{\xi,\alpha}(m)}, \quad |n| \in \mathbf{N} \setminus \{1\}, \tag{9}$$

where $\sum_{|j|=2}^{\infty} \cdots = \sum_{j=-2}^{-\infty} \cdots + \sum_{j=2}^{\infty} \cdots$ $(j = m, n)$.

Lemma 2 *The inequalities*

$$k_\beta(\lambda_1)\bigl(1-\theta(\lambda_2,m)\bigr) < \omega(\lambda_2,m) < k_\beta(\lambda_1), \quad |m| \in \mathbf{N} \setminus \{1\} \tag{10}$$

are valid, where

$$\begin{aligned}\theta(\lambda_2,m) &:= \left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2 \int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}} \frac{\ln u}{u^\lambda - 1} u^{\lambda_2-1}\,du \\ &= O\left(\frac{1}{\ln^{\lambda_2/2} A_{\xi,\alpha}(m)}\right) \in (0,1).\end{aligned} \tag{11}$$

Proof For $|m| \in \mathbf{N} \setminus \{1\}$, let

$$k^{(1)}(m,y) := \frac{\ln\ln A_{\xi,\alpha}(m) - \ln\ln[(y-\eta)(\cos\beta-1)]}{\ln^\lambda A_{\xi,\alpha}(m) - \ln^\lambda[(y-\eta)(\cos\beta-1)]}, \quad y < -\frac{3}{2},$$

$$k^{(2)}(m,y) := \frac{\ln\ln A_{\xi,\alpha}(m) - \ln\ln[(y-\eta)(\cos\beta+1)]}{\ln^\lambda A_{\xi,\alpha}(m) - \ln^\lambda[(y-\eta)(\cos\beta+1)]}, \quad y > \frac{3}{2}.$$

Then we have

$$k^{(1)}(m,-y) = \frac{\ln\ln A_{\xi,\alpha}(m) - \ln\ln[(y+\eta)(1-\cos\beta)]}{\ln^\lambda A_{\xi,\alpha}(m) - \ln^\lambda[(y+\eta)(1-\cos\beta)]}, \quad y > \frac{3}{2},$$

yields

$$\begin{aligned}\omega(\lambda_2,m) &= \sum_{n=-2}^{-\infty} \frac{k^{(1)}(m,n)\ln^{\lambda_1} A_{\xi,\alpha}(m)}{(n-\eta)(\cos\beta-1)\ln^{1-\lambda_2}[(n-\eta)(\cos\beta-1)]} \\ &\quad + \sum_{n=2}^{\infty} \frac{k^{(2)}(m,n)\ln^{\lambda_1} A_{\xi,\alpha}(m)}{(n-\eta)(1+\cos\beta)\ln^{1-\lambda_2}[(n-\eta)(1+\cos\beta)]} \\ &= \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1-\cos\beta} \sum_{n=2}^{\infty} \frac{k^{(1)}(m,-n)}{(n+\eta)\ln^{1-\lambda_2}[(n+\eta)(1-\cos\beta)]} \\ &\quad + \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1+\cos\beta} \sum_{n=2}^{\infty} \frac{k^{(2)}(m,n)}{(n-\eta)\ln^{1-\lambda_2}[(n-\eta)(1+\cos\beta)]}.\end{aligned} \tag{12}$$

In virtue of $0 < \lambda \leq 1$, $0 < \lambda_2 < 1$, and Example 1, we find that for $y > \frac{3}{2}$,

$$\frac{k^{(i)}(m,(-1)^i y)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]} > 0,$$

$$\frac{d}{dy}\frac{k^{(i)}(m,(-1)^i y)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]} < 0,$$

$$\frac{d^2}{dy^2}\frac{k^{(i)}(m,(-1)^i y)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]} > 0 \quad (i=1,2),$$

it follows that

$$\frac{k^{(i)}(m,(-1)^i y)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]} \quad (i=1,2)$$

are strictly decreasing and convex in $(\frac{3}{2},\infty)$. Then, by (5), (12) yields

