Dai et al. *Journal of Inequalities and Applications* (2018) 2018:176
https://doi.org/10.1186/s13660-018-1768-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Inequalities for the fractional convolution operator on differential forms

Zhimin Dai[1*], Huacan Li[2] and Qunfang Li[3]

*Correspondence:
daizhimin@xatu.edu.cn
[1]School of Science, Xi'an Technological University, Xi'an, China
Full list of author information is available at the end of the article

Abstract
The purpose of this paper is to derive some Coifman type inequalities for the fractional convolution operator applied to differential forms. The Lipschitz norm and BMO norm estimates for this integral type operator acting on differential forms are also obtained.

Keywords: Fractional convolution operator; Coifman type inequality; BMO norm; Lipschitz norm; Differential form

1 Introduction

As a generalization of functions, the differential form can be regarded as a special kind of vector-valued function. So, if some operators in function spaces are generalized to that in differential forms, similar properties could be obtained as in the function space. In recent years, the research on the generalization of operators from functional spaces to differential forms seems to become a new highlight in the inequalities with differential forms, see [1–6]. In this paper, we mainly consider the following convolution type fractional integrals operator acting on differential forms and develop some norm inequalities for the fractional convolution operator. Given a nonnegative, locally integrable function K_α and $\hbar_I(y)$ is a bounded function with a compactly supported set on $\mathbb{R}^n$, write $\hbar_I(y) \in L_c^\infty$. The fractional convolution operator F_α is defined by a convolution integral

$$F_\alpha \hbar(x) = \sum_I \left(\int_{\mathbb{R}^n} K_\alpha(x-y)\hbar_I(y)\,\mathrm{d}y \right) \mathrm{d}x_I, \tag{1.1}$$

provided this integral exists for almost all $\mathbb{R}^n$, where $\hbar(x) = \sum_I \hbar_I(y)\,\mathrm{d}x_I$ is a ℓ-form defined on $\mathbb{R}^n$, the summation is over all ordered ℓ-tuples I. The function K_α is also assumed to be a wide class of kernels satisfying the following growth condition (see [7]):

(1) $K_\alpha \in S_\alpha$ if there exists a constant $C > 0$ such that

$$\int_{|x|\sim s} \left|K_\alpha(x)\right| \mathrm{d}x \le Cs^\alpha; \tag{1.2}$$

(2) K_α is said to satisfy the $L^{\alpha,\varphi}$-Hörmander condition, and write $K_\alpha \in H_{\alpha,\varphi}$. If there exist $c \geq 1$ and $C > 0$ (only dependent on φ) such that, for all $y \in \mathbb{R}^n$ and $R > c|y|$,

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} \|K_\alpha(\cdot - y) - K_\alpha(\cdot)\|_{\varphi(|x|\sim 2^m R)} \leq C, \tag{1.3}$$

where φ is a Young function defined on $[0, +\infty)$, $|x| \sim s$ stands for the set $\{s < |x| \leq 2s\}$, $O(0, s)$ is a ball with the center at the origin and the radius equal to s, and the φ-mean Luxemburg norm of a function f on a cube (or a ball)O in $\mathbb{R}^n$ is given by

$$\|f\|_{\varphi(O)} = \inf\left\{\lambda > 0 : \frac{1}{|O|}\int_O \varphi\left(\frac{|f|}{\lambda}\right) \mathrm{d}x \leq 1\right\}. \tag{1.4}$$

Differential forms can be viewed as an extension of functions. When $\hbar(x)$ is a 0-form, the above-mentioned notations are in accord with those of function spaces, and the fractional convolution operator F_α we study in this paper degenerates into the operator which Bernardis discussed in [7]. Namely, for any Lebesgue measurable function $f \in L_c^\infty$, F_α is given as follows:

$$F_\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x-y) f(y)\,\mathrm{d}y. \tag{1.5}$$

This degenerated fractional convolution operator was also introduced by Riveros in [8], who presented weighted Coifman type estimates, two weight estimates of strong and weak type for general fractional operators and gave applications to fractional operators produced by a homogeneous function and a Fourier multiplier.

Now we introduce some notations and definitions. Let Θ be an open subset of $\mathbb{R}^n$ ($n \geq 2$) and O be a ball in $\mathbb{R}^n$. Let ρO denote the ball with the same center as O and $\operatorname{diam}(\rho O) = \rho \operatorname{diam}(O)(\rho > 0)$. $|\Theta|$ is used to denote the Lebesgue measure of a set $\Theta \subset \mathbb{R}^n$. Let $\bigwedge^\ell = \bigwedge^\ell(\mathbb{R}^n), \ell = 0, 1, \ldots, n$, be the linear space of all ℓ-forms $\hbar(x) = \sum_I \hbar_I(x)\,\mathrm{d}x_I = \sum_I \hbar_{i_1 i_2 \cdots i_\ell}(x)\,\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \cdots \wedge \mathrm{d}x_{i_\ell}$ in $\mathbb{R}^n$, where $I = (i_1, i_2, \ldots, i_\ell), 1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, are the ordered ℓ-tuples. Moreover, if each of the coefficients $\hbar_I(x)$ of $\hbar(x)$ is differential on Θ, then we call $\hbar(x)$ a differential ℓ-form on Θ and use $D'(\Theta, \bigwedge^\ell)$ to denote the space of all differential ℓ-forms on Θ. $C^\infty(\Theta, \bigwedge^\ell)$ denotes the space of smooth ℓ-forms on Θ. The exterior derivative $d : D'(\Theta, \bigwedge^\ell) \to D'(\Theta, \bigwedge^{\ell+1})$, $\ell = 0, 1, \ldots, n-1$, is given by

$$d\hbar(x) = \sum_I \sum_{j=1}^n \frac{\partial \hbar_{i_1 i_2 \cdots i_\ell}(x)}{\partial x_j}\,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_\ell} \tag{1.6}$$

for all $\hbar \in D'(\Theta, \bigwedge^\ell)$. $L^p(\Theta, \bigwedge^\ell)(1 \leq p < \infty)$ is a Banach space with the norm $\|\hbar\|_{p,\Theta} = (\int_\Theta |\hbar(x)|^p\,\mathrm{d}x)^{1/p} = (\int_\Theta (\sum_I |\hbar_I(x)|^2)^{p/2}\,\mathrm{d}x)^{1/p} < \infty$. Similarly, the notations $L^p_{\mathrm{loc}}(\Theta, \bigwedge^\ell)$ and $W^{1,p}_{\mathrm{loc}}(\Theta, \bigwedge^\ell)$ are self-explanatory.

From [9], $\hbar$ is a differential form in a bounded convex domain Θ, then there is a decomposition

$$\hbar = d(T\hbar) + T(d\hbar), \tag{1.7}$$

where T is called a homotopy operator. For the homotopy operator T, we know that

$$\|T\hbar\|_{p,O} \le C|O| \operatorname{diam}(O) \|\hbar\|_{p,O} \tag{1.8}$$

holds for any differential form $\hbar \in L^p_{\mathrm{loc}}(\Theta, \bigwedge^\ell)$, $\ell = 1, 2, \dots, n$, $1 < p < \infty$. Furthermore, we can define the ℓ-form $\hbar_\Theta \in D'(\Theta, \bigwedge^\ell)$ by

$$\hbar_\Theta = \begin{cases} |\Theta|^{-1} \int_\Theta \hbar(x)\,\mathrm{d}x, & \ell = 0, \\ dT(\hbar), & \ell = 1, 2, \dots, n, \end{cases} \tag{1.9}$$

for all $\hbar \in L^p(\Theta, \bigwedge^\ell)$, $1 \le p < \infty$.

A non-negative function $w \in L^1_{\mathrm{loc}}(\mathrm{d}x)$ is called a weight. We recall the definitions of the Muckenhoupt weights and the reverse Hölder condition (see [10]). For $1 < p < \infty$, we say that $w \in \mathcal{A}_p$ if there exists a constant $C > 0$ such that, for every ball $O \subset \mathbb{R}^n$,

$$\left(\frac{1}{|O|}\int_O w\,\mathrm{d}x\right)\left(\frac{1}{|O|}\int_O w^{-\frac{1}{p-1}}\,\mathrm{d}x\right)^{p-1} \le C. \tag{1.10}$$

For the case $p = 1$, $w \in \mathcal{A}_1$ if there exists a constant $C > 0$ such that, for every ball $O \subset \mathbb{R}^n$,

$$\frac{1}{|O|}\int_O w\,\mathrm{d}x \le C \operatorname*{ess\,inf}_{x\in O} w(x). \tag{1.11}$$

Also $\mathcal{A}_\infty = \bigcup_{p\ge 1} \mathcal{A}_p$. It is well known that $\mathcal{A}_p \subset \mathcal{A}_q$ for all $1 \le p \le q \le \infty$, and also that for $1 < p \le \infty$, if $w \in \mathcal{A}_p$, then there exists $\varepsilon > 0$ such that $w \in \mathcal{A}_{p-\varepsilon}$.

A function $\varphi : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex, increasing and satisfies $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Each Young function φ has an associated complementary Young function $\bar{\varphi}$ satisfying

$$t \le \varphi^{-1}(t)\bar{\varphi}^{-1}(t) \le 2t \tag{1.12}$$

for all $t > 0$, where $\varphi^{-1}(t)$ is the inverse function of $\varphi(t)$ (see [11]).

For each locally integrable function f and $0 \le \alpha < n$, the fractional maximal operator associated with the Young function φ is defined by

$$M_{\alpha,\varphi} f(x) = \sup_{x\in O} |O|^{\frac{\alpha}{n}} \|f\|_{\varphi(O)}. \tag{1.13}$$

For $\alpha = 0$, we write M_φ instead of $M_{0,\varphi}$. When $\varphi(t) = t$, then $M_{\alpha,\varphi} = M_\alpha$ is the classical fractional maximal operator. For $\alpha = 0$ and $\varphi(t) = t$, we obtain $M_{0,\varphi} = M$ is the Hardy–Littlewood maximal operator (see [8]).

2 The Coifman type inequalities for the fractional convolution operator

In [7], the inequality for the fractional convolution operator in function with the fractional maximal operator, that is, the Coifman type inequality, is proved.

Lemma 2.1 *Let φ be a Young function on $[0,+\infty)$ and f be any n-tuple function on $\mathbb{R}^n$ with $f \in L^\infty_c$. Suppose that the fractional convolution operator $F_\alpha = K_\alpha * f$ and its kernel satisfies*

$K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$, where $0 < \alpha < n$. Then there exists a constant C such that

$$\int_{\mathbb{R}^n} \left|F_\alpha f(x)\right|^p w(x)\,\mathrm{d}x \le C \int_{\mathbb{R}^n} \left[M_{\alpha,\bar{\varphi}} f(x)\right]^p w(x)\,\mathrm{d}x \tag{2.1}$$

for any $0 < p < \infty$ and $w \in A_\infty$.