$$\begin{aligned}\omega(\lambda_2,m) &< \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1-\cos\beta}\int_{\frac{3}{2}}^{\infty}\frac{k^{(1)}(m,-y)}{(y+\eta)\ln^{1-\lambda_2}[(y+\eta)(1-\cos\beta)]}\,dy \\ &\quad + \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1+\cos\beta}\int_{\frac{3}{2}}^{\infty}\frac{k^{(2)}(m,y)}{(y-\eta)\ln^{1-\lambda_2}[(y-\eta)(1+\cos\beta)]}\,dy.\end{aligned}$$

Setting $u=\frac{\ln[(y+\eta)(1-\cos\beta)]}{\ln A_{\xi,\alpha}(m)}$ ($u=\frac{\ln[(y-\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}$) in the above first (second) integral, in view of Remark 1, we obtain

$$\begin{aligned}\omega(\lambda_2,m) &< \left(\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}\right)\int_0^{\infty}\frac{\ln u}{u^{\lambda}-1}u^{\lambda_2-1}\,du \\ &= \frac{2\csc^2\beta}{\lambda^2}\int_0^{\infty}\frac{\ln v}{v-1}v^{(\lambda_2/\lambda)-1}\,dv = \frac{2\pi^2\csc^2\beta}{\lambda^2\sin^2(\frac{\pi\lambda_1}{\lambda})} = k_\beta(\lambda_1)\end{aligned}$$

by simplifications. Similarly, by (5), (12) also yields

$$\begin{aligned}\omega(\lambda_2,m) &> \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1-\cos\beta}\int_{2}^{\infty}\frac{k^{(1)}(m,-y)}{(y+\eta)\ln^{1-\lambda_2}[(y+\eta)(1-\cos\beta)]}\,dy \\ &\quad + \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1+\cos\beta}\int_{2}^{\infty}\frac{k^{(2)}(m,y)}{(y-\eta)\ln^{1-\lambda_2}[(y-\eta)(1+\cos\beta)]}\,dy \\ &\geq \left(\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}\right)\int_{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}}^{\infty}\frac{\ln u}{u^{\lambda}-1}u^{\lambda_2-1}\,du \\ &= k_\beta(\lambda_1) - 2\csc^2\beta\int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}}\frac{\ln u}{u^{\lambda}-1}u^{\lambda_2-1}\,du \\ &= k_\beta(\lambda_1)\bigl(1-\theta(\lambda_2,m)\bigr) > 0,\end{aligned}$$

where $\theta(\lambda_2,m)$ (<1) is indicated by (11). Since

$$\frac{\ln u}{u^{\lambda}-1}u^{\lambda_2/2}\to 0 \quad (u\to 0^+); \qquad \frac{\ln u}{u^{\lambda}-1}u^{\lambda_2/2}\to\frac{1}{\lambda} \quad (u\to 1),$$

there exists a positive constant C such that $\frac{\ln u}{u^{\lambda}-1}u^{\lambda_2/2}\leq C$ $(0<u\leq 1)$, and then for $A_{\xi,\alpha}(m)\geq(2+\eta)(1+\cos\beta)$, we have

$$\begin{aligned}0 < \theta(\lambda_2,m) &\leq C\left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2\int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}}u^{\frac{\lambda_2}{2}-1}\,du \\ &= \frac{2C}{\lambda_2}\left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2\left\{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}\right\}^{\frac{\lambda_2}{2}}.\end{aligned} \tag{13}$$

Hence, (10) and (11) are valid. □

Similarly, we have the following.