Theorem 2.1 *Let φ be a Young function on $[0,+\infty)$ and f, g be two functions defined on $\mathbb{R}^n$ with $|f(x)| \le |g(x)|$ for all $x \in \mathbb{R}^n$. Then for all cubes O and the Young functions φ,*

$$\|f\|_{\varphi(O)} \le \|g\|_{\varphi(O)}. \tag{2.2}$$

Proof Since φ is a Young function, it follows that

$$\begin{aligned} &\frac{1}{|O|}\int_O \varphi\left(\frac{|f|}{\|g\|_{\varphi(O)}}\right)\mathrm{d}x \le \frac{1}{|O|}\int_O \varphi\left(\frac{|g|}{\|g\|_{\varphi(O)}}\right)\mathrm{d}x \le 1 \\ &\quad\Rightarrow\quad \|g\|_{\varphi(O)} \in E = \left\{\lambda > 0 : \frac{1}{|O|}\int_O \varphi\left(\frac{|f|}{\lambda}\right)\mathrm{d}x \le 1\right\} \\ &\quad\Rightarrow\quad \|f\|_{\varphi(O)} = \inf E \le \|g\|_{\varphi(O)}. \end{aligned} \tag{2.3}$$

□

According to Theorem 2.1, we can get a similar conclusion to Lemma 2.1.

Theorem 2.2 *Let φ be a Young function on $[0,+\infty)$, $\hbar = \sum_I \hbar_I \,\mathrm{d}x_I$ be a differential form on $\Theta \subset \mathbb{R}^n$, and let all the ordered ℓ-tuples I satisfy $\hbar_I \in L_c^\infty$. Suppose that F_α is a fractional convolution operator applied to differential forms and its kernel function K_α satisfies $K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$, where $0 < \alpha < n$. Then there exists a constant C such that*

$$\int_{\mathbb{R}^n} \left|F_\alpha \hbar(x)\right|^p w(x)\,\mathrm{d}x \le C \int_{\mathbb{R}^n} \left[M_{\alpha,\bar{\varphi}} \hbar(x)\right]^p w(x)\,\mathrm{d}x \tag{2.4}$$

for any $0 < p < \infty$ and $w \in A_\infty$.

Proof By Lemma 2.1 and the following basic inequality

$$\sum_{i=1}^n |a_i|^s \le n\left(\sum_{i=1}^n |a_i|\right)^s \le n^{s+1}\sum_{i=1}^n |a_i|^s, \tag{2.5}$$

where $s > 0$ is any constant, it follows that

$$\begin{aligned} \|F_\alpha \hbar\|_{p,w,\mathbb{R}^n}^p &= \int_{\mathbb{R}^n} \left|F_\alpha \hbar(x)\right|^p w(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \left(\sum_I \left(\int_{\mathbb{R}^n} K_\alpha(x-y)\hbar_I(y)\,\mathrm{d}y\right)^2\right)^{p/2} w(x)\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^n} C_1 \sum_I \left(\int_{\mathbb{R}^n} K_\alpha(x-y)\hbar_I(y)\,\mathrm{d}y\right)^p w(x)\,\mathrm{d}x \\ &= C_1 \sum_I \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K_\alpha(x-y)\hbar_I(y)\,\mathrm{d}y\right)^p w(x)\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\le C_2 \sum_I \int_{\mathbb{R}^n} \bigl[M_{\alpha,\bar{\varphi}} \hbar_I(x)\bigr]^p w(x)\,\mathrm{d}x \\
&= C_2 \int_{\mathbb{R}^n} \sum_I \bigl[M_{\alpha,\bar{\varphi}} \hbar_I(x)\bigr]^p w(x)\,\mathrm{d}x \\
&\le C_3 \int_{\mathbb{R}^n} \Bigl(\sum_I M_{\alpha,\bar{\varphi}} \hbar_I(x)\Bigr)^p w(x)\,\mathrm{d}x. \qquad (2.6)
\end{aligned}
$$

Then, by the definition of the fractional maximal operator, notice that for any I such that $|\hbar_I| \le |\hbar|$, we obtain that

$$
\begin{aligned}
&\sum_I M_{\alpha,\bar{\varphi}} \hbar_I(x) \\
&\quad = \sum_I \sup_{x\in O} |O|^{\alpha/n} \| \hbar_I \|_{\bar{\varphi}(O)} \\
&\quad \le C_n^\ell \sup_{x\in O} |O|^{\alpha/n} \sum_I \| \hbar_I \|_{\bar{\varphi}(O)} \\
&\quad \le C_n^\ell \sup_{x\in O} |O|^{\alpha/n} C_n^\ell \| \hbar \|_{\bar{\varphi}(O)} \\
&\quad \le C_4 M_{\alpha,\bar{\varphi}} \hbar(x). \qquad (2.7)
\end{aligned}
$$

Combining (2.6) and (2.7), we have

$$
\begin{aligned}
&\|F_\alpha \hbar\|^p_{p,w,\mathbb{R}^n} \\
&\quad \le C_3 \int_{\mathbb{R}^n} \Bigl(\sum_I M_{\alpha,\bar{\varphi}} \hbar_I(x)\Bigr)^p w(x)\,\mathrm{d}x \\
&\quad \le C_3 \int_{\mathbb{R}^n} \bigl(C_4 M_{\alpha,\bar{\varphi}} \hbar(x)\bigr)^p w(x)\,\mathrm{d}x \\
&\quad \le C_5 \int_{\mathbb{R}^n} \bigl(M_{\alpha,\bar{\varphi}} \hbar(x)\bigr)^p w(x)\,\mathrm{d}x. \qquad (2.8)
\end{aligned}
$$

□

Theorem 2.3 *Let φ be a Young function on $[0,+\infty)$, $\hbar = \sum_I \hbar_I \,\mathrm{d}x_I$ be a differential form on $\Theta \subset \mathbb{R}^n$, and for all the ordered ℓ-tuples, let I satisfy $\hbar_I \in L_c^\infty$. Suppose that F_α is a fractional convolution operator on differential forms and its kernel function K_α satisfies $K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$, where $0 < \alpha < n$. Then there exists a constant C such that*

$$
\int_O \bigl|F_\alpha \hbar(x)\bigr|^p \,\mathrm{d}x \le C \int_O \bigl[M_{\alpha,\bar{\varphi}} \hbar(x)\bigr]^p \,\mathrm{d}x \qquad (2.9)
$$

for any $0 < p < \infty$ and all the balls $O \subset \mathbb{R}^n$.

Proof By the definition of the $\mathcal{A}_\infty$-weight, there exist $r_0 \ge 1$ and a constant $C < \infty$ such that, for all the balls $O \subset \mathbb{R}^n$, it follows that

$$
\biggl(\frac{1}{|O|}\int_O w(x)\,\mathrm{d}x\biggr)\biggl(\frac{1}{|O|}\int_O w(x)^{-\frac{1}{r_0-1}}\,\mathrm{d}x\biggr)^{r_0-1} \le C. \qquad (2.10)
$$

With the arbitrariness of the condition $w \in \mathcal{A}_\infty$ of Theorem 2.2, now get any ball $O_0 \subset \mathbb{R}^n$ and let

$$w(x) = \chi_{O_0}(x) = \begin{cases} 1, & x \in O_0; \\ 0, & x \notin O_0. \end{cases}$$

It is easy to check that $w(x) = \chi_{O_0}(x)$ satisfies (2.10). In fact, we have

$$\begin{aligned} &\left(\frac{1}{|O|}\int_O \chi_{O_0}(x)\,\mathrm{d}x\right)\left(\frac{1}{|O|}\int_O \chi_{O_0}(x)^{-\frac{1}{r_0-1}}\,\mathrm{d}x\right)^{r_0-1} \\ &\quad= \left(\frac{1}{|O|}|O\cap O_0|\right)\left(\frac{1}{|O|}|O\cap O_0|\right)^{r_0-1} \\ &\quad= \left(\frac{1}{|O|}|O\cap O_0|\right)^{r_0} \le 1. \end{aligned} \tag{2.11}$$

Thus

$$\begin{aligned} &\int_{O_0} \left|F_\alpha \hbar(x)\right|^p \mathrm{d}x \\ &\quad= \int_{\mathbb{R}^n} \left|F_\alpha \hbar(x)\right|^p \chi_{O_0}(x)\,\mathrm{d}x \\ &\quad\le C\int_{\mathbb{R}^n} \left[M_{\alpha,\bar{\varphi}}\hbar(x)\right]^p \chi_{O_0}(x)\,\mathrm{d}x \\ &\quad\le C\int_{O_0} \left[M_{\alpha,\bar{\varphi}}\hbar(x)\right]^p \mathrm{d}x. \end{aligned} \tag{2.12}$$

□

If the kernel function K_α and the coefficient functions $\hbar_I$ of differential forms are subject to some conditions, the following more important conclusion will be obtained.

Theorem 2.4 *Let φ be a Young function on $[0,+\infty)$, $\hbar = \sum_I \hbar_I\,\mathrm{d}x_I$ be a differential form on $\Theta \subset \mathbb{R}^n$, and let all the ordered ℓ-tuples I satisfy $\hbar_I \in L_c^\infty$. Suppose that F_α is a fractional convolution operator on differential forms and its kernel function K_α satisfies $K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$ and $K_\alpha \in C_0^\infty(\Theta)$, where $C_0^\infty(\Theta)$ stands for all the C^∞ functions with compactly supported sets in Θ and $0 < \alpha < n$. Then there exists a constant C such that*

$$\left\|F_\alpha\hbar - (F_\alpha\hbar)_O\right\|_{p,O} \le C\,\mathrm{diam}(O)|O|\left\|M_{\alpha,\bar{\varphi}}(\mathrm{d}\hbar)\right\|_{p,O} \tag{2.13}$$

for any $1 < p < \infty$ and all the balls with $O \subset \mathbb{R}^n$.

Proof By the exterior derivative operator d and the fractional convolution operator F_α, we obtain that

$$\begin{aligned} d\hbar &= \sum_I \sum_{k=1}^n \frac{\partial \hbar_I(x)}{\partial x_k}\,\mathrm{d}x_k \wedge \mathrm{d}x_I, \\ F_\alpha(d\hbar) &= \sum_I \sum_{k=1}^n \left(\int_{\mathbb{R}^n} K_\alpha(x-y)\frac{\partial \hbar_I(y)}{\partial y_k}\,\mathrm{d}y\right)\mathrm{d}x_k \wedge \mathrm{d}x_I \end{aligned} \tag{2.14}$$

and

$$dF_\alpha(\hbar) = \sum_I \sum_{k=1}^{n} \frac{\partial h_I(x)}{\partial x_k}\,\mathrm{d}x_k \wedge \mathrm{d}x_I, \tag{2.15}$$

where

$$h_I(x) = \int_{\mathbb{R}^n} K_\alpha(x-y)\hbar_I(y)\,\mathrm{d}y. \tag{2.16}$$

According to (1.7)–(1.9), it follows that

$$\bigl\|F_\alpha\hbar - (F_\alpha\hbar)_O\bigr\|_{p,O} = \bigl\|T\bigl(d(F_\alpha\hbar)\bigr)\bigr\|_{p,O} \le C_1\operatorname{diam}(O)|O|\bigl\|d(F_\alpha\hbar)\bigr\|_{p,O}. \tag{2.17}$$