Lemma 3 *For* $0 < \lambda \leq 1$, $0 < \lambda_1 < 1$, *the inequalities*

$$k_\alpha(\lambda_1)\big(1 - \tilde{\theta}(\lambda_1, n)\big) < \varpi(\lambda_1, n) < k_\alpha(\lambda_1), \quad |n| \in \mathbf{N} \setminus \{1\} \tag{14}$$

are valid, where

$$\begin{aligned}\tilde{\theta}(\lambda_1, n) &:= \left[\frac{\lambda}{\pi}\sin\left(\frac{\pi\lambda_1}{\lambda}\right)\right]^2 \int_0^{\frac{\ln[(2+\xi)(1+\cos\alpha)]}{\ln A_{\eta,\beta}(n)}} \frac{\ln u}{u^\lambda - 1} u^{\lambda_1 - 1}\,du \\ &= O\left(\frac{1}{\ln^{\lambda_1/2} A_{\eta,\beta}(n)}\right) \in (0,1).\end{aligned} \tag{15}$$

Lemma 4 *If* $(\varsigma, \gamma) = (\xi, \alpha)$ *(or* (η, β)*),* $\rho > 0$*, then we have*

$$H_\rho(\varsigma, \gamma) := \sum_{|k|=2}^{\infty} \frac{\ln^{-1-\rho} A_{\varsigma,\gamma}(k)}{A_{\varsigma,\gamma}(k)} = \frac{1}{\rho}\big(2\csc^2\gamma + o(1)\big) \quad \big(\rho \to 0^+\big). \tag{16}$$

Proof According to (5), we obtain

$$\begin{aligned}H_\rho(\varsigma, \gamma) &= \sum_{k=-2}^{-\infty} \frac{\ln^{-1-\rho}[(k-\varsigma)(\cos\gamma - 1)]}{(k-\varsigma)(\cos\gamma - 1)} + \sum_{k=2}^{\infty} \frac{\ln^{-1-\rho}[(k-\varsigma)(\cos\gamma + 1)]}{(k-\varsigma)(\cos\gamma + 1)} \\ &= \sum_{k=2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(k+\varsigma)(1-\cos\gamma)]}{(k-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(k-\varsigma)(\cos\gamma+1)]}{(k-\varsigma)(\cos\gamma+1)}\right\} \\ &< \int_{\frac{3}{2}}^{\infty}\left\{\frac{\ln^{-1-\rho}[(y+\varsigma)(1-\cos\gamma)]}{(y-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(y-\varsigma)(\cos\gamma+1)]}{(y-\varsigma)(\cos\gamma+1)}\right\}dy \\ &= \frac{1}{\rho}\left\{\frac{\ln^{-\rho}[(\frac{3}{2}+\varsigma)(1-\cos\gamma)]}{1-\cos\gamma} + \frac{\ln^{-\rho}[(\frac{3}{2}-\varsigma)(1+\cos\gamma)]}{1+\cos\gamma}\right\} \\ &= \frac{1}{\rho}\left(\frac{1}{1-\cos\gamma} + \frac{1}{1+\cos\gamma} + o_1(1)\right) = \frac{1}{\rho}\big(2\csc^2\gamma + o_1(1)\big) \quad \big(\rho \to 0^+\big),\end{aligned}$$

and

$$\begin{aligned}H_\rho(\varsigma, \gamma) &= \sum_{k=2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(k+\varsigma)(1-\cos\gamma)]}{(k-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(k-\varsigma)(\cos\gamma+1)]}{(k-\varsigma)(\cos\gamma+1)}\right\} \\ &> \int_{2}^{\infty}\left\{\frac{\ln^{-1-\rho}[(y+\varsigma)(1-\cos\gamma)]}{(y-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(y-\varsigma)(\cos\gamma+1)]}{(y-\varsigma)(\cos\gamma+1)}\right\}dy \\ &= \frac{1}{\rho}\left\{\frac{\ln^{-\rho}[(2+\varsigma)(1-\cos\gamma)]}{1-\cos\gamma} + \frac{\ln^{-\rho}[(2-\varsigma)(1+\cos\gamma)]}{1+\cos\gamma}\right\} \\ &= \frac{1}{\rho}\left(\frac{1}{1-\cos\gamma} + \frac{1}{1+\cos\gamma} + o_2(1)\right) = \frac{1}{\rho}\big(2\csc^2\gamma + o_2(1)\big) \quad \big(\rho \to 0^+\big).\end{aligned}$$