Now we will give the L^p-norm estimation of $d(F_\alpha\hbar)$. With $K_\alpha \in C_0^\infty(\Theta)$ and considering the definition of the general partial derivative (see [12]), we obtain

$$\begin{aligned}
&\bigl\|d(F_\alpha\hbar)\bigr\|_{p,O}^p \\
&\quad= \int_O \Biggl(\sum_I \sum_{k=1}^{n} \biggl|\frac{\partial h_I(x)}{\partial x_k}\biggr|^2\Biggr)^{p/2}\mathrm{d}x \\
&\quad= \int_O \Biggl(\sum_I \sum_{k=1}^{n} \biggl|\frac{\partial \int_{\mathbb{R}^n} K_\alpha(x-y)\hbar_I(y)\,\mathrm{d}y}{\partial x_k}\biggr|^2\Biggr)^{p/2}\mathrm{d}x \\
&\quad= \int_O \Biggl(\sum_I \sum_{k=1}^{n} \biggl|\int_{\mathbb{R}^n} \frac{\partial K_\alpha(x-y)}{\partial x_k}\hbar_I(y)\,\mathrm{d}y\biggr|^2\Biggr)^{p/2}\mathrm{d}x \\
&\quad= \int_O \Biggl(\sum_I \sum_{k=1}^{n} \biggl|-\int_{\mathbb{R}^n} \frac{\partial K_\alpha(x-y)}{\partial y_k}\hbar_I(y)\,\mathrm{d}y\biggr|^2\Biggr)^{p/2}\mathrm{d}x \\
&\quad= \int_O \Biggl(\sum_I \sum_{k=1}^{n} \biggl|\int_{\mathbb{R}^n} \frac{\partial \hbar_I(y)}{\partial y_k}K_\alpha(x-y)\,\mathrm{d}y\biggr|^2\Biggr)^{p/2}\mathrm{d}x \\
&\quad= \bigl\|F_\alpha(d\hbar)\bigr\|_{p,O}^p,
\end{aligned} \tag{2.18}$$

that is

$$\bigl\|d(F_\alpha\hbar)\bigr\|_{p,O} = \bigl\|F_\alpha(d\hbar)\bigr\|_{p,O}. \tag{2.19}$$

Combining (2.17) and (2.19), we obtain that

$$\begin{aligned}
&\bigl\|F_\alpha\hbar - (F_\alpha\hbar)_O\bigr\|_{p,O} \\
&\quad\le C_1\operatorname{diam}(O)|O|\bigl\|d(F_\alpha\hbar)\bigr\|_{p,O} \\
&\quad= C_1\operatorname{diam}(O)|O|\bigl\|F_\alpha(d\hbar)\bigr\|_{p,O} \\
&\quad\le C_1\operatorname{diam}(O)|O|\bigl\|M_{\alpha,\bar{\varphi}}(d\hbar)\bigr\|_{p,O}.
\end{aligned} \tag{2.20}$$

□

Since a new function is obtained when the differential form is taken as a model, we can get a global inequality in the $L^p(m)$ domain with Theorem 2.4. Now recall the definition of the $L^p(m)$ domain introduced by Staples (see [13]).

Definition 2.1 Let Θ be a real subdomain in $\mathbb{R}^n$. If, for all the functions $f \in L^p_{\mathrm{loc}}(\Theta)$, there exists a constant C such that

$$|\Theta|^{-1/p}\|f - f_{O_0}\|_{p,\Theta} \le C \sup_{O\subset\Theta} |O|^{-1/p}\|f - f_O\|_{p,O}, \tag{2.21}$$

then Θ is called an $L^p(m)$-average domain, where O_0 is a fixed ball of Θ and $p \ge 1$.

Theorem 2.5 *Let φ be a Young function on $[0, +\infty)$, $\hbar = \sum_I \hbar_I \,\mathrm{d}x_I$ be a differential form on $\Theta \subset \mathbb{R}^n$, and let all the ordered ℓ-tuples I satisfy $\hbar_I \in L^\infty_c$. Suppose that F_α is a fractional convolution operator on differential forms and its kernel function K_α satisfies $K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$ and $K_\alpha \in C^\infty_0(\Theta)$, where $0 < \alpha < n$. Then there exists a constant C such that*

$$\big\|F_\alpha\hbar - (F_\alpha\hbar)_{O_0}\big\|_{p,\Theta} \le C|\Theta| \operatorname{diam}(\Theta)\big\|M_{\alpha,\bar\varphi} d\hbar(x)\big\|_{p,\Theta} \tag{2.22}$$

for any $1 < p < \infty$ and O_0 is a fixed ball in Θ.

Proof By the definition of the $L^p(m)$-average domain and noticing that $1 - 1/p \ge 0$, we have

$$\begin{aligned}
&\big\|F_\alpha\hbar - (F_\alpha\hbar)_{O_0}\big\|_{p,\Theta}\\
&\quad\le C_1|\Theta|^{1/p} \sup_{O\subset\Theta} |O|^{-1/p}\big\|F_\alpha\hbar - (F_\alpha\hbar)_O\big\|_{p,O}\\
&\quad\le C_1|\Theta|^{1/p} \sup_{O\subset\Theta} |O|^{-1/p} C_2 \operatorname{diam}(O)|O|\big\|M_{\alpha,\bar\varphi}(d\hbar)\big\|_{p,O}\\
&\quad\le C_3|\Theta|^{1/p} \sup_{O\subset\Theta} |O|^{1-1/p} \operatorname{diam}(O)\big\|M_{\alpha,\bar\varphi}(d\hbar)\big\|_{p,O}\\
&\quad\le C_3|\Theta|^{1/p} \sup_{O\subset\Theta} |\Theta|^{1-1/p} \operatorname{diam}(\Theta)\big\|M_{\alpha,\bar\varphi}(d\hbar)\big\|_{p,\Theta}\\
&\quad= C_3\Theta| \operatorname{diam}(\Theta)\big\|M_{\alpha,\bar\varphi}(d\hbar)\big\|_{p,\Theta}.
\end{aligned} \tag{2.23}$$

□

3 The Lipschitz and BMO norm inequalities for the fractional convolution operator

It is well known that Lipschitz and BMO norms are two kinds of important norms in differential forms, which can be found in [14]. Now we recall these definitions as follows. Let $\hbar \in L^1_{\mathrm{loc}}(\Theta, \bigwedge^\ell)$, $\ell = 0, 1, \ldots, n$. We write $\hbar \in \mathrm{locLip}_k(\Theta, \bigwedge^\ell)$, $0 \le k \le 1$, if

$$\|\hbar\|_{\mathrm{locLip}_k,\Theta} = \sup_{\rho O\subset\Theta} |O|^{-(n+k)/n}\|\hbar - \hbar_O\|_{1,O} < \infty \tag{3.1}$$

for some $\rho \ge 1$.

Further, we write $\mathrm{Lip}_k(\Theta, \bigwedge^\ell)$ for those forms whose coefficients are in the usual Lipschitz space with exponent k and write $\|\hbar\|_{\mathrm{Lip}_k,\Theta}$ for this norm. Similarly, for $\hbar \in$

$L^1_{\mathrm{loc}}(\Theta, \bigwedge^\ell)$, $\ell = 0, 1, \ldots, n$, we write $\hbar \in BMO(\Theta, \bigwedge^\ell)$ if

$$\|\hbar\|_{\star,\Theta} = \sup_{\rho O \subset \Theta} |O|^{-1} \|\hbar - \hbar_O\|_{1,O} < \infty \tag{3.2}$$

for some $\rho \geq 1$.

When $\hbar$ is a 0-form, Eq. (3.2) reduces to the classical definition of $BMO(\Theta)$.

Lemma 3.1 (see [10]) *Let $0 < p, q < \infty$ and $1/s = 1/p + 1/q$. If f and g are two measurable functions on $\mathbb{R}^n$, then*

$$\|fg\|_{s,\Theta} \leq \|f\|_{p,\Theta} \|g\|_{q,\Theta} \tag{3.3}$$

for any $\Theta \subset \mathbb{R}^n$.

Theorem 3.1 *Let φ be a Young function on $[0, +\infty)$, $\hbar = \sum_I \hbar_I \, dx_I$ be a differential form on $\Theta \subset \mathbb{R}^n$, and let all the ordered ℓ-tuples I satisfy $\hbar_I \in L^\infty_c$. Suppose that F_α is a fractional convolution operator on differential forms and its kernel function K_α satisfies $K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$ and $K_\alpha \in C^\infty_0(\Theta)$, where $0 < \alpha < n$. Then, for any $1 < p < \infty$, there exists a constant C such that*

$$\|F_\alpha \hbar\|_{\mathrm{locLip}_k,\Theta} \leq C \left\| M_{\alpha,\bar{\varphi}}(d\hbar) \right\|_{p,\Theta}, \tag{3.4}$$

where k is a constant with $0 \leq k \leq 1$.

Proof By Theorem 2.4, we obtain

$$\left\| F_\alpha \hbar - (F_\alpha \hbar)_O \right\|_{p,O} \leq C|O| \operatorname{diam}(O) \left\| M_{\alpha,\bar{\varphi}} d\hbar(x) \right\|_{p,O}. \tag{3.5}$$

By Lemma 3.1 with $1 = 1/p + (p-1)/p$, for any ball with $O(O \subset \Theta)$, we have

$$\begin{aligned}
&\left\| F_\alpha \hbar - (F_\alpha \hbar)_O \right\|_{1,O} \\
&\quad = \int_O \left| F_\alpha \hbar - (F_\alpha \hbar)_O \right| \, dx \\
&\quad \leq \left(\int_O \left| F_\alpha \hbar - (F_\alpha \hbar)_O \right|^p \, dx \right)^{1/p} \left(\int_O 1^{\frac{p}{p-1}} \, dx \right)^{(p-1)/p} \\
&\quad = |O|^{(p-1)/p} \left\| F_\alpha \hbar - (F_\alpha \hbar)_O \right\|_{p,O} \\
&\quad = |O|^{1-1/p} \left\| F_\alpha \hbar - (F_\alpha \hbar)_O \right\|_{p,O} \\
&\quad \leq |O|^{1-1/p} C_1 |O| \operatorname{diam}(O) \left\| M_{\alpha,\bar{\varphi}} d\hbar(x) \right\|_{p,O} \\
&\quad \leq C_2 |O|^{2-1/p+1/n} \left\| M_{\alpha,\bar{\varphi}} d\hbar(x) \right\|_{p,O}.
\end{aligned} \tag{3.6}$$

By the definition of the Lipschitz norm and $2 - 1/p + 1/n - 1 - k/n = 1 - 1/p + 1/n - k/n > 0$, we obtain

$$\|F_\alpha \hbar\|_{\mathrm{locLip}_k,\Theta}$$

$$\begin{aligned}
&= \sup_{\rho O\subset\Theta} |O|^{-(n+k)/n}\big\|F_\alpha \hbar - (F_\alpha \hbar)_O\big\|_{1,O} \\
&= \sup_{\rho O\subset\Theta} |O|^{-1-k/n}\big\|F_\alpha \hbar - (F_\alpha \hbar)_O\big\|_{1,O} \\
&\le \sup_{\rho O\subset\Theta} |O|^{-1-k/n}C_2|O|^{2-1/p+1/n}\big\|M_{\alpha,\tilde{\varphi}}d\hbar(x)\big\|_{p,O} \\
&= \sup_{\rho O\subset\Theta} C_2|O|^{1-1/p+1/n-k/n}\big\|M_{\alpha,\tilde{\varphi}}d\hbar(x)\big\|_{p,O} \\
&\le \sup_{\rho O\subset\Theta} C_2|\Theta|^{1-1/p+1/n-k/n}\big\|M_{\alpha,\tilde{\varphi}}d\hbar(x)\big\|_{p,O} \\
&\le C_3 \sup_{\rho O\subset\Theta} \big\|M_{\alpha,\tilde{\varphi}}d\hbar(x)\big\|_{p,O} \\
&\le C_3\big\|M_{\alpha,\tilde{\varphi}}d\hbar(x)\big\|_{p,\Theta}.
\end{aligned} \tag{3.7}$$

□

Lemma 3.2 (see [14]) *If the differential form $\hbar \in \mathrm{locLip}_k(\Theta, \Lambda^\ell)$, $\ell = 0, 1, \dots, n$, $0 \le k \le 1$, is defined in a bounded convex domain Θ, then $\hbar \in BMO(\Theta, \Lambda^\ell)$ and there exists a constant C such that*

$$\|\hbar\|_{\star,\Theta} \le C\|\hbar\|_{\mathrm{locLip}_k,\Theta}. \tag{3.8}$$

By Theorem 3.1 and Lemma 3.2, we get the following conclusion.