Therefore, (16) is valid. □

3 Main results

Theorem 1 *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *we set*

$$k(\lambda_1) := k_\beta^{1/p}(\lambda_1) k_\alpha^{1/q}(\lambda_1) = \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2}. \tag{17}$$

If $a_m, b_n \geq 0$ $(|m|, |n| \in \mathbf{N} \setminus \{1\})$ *satisfy*

$$0 < \sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p < \infty, \qquad 0 < \sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q < \infty,$$

then we obtain the following equivalent inequalities:

$$\begin{aligned} I &:= \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{\ln(\ln A_{\xi,\alpha}(m)/\ln A_{\eta,\beta}(n))}{\ln^\lambda A_{\xi,\alpha}(m) - \ln^\lambda A_{\eta,\beta}(n)} a_m b_n \\ &< \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{18}$$

$$\begin{aligned} J &:= \left\{\sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \left[\sum_{|m|=2}^{\infty} \frac{\ln(\ln A_{\xi,\alpha}(m)/\ln A_{\eta,\beta}(n))}{\ln^\lambda A_{\xi,\alpha}(m) - \ln^\lambda A_{\eta,\beta}(n)} a_m\right]^p\right\}^{\frac{1}{p}} \\ &< \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}. \end{aligned} \tag{19}$$

Particularly, (i) *for* $\alpha = \beta = \frac{\pi}{2}$, $\xi, \eta \in [0, \frac{1}{2}]$, *we have the following equivalent inequalities*:

$$\begin{aligned} &\sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{\ln(|m-\xi|/|n-\eta|) a_m b_n}{\ln^\lambda |m-\xi| - \ln^\lambda |n-\eta|} \\ &\quad < \frac{2\pi^2}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} |m-\xi|}{|m-\xi|^{1-p}} a_m^p\right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} |n-\eta|}{|n-\eta|^{1-q}} b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{20}$$

$$\begin{aligned} &\left[\sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1} |n-\eta|}{|n-\eta|} \left(\sum_{|m|=2}^{\infty} \frac{\ln(|m-\xi|/|n-\eta|) a_m}{\ln^\lambda |m-\xi| - \ln^\lambda |n-\eta|}\right)^p\right]^{\frac{1}{p}} \\ &\quad < \frac{2\pi^2}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} |m-\xi|}{|m-\xi|^{1-p}} a_m^p\right]^{\frac{1}{p}}. \end{aligned} \tag{21}$$

(ii) *For* $\xi = \eta = 0$, $\alpha, \beta \in [\arccos \frac{1}{3}, \frac{\pi}{2}]$, *we have the following equivalent inequalities*:

$$\sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{\ln[\ln(|m| + m\cos\alpha)/\ln(|n| + n\cos\beta)]}{\ln^{\lambda}(|m| + m\cos\alpha) - \ln^{\lambda}(|n| + n\cos\beta)} a_m b_n < \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \times \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(|m| + m\cos\alpha)}{(|m| + m\cos\alpha)^{1-p}} a_m^p\right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}(|n| + n\cos\beta)}{(|n| + n\cos\beta)^{1-q}} b_n^q\right]^{\frac{1}{q}}, \quad (22)$$

$$\left\{\sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1}(|n| + n\cos\beta)}{|n| + n\cos\beta} \left[\sum_{|m|=2}^{\infty} \frac{\ln[\ln(|m| + m\cos\alpha)/\ln(|n| + n\cos\beta)]}{\ln^{\lambda}(|m| + m\cos\alpha) - \ln^{\lambda}(|n| + n\cos\beta)} a_m\right]^p\right\}^{\frac{1}{p}} < \frac{2\pi^2 \csc^{2/p}\beta \csc^{2/q}\alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(|m| + m\cos\alpha)}{(|m| + m\cos\alpha)^{1-p}} a_m^p\right]^{\frac{1}{p}}. \quad (23)$$