Theorem 3.2 *Let φ be a Young function on $[0, +\infty)$, $\hbar = \sum_I \hbar_I \, \mathrm{d}x_I$ be a differential form on $\Theta \subset \mathbb{R}^n$, and let all the ordered ℓ-tuples I satisfy $\hbar_I \in L_c^\infty$. Suppose that F_α is a fractional convolution operator on differential forms and its kernel function K_α satisfies $K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$ and $K_\alpha \in C_0^\infty(\Theta)$, where $0 < \alpha < n$. Then, for any $1 < p < \infty$, there exists a constant C such that*

$$\|F_\alpha \hbar\|_{\star,\Theta} \le C\big\|M_{\alpha,\tilde{\varphi}}(d\hbar)\big\|_{p,\Theta}. \tag{3.9}$$

4 Applications

With regard to the applications of the fractional convolution operator, we will point out that Theorem 2.2 has different expression forms.

Definition 4.1 (see [7]) Let $K_\alpha(x)$ be a function defined on $\mathbb{R}^n$, if there exist two constants $c \ge 1$ and $C > 0$ such that

$$\big|K_\alpha(x-y) - K_\alpha(x)\big| \le C\frac{|y|}{|x|^{n+1-\alpha}}, \quad |x| > c|y|, \tag{4.1}$$

then the kernel function K_α is said to satisfy the $H^*_{\alpha,\infty}$-condition.

Lemma 4.1 (see [7]) *Let φ be any Young function defined on $[0, +\infty)$, then $H^*_{\alpha,\infty} \subset H_{\alpha,\varphi}$.*

Theorem 4.1 *Let $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$ and $0 \le \alpha < n$, then $K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$.*

Proof Firstly prove that $K_\alpha \in S_\alpha$. By the definition of K_α, we have

$$\int_{|x|\sim s} |K_\alpha(x)| \,\mathrm{d}x \le \int_{O(0,2s)} \frac{1}{|x|^{n-\alpha}} \,\mathrm{d}x \le \sigma_n (2s)^n \cdot (s)^{\alpha-n} = 2^n \sigma_n s^\alpha, \tag{4.2}$$

where σ_n is the volume of a unit sphere n in $\mathbb{R}^n$. Thus $K_\alpha \in S_\alpha$.

Secondly prove that $K_\alpha \in H_{\alpha,\varphi}$. According to Lemma 4.1, we only need to prove that $K_\alpha \in H^*_{\alpha,\infty}$. If we choose $|x| > 2|y|(c = 2 \ge 1)$ and $y \neq O = (0,\ldots,0)$ (for $y = O$, it is clearly established), it follows that

$$|x-y|/|x| \in \begin{cases} (\frac{1}{2}, 1), & x, y \text{ each component has the same sign;} \\ (1, \frac{3}{2}), & x, y \text{ each component has the different sign,} \end{cases}$$

where $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n)$. Considering that each component of x, y is greater than zero, other cases may be considered similarly. By Lagrange's mean value theorem

$$\begin{aligned} &\bigl|K_\alpha(x-y) - K_\alpha(x)\bigr| \\ &\quad = \bigl||x-y|^{\alpha-n} - |x|^{\alpha-n}\bigr| \\ &\quad = |x|^{\alpha-n}\left|\left(\frac{|x-y|}{|x|}\right)^{\alpha-n} - 1\right| \\ &\quad \le (n-\alpha)|x|^{\alpha-n}\left(\frac{|y|}{|x|}\right)\bigl(|\xi|\bigr)^{\alpha-n-1} \\ &\quad \le 2^{n+1-\alpha}(n-\alpha)\frac{|y|}{|x|^{n+1-\alpha}}, \end{aligned} \tag{4.3}$$

where

$$|x| = \sqrt{\sum_{i=1}^n x_i^2}, \xi = (\xi_1,\ldots,\xi_n),$$

$$\frac{1}{2} < \min\left\{\frac{|x-y|}{|x|}, 1\right\} < |\xi| < \max\left\{\frac{|x-y|}{|x|}, 1\right\} < \frac{3}{2}.$$

Thus $K_\alpha \in H^*_{\alpha,\infty}$. □

Theorems 2.2 and 4.1 yield the following.

Theorem 4.2 *Let φ be any Young function defined on $[0,+\infty)$ and $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$, then the fractional convolution operator F_α in (1.1) becomes the classical Riesz potential operator*

$$I_\alpha \hbar(x) = \sum_I \left(\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \hbar_I(y) \,\mathrm{d}y\right) \mathrm{d}x_I, \tag{4.4}$$

where $\hbar = \sum_I \hbar_I \,\mathrm{d}x_I$ is a differential form in $\mathbb{R}^n$ and such that $\hbar_I \in L^\infty_c$ for all the ordered ℓ-tuples I. Then there exists a constant C such that

$$\int_{\mathbb{R}^n} \bigl|I_\alpha \hbar(x)\bigr|^p w(x) \,\mathrm{d}x \le C \int_{\mathbb{R}^n} \bigl[M_{\alpha,\bar{\varphi}} \hbar(x)\bigr]^p w(x) \,\mathrm{d}x \tag{4.5}$$

for any $0 < p < \infty$ and $w \in A_\infty$.