Proof According to Hölder's inequality with weight (cf. [20]) and (9), we find

$$\begin{aligned}
\left(\sum_{|m|=2}^{\infty} k(m,n) a_m\right)^p &= \left\{\sum_{|m|=2}^{\infty} k(m,n) \left[\frac{(A_{\xi,\alpha}(m))^{\frac{1}{q}} \ln^{\frac{1-\lambda_1}{q}} A_{\xi,\alpha}(m)}{\ln^{\frac{1-\lambda_2}{p}} A_{\eta,\beta}(n)} a_m\right] \left[\frac{\ln^{\frac{1-\lambda_2}{p}} A_{\eta,\beta}(n)}{(A_{\xi,\alpha}(m))^{\frac{1}{q}} \ln^{\frac{1-\lambda_1}{q}} A_{\xi,\alpha}(m)}\right]\right\}^p \\
&\leq \sum_{|m|=2}^{\infty} k(m,n) \frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}} \ln^{\frac{(1-\lambda_1)p}{q}} A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2} A_{\eta,\beta}(n)} \times a_m^p \left[\sum_{|m|=2}^{\infty} k(m,n) \frac{\ln^{\frac{(1-\lambda_2)q}{p}} A_{\eta,\beta}(n)}{A_{\xi,\alpha}(m) \ln^{1-\lambda_1} A_{\xi,\alpha}(m)}\right]^{p-1} \\
&= \frac{(\varpi(\lambda_1, n))^{p-1} A_{\eta,\beta}(n)}{\ln^{p\lambda_2-1} A_{\eta,\beta}(n)} \sum_{|m|=2}^{\infty} k(m,n) \frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}} \ln^{\frac{(1-\lambda_1)p}{q}} A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n) \ln^{1-\lambda_2} A_{\eta,\beta}(n)} a_m^p.
\end{aligned}$$

Then, by (14), it yields

$$\begin{aligned}
J &< k_\alpha^{1/q}(\lambda_1) \left[\sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} k(m,n) \frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}} \ln^{\frac{(1-\lambda_1)p}{q}} A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n) \ln^{1-\lambda_2} A_{\eta,\beta}(n)} a_m^p\right]^{\frac{1}{p}} \\
&= k_\alpha^{1/q}(\lambda_1) \left[\sum_{|m|=2}^{\infty} \sum_{|n|=2}^{\infty} k(m,n) \frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}} \ln^{\frac{(1-\lambda_1)p}{q}} A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n) \ln^{1-\lambda_2} A_{\eta,\beta}(n)} a_m^p\right]^{\frac{1}{p}} \\
&= k_\alpha^{1/q}(\lambda_1) \left[\sum_{|m|=2}^{\infty} \omega(\lambda_2, m) \frac{n^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}. \quad (24)
\end{aligned}$$

Combining (10) and (17), we obtain (19).

Using Hölder's inequality again, we obtain

$$I = \sum_{|n|=2}^{\infty}\left[\frac{(A_{\eta,\beta}(n))^{\frac{-1}{p}}}{\ln^{\frac{1}{p}-\lambda_2} A_{\eta,\beta}(n)} \sum_{|m|=2}^{\infty} k(m,n)a_m\right]\left[\frac{\ln^{\frac{1}{p}-\lambda_2} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{\frac{-1}{p}}} b_n\right]$$
$$\leq J\left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}}. \tag{25}$$

Then, according to (19), we obtain (18).

On the other hand, assuming that (18) is valid, we let

$$b_n := \frac{\ln^{p\lambda_2-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\left(\sum_{|m|=2}^{\infty} k(m,n)a_m\right)^{p-1}, \quad |n| \in \mathbf{N}\setminus\{1\}.$$

According to (24), it follows that $J < \infty$. If $J = 0$, then (20) is trivially valid; if $J > 0$, then we have

$$0 < \sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q$$
$$= J^p = I$$
$$< k(\lambda_1)\left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}},$$
$$J = \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{p}}$$
$$< k(\lambda_1)\left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}.$$

Thus (19) is valid, which is equivalent to (18). □

Theorem 2 *With regards to the assumptions in Theorem 1, $k(\lambda_1)$ is the best possible constant factor in (18) and (19).*

Proof For $0 < \varepsilon < \min\{q(1-\lambda_1), q\lambda_2\}$, we let $\tilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ $(\in (0,1))$, $\tilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ $(\in (0,1))$, and

$$\tilde{a}_m := \frac{\ln^{\lambda_1-\frac{\varepsilon}{p}-1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} = \frac{\ln^{\tilde{\lambda}_1-\varepsilon-1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \quad \big(|m| \in \mathbf{N}\setminus\{1\}\big),$$
$$\tilde{b}_n := \frac{\ln^{\lambda_2-\frac{\varepsilon}{q}-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} = \frac{\ln^{\tilde{\lambda}_2-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \quad \big(|n| \in \mathbf{N}\setminus\{1\}\big).$$

Then (16) and (14) yield

$$
\begin{aligned}
\tilde{I}_1 &:= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} \tilde{a}_m^p\right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} \tilde{b}_n^q\right]^{\frac{1}{q}} \\
&= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{-1-\varepsilon} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}\right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{-1-\varepsilon} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\right]^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left(2\csc^2\alpha + o(1)\right)^{\frac{1}{p}}\left(2\csc^2\beta + \tilde{o}(1)\right)^{\frac{1}{q}} \quad \left(\varepsilon \to 0^+\right), \\
\tilde{I} &:= \sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty} k(m,n)\tilde{a}_m\tilde{b}_n = \sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty} k(m,n)\frac{\ln^{\tilde{\lambda}_1-\varepsilon-1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}\frac{\ln^{\tilde{\lambda}_2-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \\
&= \sum_{|m|=2}^{\infty} \omega(\tilde{\lambda}_2, m)\frac{\ln^{-\varepsilon-1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} > k_\beta(\tilde{\lambda}_1)\sum_{|m|=2}^{\infty}\left(1-\theta(\tilde{\lambda}_2, m)\right)\frac{\ln^{-\varepsilon-1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \\
&= k_\beta(\tilde{\lambda}_1)\left[\sum_{|m|=2}^{\infty}\frac{\ln^{-\varepsilon-1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} - \sum_{|m|=2}^{\infty}\frac{O(\ln^{-(\frac{\varepsilon}{p}+\frac{\lambda_2}{2})-1} A_{\xi,\alpha}(m))}{A_{\xi,\alpha}(m)}\right] \\
&= \frac{1}{\varepsilon}k_\beta(\tilde{\lambda}_1) \quad \left(2\csc^2\alpha + o(1) - \varepsilon O(1)\right).
\end{aligned}
$$

If there exists a positive number $K \leq k(\lambda_1)$ such that (18) is still valid when replacing $k(\lambda_1)$ by K, then we obtain

$$
\varepsilon\tilde{I} = \varepsilon\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty} k(m,n)\tilde{a}_m\tilde{b}_n < \varepsilon K\tilde{I}_1.
$$

Hence, in view of the above results, it follows that

$$
k_\beta\left(\lambda_1 + \frac{\varepsilon}{q}\right)\left(2\csc^2\alpha + o(1) - \varepsilon O(1)\right) < K\left(2\csc^2\alpha + o(1)\right)^{\frac{1}{p}}\left(2\csc^2\beta + \tilde{o}(1)\right)^{\frac{1}{q}},
$$

and then

$$
\frac{4\pi^2}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}\csc^2\beta\csc^2\alpha \leq 2K\csc^{\frac{2}{p}}\alpha\csc^{\frac{2}{q}}\beta \quad \left(\varepsilon \to 0^+\right),
$$

namely

$$
k(\lambda_1) = \frac{2\pi^2}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha \leq K.
$$

Hence, $K = k(\lambda_1)$ is the best possible constant factor in (18).