Lemma 4.2 (see [7]) *Denote by S^{n-1} the unit sphere of $\mathbb{R}^n$, Ω is a homogeneous function defined on S^{n-1} with $\Omega(x) = \Omega(x')$ and the kernel function $K_\alpha(x) = \Omega(x)/|x|^{n-\alpha}(x \neq 0)$, where $x' = x/|x|(x \neq 0)$. Given a Young function φ, we define the $Ł^\varphi$-modulus of continuity of Ω as*

$$\varpi_\varphi(t) = \sup_{|y|\le t} \|\Omega(\cdot + y) - \Omega(\cdot)\|_{\varphi(S^{n-1})} \tag{4.6}$$

and write $\Omega \in Ł^\varphi(S^{n-1})$. If

$$\int_0^1 \varpi_\varphi(t)\frac{\mathrm{d}t}{t} < \infty, \tag{4.7}$$

then $K_\alpha \in S_\alpha \cap H_{\alpha,\varphi}$.

By Theorem 2.2 and Lemma 4.2, we have the following.

Theorem 4.3 *Let φ be a Young function, Ω be a homogeneous function in S^{n-1} with $\Omega(x) = \Omega(x')$ and $\Omega \in Ł^\varphi(S^{n-1})$. Suppose that F_α is the fractional convolution operator with its kernel function $K_\alpha(x) = \Omega(x)/|x|^{n-\alpha}$. Let $\hbar = \sum_I \hbar_I \,\mathrm{d}x_I$ be a differential form in $\mathbb{R}^n$ with $\hbar_I \in L_c^\infty$ for all the ordered ℓ-tuples I. If $\int_0^1 \varpi_\varphi(t)\frac{\mathrm{d}t}{t} < \infty$, then there exists a constant C such that*

$$\int_{\mathbb{R}^n} |F_\alpha \hbar(x)|^p w(x)\,\mathrm{d}x \le C\int_{\mathbb{R}^n} [M_{\alpha,\bar{\varphi}}\hbar(x)]^p w(x)\,\mathrm{d}x \tag{4.8}$$

for any $0 < p < \infty$ and $w \in A_\infty$.

Funding
This work is supported by Xi'an Technological University president Fund Project (XAGDXJJ16019). The authors wish to thank the anonymous referees for their time and thoughtful suggestions.

Competing interests
The authors declare that there are no competing interests regarding the publication of this article.

Authors' contributions
All results and investigations of this manuscript were due to the joint efforts of all authors. ZD finished the proof and the writing work. HL gave ZD some excellent advices in the proof and writing. QL took part in the original conceiving and discussion, and carefully checked the second amendment of the English writing and the proof in this article. All authors read and approved the final manuscript.

Author details
[1]School of Science, Xi'an Technological University, Xi'an, China. [2]School of Science, Jiangxi University of Science and Technology, Ganzhou, China. [3]Department of Mathematics, Ganzhou Teachers' College, Ganzhou, China.

Publisher's Note

Received: 9 December 2017 Accepted: 11 June 2018 Published online: 16 July 2018

References
1. Bi, H.: Weighted inequalities for potential operators on differential forms. J. Inequal. Appl. **2010**, Article ID 713625 (2010). https://doi.org/10.1155/2010/713625
2. Tang, Z., Zhu, J.: Carathéodory operator of differential forms. J. Inequal. Appl. **2011**, Article ID 88 (2011). https://doi.org/10.1186/1029-242X-2011-88
3. Tang, Z., Zhu, J.: Estimates for the composition of the Carathéodory and homotopy operators. J. Inequal. Appl. **2012**, Article ID 188 (2012). https://doi.org/10.1186/1029-242X-2012-188
4. Dai, Z., Xing, Y., Ding, S., Wang, Y.: Inequalities for the composition of Green's operator and the potential operator. J. Inequal. Appl. **2012**, Article ID 271 (2012). https://doi.org/10.1186/1029-242X-2012-271

5. Dai, Z., Xing, Y., Ding, S.: BMO and Lipschitz norm estimates for the composition of Green's operator and the potential operator. J. Inequal. Appl. **2013**, Article ID 26 (2013). https://doi.org/10.1186/1029-242X-2013-26
6. Wang, Y., Xing, Y.: Inequalities for two type potential operators on differential forms. J. Math. Anal. Appl. **417**, 774–786 (2014). https://doi.org/10.1016/j.jmaa.2014.03.048
7. Bernardis, A.L., Lorente, M., Riveros, M.S.: Weighted inequalities for fractional integral operators with kernel satisfying Hörmander type conditions. Math. Inequal. Appl. **14**(4), 881–895 (2011). https://doi.org/10.7153/mia-14-73
8. Riveros, M.S.: Weighted inequalities for generalized fractional operators. Rev. Unión Mat. Argent. **49**(2), 29–38 (2008)
9. Iwaniec, T., Lutoborski, A.: Integral estimates for null Lagrangians. Arch. Ration. Mech. Anal. **125**, 25–79 (1993). https://doi.org/10.1007/BF00411477
10. Agarwal, R.P., Ding, S., Nolder, C.A.: Inequalities for Differential Forms. Springer, New York (2009). https://doi.org/10.1007/978-0-387-68417-8
11. Rao, M., Ren, Z.: Theory of Orlicz Spaces. Monographs and Textbooks in Pure and Applied Mathematics, vol. 146. Dekker, New York (1991)
12. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1997). https://doi.org/10.1007/978-3-642-61798-0
13. Staples, S.G.: L^p-Averaging domains and the Poincaré inequality. Ann. Acad. Sci. Fenn., Ser. A 1 Math. **14**, 103–127 (1989)
14. Ding, S.: Lipschitz and BMO norm inequalities for operators. Nonlinear Anal., Theory Methods Appl. **71**, e2350–e2357 (2009). https://doi.org/10.1016/j.na.2009.05.032