$k(\lambda_1)$ in (19) is still the best possible constant factor. Otherwise we would reach a contradiction by (25) that $k(\lambda_1)$ in (18) is not the best possible constant factor. □

4 Operator expressions and a remark

Let $\varphi(m) := \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}$ ($|m| \in \mathbf{N} \setminus \{1\}$), and $\psi(n) := \frac{\ln^{q(1-\lambda_2)-1} A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}$, wherefrom

$$\psi^{1-p}(n) := \frac{\ln^{p\lambda_2-1} A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \quad \bigl(|n| \in \mathbf{N} \setminus \{1\}\bigr).$$

We define the real weighted normed function spaces as follows:

$$l_{p,\varphi} := \left\{ a = \{a_m\}_{|m|=2}^{\infty}; \|a\|_{p,\varphi} = \left(\sum_{|m|=2}^{\infty} \varphi(m)|a_m|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_n\}_{|n|=2}^{\infty}; \|b\|_{q,\psi} = \left(\sum_{|n|=2}^{\infty} \psi(n)|b_n|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ c = \{c_n\}_{|n|=2}^{\infty}; \|c\|_{p,\psi^{1-p}} = \left(\sum_{|n|=2}^{\infty} \psi^{1-p}(n)|c_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

For $a = \{a_m\}_{|m|=2}^{\infty} \in l_{p,\varphi}$, we let $c_n = \sum_{|m|=2}^{\infty} k(m,n)a_m$ and $c = \{c_n\}_{|n|=2}^{\infty}$, it follows by (19) that $\|c\|_{p,\psi^{1-p}} < k(\lambda_1)\|a\|_{p,\varphi}$, namely $c \in l_{p,\psi^{1-p}}$.

Further, we define a Mulholland-type operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For $a_m \geq 0$, $a = \{a_m\}_{|m|=2}^{\infty} \in l_{p,\varphi}$, there exists a unique representation $Ta = c \in l_{p,\psi^{1-p}}$. We also define the following formal inner product of Ta and $b = \{b_n\}_{|n|=2}^{\infty} \in l_{q,\psi}$ ($b_n \geq 0$):

$$(Ta, b) := \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} k(m,n)a_m b_n. \tag{26}$$

Hence, we can respectively rewrite (18) and (19) as the following operator expressions:

$$(Ta, b) < k(\lambda_1)\|a\|_{p,\varphi}\|b\|_{q,\psi}, \tag{27}$$

$$\|Ta\|_{p,\psi^{1-p}} < k(\lambda_1)\|a\|_{p,\varphi}. \tag{28}$$

It follows that the operator T is bounded with

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\varphi}} \frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\varphi}} \leq k(\lambda_1). \tag{29}$$

Since $k(\lambda_1)$ in (19) is the best possible constant factor, we obtain

$$\|T\| = k(\lambda_1) = \frac{2\pi^2 \csc^{2/p} \beta \csc^{2/q} \alpha}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2}. \tag{30}$$

Remark 2 (i) For $\xi = \eta = 0$ in (20), we have the following new inequality:

$$\begin{aligned} &\sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{\ln(\ln|m|/\ln|n|)a_m b_n}{\ln^{\lambda}|m| - \ln^{\lambda}|n|} \\ &\quad < \frac{2\pi^2}{[\lambda \sin(\frac{\pi\lambda_1}{\lambda})]^2} \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}|m|}{|m|^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}|n|}{|n|^{1-q}} b_n^q \right]^{\frac{1}{q}}. \end{aligned} \tag{31}$$

It follows that (20) is an extension of (31). In particular, for $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, we have the following simple Mulholland-type inequality in the whole plane with the best possible constant factor $\frac{2\pi^2}{\sin^2(\frac{\pi}{p})}$:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{\ln(\ln|m|/\ln|n|)}{\ln(|m|/|n|)}a_m b_n < \frac{2\pi^2}{\sin^2(\frac{\pi}{p})}\left(\sum_{|m|=2}^{\infty}\frac{a_m^p}{|m|^{1-p}}\right)^{\frac{1}{p}}\left(\sum_{|n|=2}^{\infty}\frac{b_n^q}{|n|^{1-q}}\right)^{\frac{1}{q}}. \tag{32}$$

(ii) If $a_{-m} = a_m$, $b_{-n} = b_n$ $(m, n \in \mathbf{N} \setminus \{1\})$, then (20) reduces to

$$\begin{aligned}&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\left\{\frac{\ln[\ln(m-\xi)/\ln(n-\eta)]}{\ln^{\lambda}(m-\xi)-\ln^{\lambda}(n-\eta)}+\frac{\ln[\ln(m-\xi)/\ln(n+\eta)]}{\ln^{\lambda}(m-\xi)-\ln^{\lambda}(n+\eta)}\right.\\&\qquad\left.+\frac{\ln[\ln(m+\xi)/\ln(n-\eta)]}{\ln^{\lambda}(m+\xi)-\ln^{\lambda}(n-\eta)}+\frac{\ln[\ln(m+\xi)/\ln(n+\eta)]}{\ln^{\lambda}(m+\xi)-\ln^{\lambda}(n+\eta)}\right\}a_m b_n\\&\quad<\frac{2\pi^2}{[\lambda\sin(\frac{\pi\lambda_1}{\lambda})]^2}\left\{\sum_{m=2}^{\infty}\left[\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}+\frac{\ln^{p(1-\lambda_1)-1}(m+\xi)}{(m+\xi)^{1-p}}\right]a_m^p\right\}^{\frac{1}{p}}\\&\qquad\times\left\{\sum_{n=2}^{\infty}\left[\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}+\frac{\ln^{q(1-\lambda_2)-1}(n+\eta)}{(n+\eta)^{1-q}}\right]b_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{33}$$

In particular, for $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, $\xi = \eta \in [0, \frac{1}{2}]$, we obtain

$$\begin{aligned}&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\left\{\frac{\ln[\ln(m-\xi)/\ln(n-\xi)]}{\ln[(m-\xi)/(n-\xi)]}+\frac{\ln[\ln(m-\xi)/\ln(n+\xi)]}{\ln[(m-\xi)/(n+\xi)]}\right.\\&\qquad\left.+\frac{\ln[\ln(m+\xi)/\ln(n-\xi)]}{\ln[(m+\xi)/(n-\xi)]}+\frac{\ln[\ln(m+\xi)/\ln(n+\xi)]}{\ln[(m+\xi)/(n+\xi)]}\right\}a_m b_n\\&\quad<\frac{2\pi^2}{\sin^2(\frac{\pi}{p})}\left\{\sum_{m=2}^{\infty}\left[\frac{1}{(m-\xi)^{1-p}}+\frac{1}{(m+\xi)^{1-p}}\right]a_m^p\right\}^{\frac{1}{p}}\\&\quad\times\left\{\sum_{n=2}^{\infty}\left[\frac{1}{(n-\xi)^{1-q}}+\frac{1}{(n+\xi)^{1-q}}\right]b_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{34}$$

For $\xi = 0$, (34) reduces to the following simple Mulholland-type inequality with the best possible constant factor $\frac{\pi^2}{\sin^2(\frac{\pi}{p})}$:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\ln(\ln m/\ln n)}{\ln(m/n)}a_m b_n < \frac{\pi^2}{\sin^2(\frac{\pi}{p})}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m^{1-p}}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n^{1-q}}\right)^{\frac{1}{q}}. \tag{35}$$

5 Conclusions

In this paper, we present a new discrete Mulholland-type inequality in the whole plane with a best possible constant factor that is similar to that in (4) via introducing multi-parameters, applying weight coefficients, and using Hermite–Hadamard's inequality in Theorem 1 and Theorem 2. Moreover, the equivalent forms, some particular cases, and the operator expressions are considered. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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